USING ALL THE DIGITS.

A mathematical vignette

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In order to get students engaged in the properties and processes of ordinary arithmetic, so that they can get an intimate feel for the underlying mechanisms, it is useful to assign tasks that involve trying things out. It is also useful if the answer is not unique, so that you can introduce a competitive aspect, either students vying with each other answers that are different or optimal in some way, or with themselves, as they try to improve on what they have already found out.

Sums involving all the digits.

(1) Using each of the nine nonzero digits exactly once, form three numbers A, B and C for which $A + B = C$.

The three numbers have nine digits altogether. We will suppoae that A is the smallest number. It can be seen that the sum C cannot have more than four digits and that in this case, the first digit must be 1. When you consider that, for a four-digit sum, the second digit must be 0, you see that this case is impossible. Therefore, you conclude that C, as well as A and B must have three digits.

In the appendix, we will give some examples, with no claim that we have found all the possibilities. What are the largest and smallest possible sums?

(2) Using each of the ten digits exactly once, form three numbers D, E, F for which $D + E = F$.

Here, the sum F must have exactly four digits. (Why cannot it have three or five digits?) However, there are two possibilities for the number of digits in the addends. Either D has two and E four digits, or D and E both have three digits.

If both D and E have three digits, then the sum F must lies between 1023 and 1100. If D has two and E has four digits, then the second digit of E must be 9 and the second digit of F must be 0. In this case F lies between 2000 and 8100.

Some examples are given in the appendix.

Numbers with a rolling divisibility property.

The number 123654 has the property that the number formed by its first digit, 1, is divisible by 1; by its first two digits is divisible by 2; by it first three digits divisible by 3; its first four digits divisible by 4; its first five digits divisible by 5; its first six digits divisible by 6; counting from the left. Moreover, each of the smalles six nonzero digits is used exactly once. What other numbers have the analogous property? Can you find a nine-digit number using all the nine nonzero digits for which the number formed by the first k digits is divisible by k for $1 \leq k \leq 9$?

The number 120 has the property that the number formed by its first digit, 1, is divisible by 1; the number formed by its first two digits 12 is divisible by 2; the number formed by its first three digits 120 is divisible by 3. Moreover, it is formed by the smallest three digits 0, 1, 2, each used exactly once. What other numbers have the analogous property? Can you find a ten-digit number?

You can make some observations to help fill in the digits. Divisibility by 1 is always possible. Any multiple of an even number is even. Any multiple of 5 ends in 5 or 0; if you are looking for a 10-digit number, the fifth and tenth digits are determined. For the ten-digit number, the number formed by the first nine digits is always divisible by 9. A number is a multiple of 4 if and only if the number formed by the last two digits is divisible by 4. A number is divisible by 8 if and only if the number formed by its last three digits is divisible by 8. A number is divisible by 3 if and ony if the sum of its digits is divisible by 3.

Appendix

Examples for $A + B = C$ with nine digits.

$$\begin{aligned}(A, B; C) &= (173, 286; 459), (173, 295; 468), (129, 357; 486), (127, 368; 495),\\ &= (182, 394; 576), (182, 493; 675), (124, 659; 783), (317, 628; 945), (216, 738; 954).\end{aligned}$$

Examples of $D + E = F$ with ten digits.

$$\begin{aligned}(D, E; F) = (437, 589; 1026), (246, 789; 1035), (264, 789; 1053), (432, 657; 1089), (423, 675; 1098),\\ (56, 1987; 2043), (65, 1978; 2043), (26, 4987; 5013), (34 + 5978; 6012).\end{aligned}$$

The only ten-digits number with the divisibility propoerty is 3816547290.