Round-robin Tournaments.

A mathematical vignette

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1. Round-robin tournaments.

A *round-robin* tournament is a set of matches between pairs of n teams, in which each team plays each other team exactly once. We will suppose that, at each match, one team wins and the other loses; there are no ties. Tennis matches are an example.

Suppose that the ith team has w_i wins and d_i losses; it is clear that $w_i+d_i = n-1$ and that

$$\sum_{i=1}^n w_i = \sum_{i=1}^n d_i = \binom{n}{2} = \frac{n(n-1)}{2},$$

the number of matches.

In fact, there is the striking result that

$$\sum_{i=1}^n w_i^2 = \sum_{i=1}^n d_i^2.$$

To see this, note that

$$\begin{aligned}\sum_{i=1}^n w_i^2 &= \sum_{i=1}^n [(n-1)-d_i]^2 \\ &= n(n-1)^2 - 2(n-1)\sum_{i=1}^n d_i + \sum_{i=1}^n d_i^2 \\ &= n(n-1)^2 - 2(n-1)\left(\frac{n(n-1)}{2}\right) + \sum_{i=1}^m d_i^2 = \sum_{i=1}^n d_i^2.\end{aligned}$$

There are a number of things that can be investigated:

(1) If we have set $\{w_i\}$ and $\{d_i\}$ of positive integers for which $w_i + d_i = n-1$ and $\sum w_i = \sum d_i = \binom{n}{2}$, does this necessarily correspond to a tournament?

(2) Is there anything that can be said about the common sum of the w_i^2 and d_i^2. For example, what are their minimum and maximum values?

(3) How many different set $\{w_i\}$ arise out of a tournament. For how many of them are the sets $\{w_i\}$ and $\{d_i\}$ the same?

2. A familiar relationship.

The conditions $\sum_{i=1}^n w_i = \sum_{i=1}^n d_i = n-1$ and $\sum_{i=1}^n w_i = \sum_{i=1}^n d_i = n(n-1)/2$, alone, are enough to ensure that $\sum_{i=1}^n w_i^2 = \sum_{i=1}^n d_i^2$. The w_i and d_i do not have to arise from a tournament, nor even all be positive integers.

Consider the following situation, where for convenience we replace n by $n+1$. Let

$$w_0 = \binom{n+1}{2},$$

and

$$d_0 = n - \binom{n+1}{2} = -\binom{n}{2}.$$

For $1 \leq i \leq n$, let $w_i = 0$ and $d_i = n$. Then $\sum w_i = \sum d_i = \binom{n+1}{2}$ and $w_i + d_i = n$. Then the equality of the square sums yields

$$\sum_{i=1}^n \binom{n+1}{2}^2 = \sum_{i=1}^n \binom{n}{2}^2 + n^3,$$

for $n \geq 1$. This is equivalent to the identity

$$\sum_{k=1}^n k^3 = \binom{n+1}{2}^2 = (1+2+\cdots+n)^2.$$

3. Appendix: sets of win numbers

The following list gives possible sets of winning numbers for n teams, along with the sums of their square. Where two are listed in a bracket, the second is the first when losses are converted to wins.

$n = 3$: $(2,1,0;5), (1,1,1;3)$

$n = 4$: $(3,2,1,0;14), (3,1,1,1;2,2,2,0;12), (2,2,1,1;10)$

$n = 5$: $(4,3,2,1,0;30), (4,3,1,1,1;3,3,3,1,0;28), (4,2,2,2,0;28)$
$(4,2,2,1,1;3,3,2,2,0;26), (3,3,2,1,1;24), (3,2,2,2,1;22), (2,2,2,2,2;20)$

$n = 6$: $(5,4,3,2,1,0;55), (5,4,3,1,1,1;4,4,4,2,1;53), (5,4,2,2,2,0;5,3,3,1,0;53),$
$(5,4,2,2,1,1;4,4,3,3,1,0;51), (5,3,3,2,2,0;51), (4,4,4,1,1,1;51), (5,3,3,2,1,1;4,4,3,2,2,0;49)$
$(5,3,2,2,2,1;4,3,3,3,2,0;47), (5,2,2,2,2,2;3,3,3,3,3,0;45), (4,3,3,2,2,1;43)$
$(4,3,2,2,2,2;3,3,3,3,2,1;41), (3,3,3,2,2,2;39)$