THE UNIVERSITY OF TORONTO
UNDERGRADUATE MATHEMATICS COMPETITION

In Memory of Robert Barrington Leigh

March 9, 2008

Time: $3\frac{1}{2}$ hours

`No aids or calculators permitted.`

It is not necessary to do all the problems. Complete solutions to fewer problems are preferred to partial solutions to many.

1. Three angles of a hepatagon (7-sided polygon) inscribed in a circle are equal to $120°$. Prove that at least two of its sides are equal.

2. (a) Determine a real-valued function g defined on the real numbers that is decreasing and for which $g(g(x)) = 2x + 2$.

   (b) Prove that there is no real-valued function f defined on the real numbers that is decreasing and for which $f(f(x)) = x + 1$.

3. Suppose that a is a real number and the sequence $\{a_n\}$ is defined recursively by $a_0 = a$ and

$$a_{n+1} = a_n(a_n - 1)$$

   for $n \geq 0$. Find the values of a for which the sequence $\{a_n\}$ converges.

4. Suppose that u, v, w, z are complex numbers for which $u + v + w + z = u^2 + v^2 + w^2 + z^2 = 0$. Prove that

$$(u^4 + v^4 + w^4 + z^4)^2 = 4(u^8 + v^8 + w^8 + z^8) \ .$$

5. Suppose that $a, b, c \in \mathbf{C}$ with $ab = 1$. Evaluate the determinant of

$$\begin{pmatrix} c & a & a^2 & \cdots & a^{n-1} \\ b & c & a & \cdots & a^{n-2} \\ b^2 & b & c & \cdots & a^{n-3} \\ \vdots & \vdots & \vdots & & \vdots \\ b^{n-1} & b^{n-2} & & \cdots & c \end{pmatrix}$$

6. 2008 circular coins, possibly of different diameters, are placed on the surface of a flat table in such a way that no coin is on top of another coin. What is the largest number of points at which two of the coins could be touching?

7. Let G be a group of finite order and identity e. Suppose that ϕ is an automorphism of G onto itself with the following properties: (1) $\phi(x) = x$ if and only if $x = e$; (2) $\phi(\phi(x)) = x$ for each element x of G. (The mapping ϕ has the property that it is one-one onto and that $\phi(xy) = \phi(x)\phi(y)$ for each pair x, y of elements of G.)

   (a) Give an example of a group and automorphism for which these conditions are satisfied.

   (b) Prove that G is commutative (*i.e.*, $xy = yx$ for each pair x, y of elements in G).

8. Let $b \geq 2$ be an integer base of numeration and let $1 \leq r \leq b - 1$. Determine the sum of all $r-$digit numbers of the form

$$\overline{a_{r-1}a_{r-2} \cdots a_2 a_1 a_0} \equiv a_{r-1}b^{r-1} + a_{r-2}b^{r-2} + \cdots + a_1 r + a_0$$

whose digits increase strictly from left to right: $1 \le a_{r-1} < a_{r-2} < \cdots < a_1 < a_0 \le b-1$.

9. For each positive integer n, let

$$S(n) = \sum_{k=1}^{n} \frac{2^k}{k^2} \ .$$

Prove that $S(n+1)/S(n)$ is not a rational function of n. [A *rational function* is one that can be written as a ratio of two polynomials.]

10. A point is chosen at random (with the uniform distribution) on each side of a unit square. What is the probability that the four points are the vertices of a quadrilateral with area exceeding $\frac{1}{2}$?

END

Solutions

1. Three angles of a hepatagon (7-sided polygon) inscribed in a circle are equal to $120°$. Prove that at least two of its sides are equal.

Solution. Consider two adjacent sides of the heptagon for which the angle between them is $120°$. The chord of the circle joining the endpoints that are not common to the chords subtends an angle of $60°$ at the circumference of the circle and therefore an angle of $120°$ at the centre of the circle. If the three pairs of adjacent sides forming angles of $120°$ are mutually disjoint from each other, then they constitute six sides of the heptagon which subtend in total angles totalling $360°$ at the centre of the circle, leaving no positive angle for the seventh side to subtend. Hence, two of the pairs of sides must have an edge in common. However, since each pair subtends the same angle at the centre, the edges that the pairs do not have in common must be of equal length and the result follows.

