BINARY EQUALITIES AND HARMONIOUS QUARTETS

A mathematical vignette

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§1. Harmonious quartets

The quartet $(a, b; c, d) = (2, 4; 4, 2)$ is particularly harmonious. Its first two and last two entries have the same sum, product and exponential; in particular, the exponential operator turns out to be commutative in this case $2^4 = 4^2$. Such harmony is generally not attainable by quartets of integers, by we can nevertheless encounter some interesting tunes.

Specifically, we are going to consider 4-tples $(a, b; c, d)$ of positive integers for which $a < c$, $b > d$ and satify at least two of the three following properties:

A: $a + b = c + d$;

M: $ab = cd$;

E: $a^b = c^d$.

Define an **AM** quartet to be a 4-tple $(a, b; c, d)$ that satifies **A** and **M** simultaenously, and **AE** and **ME** quartets similarly. All quartets of these types make up the class of *harmonious quartets*. There are trivial **AM** quartets found by taking c and d to be a and b in some order. We exclude these from further consideration. Such a device is not generally for harmonious quartets involving exponentiation since the operation is not commutative.

Exercise 1. Show that, in any hrmonious quartet, $a > 1$.

Exercise 2. It is quite straightforward to determine all the **AM** quartets. Multiple the equation **A** by a and use **M** to obtain the equation $(a - c)(b - d) = 0$. Alternatively, observe that the pairs (a, b) and (c, d) satisfy the same quadratic equation.

Exercise 3. Suppose that a, b, c, d satisfy equation **E**. Prove that there is are positive integers m, r, s for which the greatest common divisor of r and s is 1, $r < s$ and

$$a = m^r; \qquad b = m^s; \qquad rb = sd.$$

Exercise 4. Before going further, we check how much leeway we have for commutativity of exponentiation. Suppose that $a^b = b^a$ with $a < b$. Apply Exercise 4 to obtain $a = m^r$, $b = m^s$ and obtain $m^{s-r} = s/r$. What are the possible values for m, r and s?

Exercise 5. Suppose that $(a, b; c, d)$ is a **ME** quartet. Determine the triple $(m; r, s)$ as in Exercise 2 and show that $rm^{s-r} = s$, Deduce that $r = 1$ and show that $(a, b; c, d)$ must have the form $(m, sd; m^s, d)$ and in addition satisfy $m^{s-1} = s$.

Observe that $2^{s-1} \leq m^{s-1}$, check that $s-1 \leq 2^{s-1}$ for all values of $s \geq 2$, and find all of the **ME** quartets. ♠

It remains to investigate **AE** quartets. As Exercise 4 indicates, we may take $a = m^r$ and $c = m^s$ where r and s are coprime and $1 \leq r < s$. Equation **E** implies that $br = ds$; let k be the common value.

Exercise 6. From equation **A**, deduce that

$$k(s-r) = rsm^r(m^{s-r} - 1).$$

Therefore, any **AE** quartet must have the form

$$(a, b; c, d) = (m^r, (s-r)^{-1}sm^r(m^{s-r} - 1); m^s, (s-r)^{-1}rm^r(m^{s-r} - 1)),$$

where $s-r$ is a divisor of $m^r(m^{s-r} - 1)$.

Conversely, verify that for any choice of $(m; r, s)$ for which $s-r$ divides $m^r(m^{s-r} - 1)$, we obtain a **AE** quartet. ♠

In particular, when $s = r+1$, we obtain a **AE** quartet, so that there are infinitely many solutions to this equation. However, there are multitudes of solutions where $s - r$ exceeds 1.

Exercise 7. As we see in Exercise 6, we can generate many **AE** sets according to pairs (m, n) for which n is a dividor of $m^n - 1$. Prove that, if m is odd, and n divides $m^n - 1$, then $2n$ must divide $m^{2n} - 1$. Determine infinitely may values of n for which n divides $3^n - 1$ and use this to generate infinitely many **AE** quartets for which $m = 3$.

Exercise 8. Determine all the **AE** quartets $(a, b; c, d)$ for which $a+b = c+d \leq 100$. ♠

§2. Binary equalities

Exercise 9. Sketch the graph of all those real points (x, y) for which $x + y = xy$.

Exercise 10. Sketch the graph of all those positive real points (x, y) for which $x + y = x^y$.

Exercise 11. Sketch the graph of all those positive real points (x, y) for which $xy = x^y$.

Exercise 12. Sketch the graph of all those positive real points (x, y) for which $x^y = y^x$.

Notes. In Exercise 3, note that a and c are divisible by exactly the same set P of primes, so that $a = \prod p^i$ and $c = \prod p^j$, where the products are taken over P. Equation **E** and the uniqueness of prime factorization applied to $\prod p^{ib} = a^b = c^d = \prod p^{jd}$ forces $bi = dj$ for every pair (i, j) of exponents. Thus for every prime $p \in P$, the exponents are in the ratio $d : b$. Let $r : s$ be the proportional ratio in lowest terms (r and s are coprime). Then i/r and j/s are equal integers. Let

$$m = \prod_P p^{i/r} = \prod_P p^{j/s}.$$

Then $a = m^r$ and $b = m^s$.

For Exercise 5, since $s \leq m^{s-1}$ with equality if and only if $m = 2$ and $s = 2$, the only ME quartets are of the form $(a, b; c, d) = (2, 2d; 4, d)$ where d is a positive integer.

For Exercise 8, $c = m^s$ in particular must be less than 100. Since $s \geq 2$, this forces m to be less than 10. If $5 \leq m \leq 9$, then $(r, s) = (1, 2)$ Since $a + b = m(1 + 2(m - 1)) < 100$, this forces $m \leq 7$.

Here are the required **AE** quartets:

$(m; r, s)$	$(a, b; c, d)$	$a + b = c + d$
(2; 1, 2)	(2, 4; 4, 2)	6
(2; 1, 3)	(2, 9; 8, 3)	11
(3; 1, 2)	(3, 12; 9, 6)	15
(2; 2, 3)	(4, 12; 8, 8)	16
(2; 2, 4)	(4, 24; 16, 12)	28
(3; 1, 3)	(3, 36; 27, 12)	39
(2; 3, 4)	(8, 32; 16, 24)	40
(5; 1, 2)	(5, 40; 25, 20)	45
(3; 2, 3)	(9, 54; 27, 36)	63
(6; 1, 2)	(6, 60; 36, 30)	66
(2; 3, 5)	(8, 60; 32, 36)	68
(7; 1, 2)	(7, 84; 49, 42)	91
(4; 1, 3)	(4, 90; 64, 30)	94
(2; 4, 5)	(16, 80; 32, 64)	96

The equation $xy = x + y$ in Exercise 9 can be rewritten $(x - 1)(y - 1) = 1$, so that the locus is a rectangular hyperbola with centre $(1, 1)$ and axes given by $|y| = |x|$.

In Exercise 11, the equation can be rewritten as $x = \exp((y - 1)^{-1} \log y)$.

In Exercise 12, the equation can be rewritten as

$$\frac{\log y}{y} = \frac{\log x}{x}.$$

The locus includes the line $y = x$ and also real points (x, y) where one coordinate lies in the intercal $(1, e)$ and the other in (e, ∞). This can be seen by examining the graph of the function $t^{-1} \log t$ and checking for where it takes the same value at two distinct points.