CONSECUTIVE MULTIPLES AFTER CONSECUTIVE SQUARES.

A mathematical vignette

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$36 = 6^2$ and $49 = 7^2$ are two consecutive squares. If we add 3 to each of the numbers, we get 39 and 52, which are two consecutive multiple of 13. We ask the question:

Is it possible, given two consecutive squares, by adding the same small number to each of them, we can get two consecutive multiples of some number?

Experimenting with other pairs of consecutive squares, we can find other examples, such as $4^2 + 2 = 2 \times 9$ and $5^2 + 2 = 3 \times 9$.

Think about this for for a while before reading on.

This is one of those situations that become clear once we make a couple of key observations. This first is that the difference between the multiples has to be the same as the difference between the squares. Call this difference X. The second is that the difference between the multiples, since they are consecutive, must be X. So the problem reduces to find a multiple of X between the two squares?

How do we know that there is such a multiple? There are $X + 1$ consecutive numbers between the two squares (counting the squares themselves); so one of these has to be a multiple of X. Going back to our initial example, since the difference between 49 and 36 is 13, we look for a multiple of 13 between 36 and 49; this is $39 = 36 + 3$, Sure enough, when we add 3 to 49, we obtain $52 = 49 + 3$, the next higher multiple of 13.

There is another way of approach this as well. If we add -36 to both 36 and 49, we get 0 and 13, two conseutive multiples of 13. Adding 13 to the number we add to the squares gives again two consecutive multiples of 13, namely $36 - 23 = 13$ and $49 - 23 + 26$. Adding 13 again gives us $36 - 10 = 26$ and $49 - 10 = 39$; we can thus work our way up.

Having satisfied ourselves that we take any consecutive pair of squares and add a number to get consecutive multiples, we can ask if there is a convenient formula for the number we add.

We can ask the same question for cubes, or higher powers, instead of squares. Indeed, we can investigate the situation for any sequence of numbers.

The mathematics involved this will give students practice in criticially examining information to arrive at some formula and in doing the manipulations to check that the formula works.

Appendix.

For consecutive squares, for obtaining a convenient form, we have to consider parity. Thus, we find that, when the smaller square is even,

$$(2a)^2 + a = (4a+1)a$$

and

$$(2a+1)^2 + a = (4a+1)(a+1).$$

When the smaller square is odd, we have that

$$(2a-1)^2 + (3a-1) = (4a-1)a$$

and

$$(2a)^2 + (3a-1) = (4a-1)(a+1).$$

For cubes, it is convenient to consider three cases:

$$(3a+1)^3 - (3a)^3 = 27a^2 + 9a + 1;$$
$$(3a)^3 + (9a^2 + a) = (27a^2 + 9a + 1)a;$$
$$(3a+1)^3 + (9a^2 + a) = 27a^3 + 36a^2 + 10a + 1 = (27a^2 + 9a + 1)(a+1).$$

$$(3a+2)^3 - (3a+1)^3 = 27a^2 + 27a + 7;$$
$$(3a+1)^3 + (27a^2 + 25a + 6) = 27a^3 + 54a^2 + 34a + 7 = (27a^2 + 27a + 7)(a+1);$$
$$(3a+2)^3 + (27a^2 + 25a + 6) = 27a^3 + 81a^2 + 61a + 14 = (27a^2 + 27a + 7)(a+2).$$

$$(3x+3)^2 - (3x+2)^2 = 27a^2 + 45a + 19;$$
$$(3a+2)^3 + (18a^2 + 28a + 11) = 27a^2 + 72a^2 + 64a + 19 = (27a^2 + 45a + 19)(a+1);$$
$$(3a+3)^3 + (18a^2 + 28a + 11) = 27a^3 + 99a^2 + 109a + 38 = (27a^2 + 45a + 19)(a+2).$$