2. (a) Determine a real-valued function g defined on the real numbers that is decreasing and for which $g(g(x)) = 2x + 2$.

(b) Prove that there is no real-valued function f defined on the real numbers that is decreasing and for which $f(f(x)) = x + 1$.

Solution 1. (a) Let $r > 0$, $r \neq 1$. We determine a decreasing composite square root of $r^2x + r^2$. Let $g(x) = -rx + b$. Then $g(g(x)) = r^2 + b(1 - r)$. When $b = r^2(1 - r)^{-1}$, g is a decreasing function for which $g(g(x)) = r^2x + r^2$. The particular case in the problem can be thus dealt with; an answer is

$$g(x) = -\sqrt{2}x - 2(\sqrt{2} + 1) = -\sqrt{2}(x + 2) - 2 \ .$$

(b) [C. Ochanine] Suppose that $f(f(x)) = x + 1$ ($f(x) = x + \frac{1}{2}$ defines such a function). Then

$$f(x + 1) = f(f(f(x))) = f(x) + 1$$

for all x, whence $f(x + 1) > f(x)$. Therefore, f cannot be decreasing.

Solution 2. (b) If there is such a function f, then it must be one-one and onto **R**. Observe that $u <> f(u)$ if and only if $v = f(u) >< f(f(u)) = f(v)$. It follows that $x - f(x)$ can take both positive and negative values. By the intermediate value theorem, there is a number z for which $z = f(z)$. But then $z = f(f(z)) = z + 1$, a contradiction. Therefore there is no such function f

Solution 3. (b) Suppose, if possible, there is a decreasing continuous function f for which $f(f(x)) = x+1$. Since $x + 1$ has no fixpoint, neither does the function $f(x)$. Therefore $f(x) - x$ never vanishes. As f is coninuous on **R**, either $f(x) > x$ for all real x or $f(x) < x$ for all real x.

Suppose the former. Let M exceed the maximum of 0 and $g(0)$. Then $f(0) > f(M) > M > f(0)$, a contradiction. Therefore, $f(x) > x$ for all x cannot occur. Similarly, $f(x) < x$ for all x cannot occur. Therefore, no such function f exists and $x + 1$ has no continuous decreasing composite square root.

3. Suppose that a is a real number and the sequence $\{a_n\}$ is defined recursively by $a_0 = a$ and

$$a_{n+1} = a_n(a_n - 1)$$

for $n \geq 0$. Find the values of a for which the sequence $\{a_n\}$ converges.

Solution. When the sequence converges, the limit b must satisfy the equation $b = b(b - 1)$ so that $b = 0$ or $b = 2$. If is clear that the sequence converges then $a = -1, 0, 1, 2$. If $a > 2$, then an induction argument show that $\{a_n\}$ is increasing and unbounded. If $a < -1$, then $a_1 > 2$ and the sequence again diverges.

If $-1 < a < 2$, we show that the sequence will have an entry in the interval $[0,1]$. Suppose that $1 < a < 2$, then $a(a-1) < a$, so that a_n will initially decrease until it arrives in the interval $(0,1]$. If $-1 < a < 0$, then $a_1 = (-a)(1-a)$ will lie in $(0,2)$.

Hence, wolog, we may analyze the situation that $0 < a < 1$. Then $-1/4 < a_1 < 0$ and $0 < a_2 < 5/16 < 1$. Thus, the sequence alternates between the intervals $(-1/4, 0)$ and $(0,1)$. Observe that

$$\begin{aligned} a_{n+1} &= a_{n+1}(a_{n+1} - 1) = a_n(a_n - 1)(a_n^2 - a_n - 1) \\ &= a_n(1 - a_n^2(2 - a_n)) \ . \end{aligned}$$

When n is even, then a_n and a_{n+2} are both positive and $a_{n+2} < a_n$. When n is odd, then a_n and a_{n+2} are both negative and $1 - a_n^2(2 - a_n) \in (0,1)$, so that the sequence $\{a_{2m}\}$ decreases to a limit u and $\{a_{2m+1}\}$ increases to a limit v. We must have that $u = v(v-1)$ and $v = u(u-1)$ so that $u = u(u-1)(u^2 - u - 1)$ or $u^3(2-u) = 0$. Since $u \neq 2$, we must have $u = v = 0$.

Hence the sequence converges if and only if $-1 \leq a \leq 2$; the limit is 2 when $a = -1, 2$ and 0 when $-1 < a < 2$.

Comment. The situation can also be analyzed using a cobweb diagram on the graph of the function $y = x(x-1)$.

4. Suppose that u, v, w, z are complex numbers for which $u + v + w + z = u^2 + v^2 + w^2 + z^2 = 0$. Prove that
$$(u^4 + v^4 + w^4 + z^4)^2 = 4(u^8 + v^8 + w^8 + z^8) \ .$$

Solution. Let u, v, w, z be the zeros of a monic quartic polynomial $f(t)$. Since $u + v + w + z = 0$ and

$$uv + uw + uz + vw + vz + wz = \frac{1}{2}[(u+v+w+z)^2 - (u^2 + v^2 + w^2 + z^2)] = 0 \ ,$$

we must have that $f(t) = t^4 - at - b$ for some complex numbers a and b.

Suppose that $s_n = u^n + v^n + w^n + z^n$ for positive integers n. Then $s_4 = 4b$ and $s_{k+4} = as_{k+1} + bs_k$ for $k \geq 1$. Therefore, $s_5 = 0$ and $s_8 = as_5 + bs_4 = 4b^2$, so that $4s_8 = (4b)^2 = s_4^2$, as desired.

5. Suppose that $a, b, c \in \mathbf{C}$ with $ab = 1$. Evaluate the determinant of

$$\begin{pmatrix} c & a & a^2 & \cdots & a^{n-1} \\ b & c & a & \cdots & a^{n-2} \\ b^2 & b & c & \cdots & a^{n-3} \\ \vdots & \vdots & \vdots & & \vdots \\ b^{n-1} & b^{n-2} & & \cdots & c \end{pmatrix}$$

Solution 1. With $\mathbf{u} = (1, b, b^2, \cdots, b^{n-1})$ and $\mathbf{v} = (1, a, a^2, \cdots, a^{n-1})$, we have to find $\det(\mathbf{u^t v} + (c-1)I_n)$ with $\mathbf{uv^t} = n$. Since $\mathbf{u^t v}$ is similar to

$$\begin{pmatrix} n & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}$$

we are led to the answer $(n + c - 1)(c - 1)^{n-1}$.

Solution 2. Setting $c = 1$ yields n columns that are proportional. Hence $(c-1)^{n-1}$ is a factor of the determinant. The leading terms is c^n, so that the determinant must be $(c+k)(c-1)^{n-1}$. Since the coefficient of c^{n-1} vanishes, $k = n - 1$.

Solution 3. [C. Ochanine] Denote the determinant of the $n \times n$ matrix by $\Delta_n(a,b,c)$. By subtracting a suitable multiple of the first row, we find that $\Delta_n(a,b,c)$ is equal to

$$\begin{pmatrix} c & a & a^2 & \cdots & a_{n-1} \\ 0 & c-\frac{1}{c} & a-\frac{a}{c} & \cdots & a^{n-2}-\frac{a^{n-2}}{c} \\ 0 & b-\frac{b}{c} & c-\frac{1}{c} & \cdots & a^{n-3}-\frac{a^{n-3}}{c} \\ 0 & \vdots & \vdots & & \vdots \end{pmatrix}$$

Pulling out the factor $1-\frac{1}{c}$ from the last $n-1$ rows yields

$$\Delta_n(a,b,c) = c\left(1-\frac{1}{c}\right)^{n-1}\Delta_{n-1}(a,b,c+1)\ .$$

An induction argument shows that $\Delta_n(a,b,c) = (n+c-1)(c-1)^{n-1}$.

Solution 4. [S. Wong] Then eigenvalues of the given matrix are those numbers λ for which

$$\begin{aligned} 0 &= \det \begin{pmatrix} c-\lambda & a & a^2 & \cdots & a^{n-1} \\ b & c-\lambda & a & \cdots & a^{n-2} \\ b^2 & b & c-\lambda & \cdots & a^{n-3} \\ \vdots & \vdots & \vdots & & \vdots \\ b^{n-1} & b^{n-2} & & \cdots & c-\lambda \end{pmatrix} \\ &= \det \begin{pmatrix} 1-(\lambda-c+1) & a & a^2 & \cdots & a^{n-1} \\ b & 1-(\lambda-c+1) & a & \cdots & a^{n-2} \\ b^2 & b & 1-(\lambda-c+1) & \cdots & a^{n-3} \\ \vdots & \vdots & \vdots & & \vdots \\ b^{n-1} & b^{n-2} & & \cdots & 1-(\lambda-c+1) \end{pmatrix} \end{aligned}$$

Hence λ is is an eigenvalue of the given matrix if and only if $\lambda - c + 1$ is an eigenvalue with the same multiplicity of the matrix

$$M \equiv \begin{pmatrix} 1 & a & a^2 & \cdots & a^{n-1} \\ b & 1 & a & \cdots & a^{n-2} \\ b^2 & b & 1 & \cdots & a^{n-3} \\ \vdots & \vdots & \vdots & & \vdots \\ b^{n-1} & b^{n-2} & & \cdots & 1 \end{pmatrix} .$$

Since all the columns of M are in proportion, the rank of M is 1 and so 0 is an eigenvalue of multiplicity $n-1$. By examining the trace (which is the sum of the eigenvalues), we see that the remaining eigenvalue is n (an eigenvector is $(1,b,b^2,\cdots,b^{n-1})^{\mathbf{t}}$). Thus, the eigenvalues of the given matrix is $c-1$ with multiplicity $n-1$ and $n+c-1$ with multiplicity 1. Since the determinant of the given matrix is the product of its eigenvalues, we find that the required answer is $(n+c-1)(c-1)^{n-1}$.

Solution 5. [E. Flat] Denote the determinant to be found by D_n. Observe that $D_1 = c$ and that $D_2 = c^2 - 1 = (c-1)(c+1)$. By taking b times each row from the next, we find that

$$D_n = \begin{pmatrix} c & a & a^2 & \cdots & a^{n-2} & a^{n-1} \\ b(1-c) & c-1 & 0 & \cdots & 0 & 0 \\ 0 & b(1-c) & c-1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & c-1 & 0 \\ 0 & 0 & 0 & \cdots & b(1-c) & c-1 \end{pmatrix}$$

Expanding along the last column yields, for $n \geq 2$,

$$D_n = (c-1)D_{n-1} + (-1)^{n-1}a^{n-1}[b(1-c)]^{n-1} = (c-1)D_{n-1} + (c-1)^{n-1} .$$

An induction argument then establishes that

$$D_n = (c-1)^{n-1}(c+n-1) .$$

Solution 6. [J. Kramar, O. Ivrii] The (i,j)th term in the given matrix is $c^{\delta(i,j)}a^{i-j}$, where $\delta(i,j)$ is the Kronecker delta that takes the value 1 when $i=j$ and 0 otherwise. The expansion of the determinant of this matrix is the sum of $n!$ terms of the form

$$\pm c^{\epsilon(\pi)} \prod_{i=1}^{n} a^{i-\pi(i)} = \pm c^{\epsilon\pi} \prod_{i=1}^{n} a^{i} \prod_{i=1}^{n} a^{-\pi(i)} = \pm c^{\epsilon(\pi)} .$$

where π is a permutation of $\{1,2,\cdots,n\}$ and $\epsilon(\pi)$ is the number of fixed elements of π. Thus, the value of the determinant is independent of a and b, so that, wolog, we may restrict ourselves to the case that $a=b=1$.

But then the required determinant is the product of the eignevalues (counting multiplicity) of

$$\begin{pmatrix} 1 & a & a^2 & \cdots & a^{n-1} \\ b & 1 & a & \cdots & a^{n-2} \\ b^2 & b & 1 & \cdots & a^{n-3} \\ \vdots & \vdots & \vdots & & \vdots \\ b^{n-1} & b^{n-2} & & \cdots & 1 \end{pmatrix} .$$

The $(n-1)$-dimensional subspace

$$\{(x_1, x_2, \cdots, x_n)^{\mathbf{t}} : x_1 + x_2 + \cdots + x_n = 0\}$$

consists of eigenvectors with eigenvalue $c-1$. A complement of this subspace is the span of the vector $(1,1,\cdots,1)^{\mathbf{t}}$, which is an eignevector with eigenvalue $c+n-1$. Therefore the product of the eigenvalues and determinant of the matrix is $(c+n-1)(c-1)^{n-1}$.

6. 2008 circular coins, possibly of different diameters, are placed on the surface of a flat table in such a way that no coin is on top of another coin. What is the largest number of points at which two of the coins could be touching?

Solution. Suppose that there are n coins where $n \geq 3$. We show that the answer is $3n-6$. When $n = 2008$, this number is equal to 6018. We can achieve this as follows. Start with three mutually touching coins. Insert a coin of suitable diameter in the middle so that it touches all three coins. Continue adding coins, each in the middle of three mutually touching coins that have already been placed.

To show that this is an upper bound, construct a graph whose vertices are the centres of the coins, with two vertices conncected by an edge if and only if the their two coins touch. This is a planar graph with n vertices, with the number of edges equal to the number of points where two coins touch. We use the following result from graph theory: *Any planar graph with V vertices, E edges and F faces (including the unbounded face) has at most $3V-6$ edges.*

Since each face has at least 3 edges, and each edge belongs to two faces, we must have $2E \geq 3F$. Hence, using Euler's relation,

$$2 = V - E + F \leq V - E + \frac{2}{3}E = V - \frac{1}{3}E ,$$

whence $E \leq 3V-6$.

This problem was contributed by David Arthur.

7. Let G be a group of finite order and identity e. Suppose that ϕ is an automorphism of G onto itself with the following properties: (1) $\phi(x) = x$ if and only if $x = e$; (2) $\phi(\phi(x)) = x$ for each element x of G. (The mapping ϕ has the property that it is one-one onto and that $\phi(xy) = \phi(x)\phi(y)$ for each pair x, y of elements of G.)

   (a) Give an example of a group and automorphism for which these conditions are satisfied.

   (b) Prove that G is commutative (*i.e.*, $xy = yx$ for each pair x, y of elements in G).

Solution. (a) An example is $\mathbf{Z}_3 \equiv \{0, 1, 2\}$, the integers modulo 3 with mod 3 addition as the group operation, the automorphism that which switches 1 and 2.

(b) Suppose for each element x, we denote $\phi(x)$ by x'. We define the mapping $\alpha : G \to G$ by $\alpha(x) = x'x^{-1}$. We show that α is one-one.

Suppose that $\alpha(x) = \alpha(y)$. The $x'x^{-1} = y'y^{-1}$, so that

$$x^{-1}y = x'^{-1}y' = (x^{-1})'y' = (x^{-1}y)' \ .$$

Hence $x^{-1}y = e$, so that $y = x$. Since α is one-one and G is finite, α is onto.

For any element $z \in G$, select x so that $z = x'x^{-1}$. Then $z' = x(x^{-1})'$, so that

$$zz' = x'x^{-1}x(x')^{-1} = e = z'z \ ,$$

whence $z' = z^{-1}$. Therefore

$$\phi(z) = z^{-1}$$

for each $z \in G$.

Let x, y be any two elements of G. Then

$$xy = [(xy)']' = (x'y')' = (x^{-1}y^{-1})^{-1} = yx \ .$$

The desired result follows.

8. Let $b \geq 2$ be an integer base of numeration and let $1 \leq r \leq b - 1$. Determine the sum of all $r-$digit numbers of the form

$$\overline{a_{r-1}a_{r-2}\cdots a_2a_1a_0} \equiv a_{r-1}b^{r-1} + a_{r-2}b^{r-2} + \cdots + a_1r + a_0$$

   whose digits increase strictly from left to right: $1 \leq a_{r-1} < a_{r-2} < \cdots < a_1 < a_0 \leq b - 1$.

Solution. The answer is $\overline{12\cdots r}\binom{b}{r+1}$.

Let n_s denote the $s-$digit number $\overline{12\cdots s}$ for each positive integer s. We use a double induction and show that for $b - 1 \geq k \geq r$, the sum of all the $r-$digit numbers whose units digit is k is $n_r\binom{k}{r}$.

When $r = 1$, there is only one number ending in k, namely k itself, and the desired sum is $k = 1 \times \binom{k}{1}$.

Suppose that the result holds up to $r - 1$ and up to k. The $r-$digit numbers that end in $k + 1$ fall into two classes: (a) those whose second last digit does not exceed $k - 1$ and (b) those whose last two digits are k and $k + 1$. There are $\binom{k-1}{r-1}$ numbers in class (a) and $\binom{k-1}{r-2}$ numbers in class (b).

Since the numbers in class (a) can be formed by taking all the numbers with $r-$digits and last digit k and adding 1 to each, the sum of the numbers in class (a) is equal to

$$n_r\binom{k}{r} + \binom{k-1}{r-1} \ .$$

The numbers in class (b) can be found by taking all the $(r-1)-$digit numbers ending in k, multiplying them by b and adding to each $k+1$. The sum of these numbers is

$$bn_{r-1}\binom{k}{r-1} + \binom{k-1}{r-2}(k+1) = (n_r - r)\binom{k}{r-1} + \binom{k-1}{r-2}(k+1) \ .$$

Therefore the sum of all the numbers in classes (a) and (b) is

$$\begin{aligned} n_r\binom{k}{r} &+ \binom{k-1}{r-1} + n_r\binom{k}{r-1} - r\binom{k}{r-1} + (k+1)\binom{k-1}{r-2} \\ &= n_r\left[\binom{k}{r} + \binom{k}{r-1}\right] + \left[\binom{k-1}{r-1} + (k+1)\binom{k-1}{r-2} - r\binom{k}{r-1}\right] \\ &= n_r\binom{k+1}{r} + 0 = n_r\binom{k+1}{r} \ . \end{aligned}$$

Therefore the sum of all the $r-$digit numbers with increasing digits is

$$n_r\sum_{k=r}^{b-1}\binom{k}{r} = n_r\binom{b}{r+1} \ .$$

9. For each positive integer n, let

$$S(n) = \sum_{k=1}^{n}\frac{2^k}{k^2} \ .$$

Prove that $S(n+1)/S(n)$ is not a rational function of n. [A *rational function* is one that can be written as a ratio of two polynomials.]

Solution 1. Assume that S_{n+1}/S_n is a rational function. Then

$$\frac{S_n}{2^n n^{-2}} = \frac{S_n}{S_n - S_{n-1}} = \frac{1}{1 - \frac{S_{n-1}}{S_n}}$$

is a rational function. From this, it follows that $S_n = r(n)2^n n^{-2}$ for some rational function r. Since

$$\frac{2^{n+1}}{(n+1)^2} = S_{n+1} - S_n = r(n+1)\frac{2^{n+1}}{(n+1)^2} - r(n)\frac{2^n}{n^2} \ ,$$

we have that

$$2n^2 r(n+1) - (n+1)^2 r(n) = 2n^2 \ .$$

Since this equation holds for infinitely many values of n, we have the corresponding polynomial identity. Letting $r(x) = f(x)/g(x)$, where f and g are coprime polynomials, we obtain the equation

$$2x^2 f(x+1)g(x) - (x+1)^2 f(x)g(x+1) = 2x^2 g(x)g(x+1) \ .$$

The polynomial $g(x)$ cannot be a constant (look at the leading coefficients).

There exists an integer n, namely 0, for which the polynomials $g(x)$ and $g(x+n)$ have a common divisor of positive degree. However, if n exceeds the maximum absolute value of a root of $g(x)$, then $g(x)$ and $g(x+n)$ are coprime. Let N be the largest integer for which $g(x)$ and $g(x+N)$ have a common irreducible divisor $u(x)$ of positive degree. Then $u(x-N)$ divides $g(x)$ and so divides $(x+1)^2 f(x)g(x+1)$. Since f

and g are coprime, $u(x-N)$ divides $(x+1)^2 g(x+1)$. Suppose that $u(x-N)$ divides $g(x+1)$; then $u(x)$ must divide $g(x+N+1)$, which contradicts the determination of N. Therefore, $u(x-N)$ divides $(x+1)^2$, and so $u(x+1)$ divides $(x+N+2)^2$.

Since $u(x+1)$ divides $g(x+1)$, $u(x+1)$ must divide $2x^2 f(x+1)g(x)$. Since $u(x+1)$ does not divide $f(x+1)$. Since $u(x+1)$ divides $g(x+N+1)$, $u(x+1)$ cannot divide $g(x)$. Therefore, $u(x+1)$ must divided x^2, as well as $(x+N+2)^2$. But this is an impossibility. This contradiction yields the result of the problem.

Solution 2. [C. Ochanine] Since

$$\frac{S_{n+1}}{S_n} - 1 = \frac{S_{n+1} - S_n}{S_n} = \frac{2^{n+1}}{(n+1)^2 S_n} ,$$

it is enought to show that $S_n/2^n$ is not a rational functions. Let $f(n) = S_n/2^n$. Then

$$f(n+1) = \frac{1}{2} f(n) + \frac{1}{(n+1)^2}$$

for every positive integer n.

If f were rational, then we would have

$$f(x+1) = \frac{1}{2} f(x) + \frac{1}{(x+1)^2}$$

for all real x. Since, by the definition, $f(1)$ is finite, so also is $f(0)$. Substituting $x = -1$ is the foregoing equation, we see that -1 is a pole of $f(x)$. Since

$$f(x+2) = \frac{1}{2} f(x+1) + \frac{1}{(x+2)^2} = \frac{1}{4} f(x) + \frac{1}{2(x+1)^2} + \frac{1}{(x+2)^2} ,$$

we see that -2 is also a pole of $f(x)$. Continuing on, we find that every negative integer is a pole of $f(x)$, contradicting its rationality.

The desired result follows.

This problem was contributed by Franklin Vera Pacheco.

10. A point is chosen at random (with the uniform distribution) on each side of a unit square. What is the probability that the four points are the vertices of a quadrilateral with area exceeding $\frac{1}{2}$?

Solution 1. The desired probability is $\frac{1}{2}$.

Suppose that the points are located in a counterclockwise direction distance x, y, z, w from the four vertices in order. Then the area of that part of the square lying outside of the inner quadrilateral is

$$\frac{1}{2}[x(1-w) + y(1-x) + z(1-y) + w(1-z)] = \frac{1}{2}[(x+y+z+w) - (xw + yx + zy + wz)] .$$

The condition that the area of the inner quadrilateral exceeds $\frac{1}{2}$ is that

$$(x+y+z+w) - (xw + yx + zy + wz) < 1.$$

The area is equal to $\frac{1}{2}$ if and only if $x + z = 1$ or $w(1-x-z) = 1 - x - y - x + xy + yz$. both of which occur with probability 0.

On the other hand, suppose that the points are located at distances x, y from one vertex of the square and distance z, w from the diagonally opposite vertex. Then the area of that part of the square ying outside the inner quadrilateral is

$$\frac{1}{2}[xy + (1-y)(1-z) + zw + (1-w)(1-x)] = \frac{1}{2}[(xy + yz + zw + wx) - (x+y+z+w) + 2] ,$$

so that the condition that the area of the inner quadrilateral exceeds $\frac{1}{2}$ is that

$$(x + y + z + w) - (xw + yx + zy + wz) > 1.$$

Since the x, y, z, w are chosen independently and uniformly in each case, the probabilities of either inequality occurring is the same as the probability of the other. Therefore, the desired probability is $\frac{1}{2}$.

Solution 2. Let the square have vertices $(0,0)$, $(1,0)$, $(1,1)$, $(0,1)$ in the cartesian plne and let the four selected points by $(a,0)$, $(1,b)$, $(c,1)$ and $(0,d)$, where $0 \leq a,b,c,d \leq 1$. Then the area of the inner quadrilateral is equal to

$$1 - \frac{1}{2}(1-a)b - \frac{1}{2}(1-c)(1-b) - \frac{1}{2}(1-d)c - \frac{1}{2}ad = \frac{1}{2}[1 + (d-b)(c-a)] \ .$$

The area of the quadrilateral is exactly $\frac{1}{2}$ when $b = d$ or $a = c$, which occurs with probability 0. The area exceeds $\frac{1}{2}$ if and only if, either $b < d$ and $a < c$, or $b > d$ and $a > c$. The area is less than $\frac{1}{2}$ if and only if, either $b < d$ and $c < a$, or $b > d$ and $c > a$. Since a,b,c,d are selected randomly and independently from $[0,1]$, by symmetry, the two latter events have equal probability of $\frac{1}{2}$.