THE UNIVERSITY OF TORONTO
UNDERGRADUATE MATHEMATICS COMPETITION

In Memory of Robert Barrington Leigh

March, 2021

Time: $3\frac{1}{2}$ hours

`No aids or calculators permitted.`

The grading is designed to encourage only the stronger students to attempt more than five problems. Each solution is graded out of 10. If the sum of the scores for the solutions to the five best problems does not exceed 30, this sum will be the final grade. If the sum of these scores does exceed 30, then all solutions will be graded for credit.

1. Solve the equation
$$x^3 + x^2 + x = -\frac{1}{3}.$$

2. Let n be a positive integer and S be a set $2n-1$ integers, each equal to one of 0, 1, and 2. Each number occurs less than n times. Prove that there is a subset T of n of them whose sum is equal to n.

3. Let a, b, c, d be the lengths of the sides of a quadrilateral. Suppose that the sum of any three sidelengths is an integer multiple of the fourth side. Prove that two of the sidelengths are equal.

4. Let $p(z)$ be a polynomial over the complex numbers with positive degree n. Prove that there are complex numbers c_0, c_1, $\ldots$, c_n, not all zero, for which $p(z)$ divides the polynomial
$$\sum_{k=0}^{n} c_k z^{k^2}.$$

5. Let n be an integer, and let a, b, c be nonnegative reals for which $ab + bc + ca = 3$. Prove that
$$a^n + b^n + c^n \geq 3.$$

6. Suppose that AB is a chord of a circle with centre O. The radius OY perpendicular to AB meets AB at Q. An arbitrary point P on the circumference of the circle distinct from A and B and on the same side of AB as O is chosen. PY intersects AB at X and PQ intersects the circle again at R. Prove that $XY > QR$.

7. Let n be a positive integer exceeding 1, and let $G = (V, E)$ be the graph with a set V of n vertices whose set S of $\binom{n}{2}$ edges consists of one edge joining each pair of vertices. Suppose that $f : V \longrightarrow \{0, 1\}$ is a function defined on V that takes only the values 0 and 1.

   Define $g : E \longrightarrow \{0, 1\}$ as follows: when the edge e connects the vertices x and y, then we select that value for g for which
$$g(e) \equiv f(x) + f(y) \mod 2.$$

   For which values of n is it possible to find a function f for which the corresponding function g assumes each of the values 0 and 1 equally often?

8. Prove that
$$\int_0^x t^{2021} e^t \sin t \, dt = p(x)e^x \sin x + q(x)e^x \cos x + C,$$

for some polynomials $p(x)$ and $q(x)$ and constant C, and determine the value of C.

9. A middle school student presented the following procedure for trisecting an acute angle with straightedge and compasses:

   Suppose the arms of the angle to be trisected meet at O. From a point A on one arm of the angle, drop a perpendicular to the other arm intersecting it at B. Construct an equilateral triangle ABC with O and C on opposite sides of AB. Then the angle AOB is trisected by the line OC.

   (a) Provide an argument that this procedure does not work. Your solution will be graded on how elementary your argument is; in particular, it should involve mathematics accessible to a secondary student.

   (b) Is there any acute angle for which the procedure works?

10. Determine all bilateral sequences $\{x_n : n \in \mathbf{Z}\}$ whose entries are nonzero integers, that satisfy the recursion

$$x_{n+1} = \frac{x_n + x_{n-1} + 1}{x_{n-2}}$$

   for each integer n.

Solutions

1. Solve the equation

$$x^3 + x^2 + x = -\frac{1}{3}.$$

   Solution 1. Rewrite the equation as

$$2x^3 + (x+1)^3 = 3x^3 + 3x^2 + 3x + 1 = 0.$$

Then

$$\frac{(x+1)^3}{x^3} = -2.$$

Let $\theta = 2^{1/3}$ and $\omega = \frac{1}{2}(-1 + i\sqrt{3}) = \cos(2\pi/3) + i\sin(2\pi/3)$. Then

$$x = \frac{-1}{1+\theta}, \quad \frac{-1}{1+\theta\omega}, \quad \frac{-1}{1+\theta\bar{\omega}}.$$

Solution 2. Using the same notation as in Solution 1, we have that

$$\begin{aligned} 0 = 2x^3 + (x+1)^3 &= (\theta x + (x+1))((\theta^2 x^2 - \theta x(1+x) + (x+1)^2) \\ &= (\theta x + (x+1))((\theta^2 - \theta + 1)x^2 + (2-\theta)x + 1), \end{aligned}$$

whence $x = -1/(1+\theta)$ or $x = [(\theta - 2) \pm i\theta\sqrt{3}]/2(\theta^2 - \theta + 1)$.

Notes: Some competitors used one of the standard methods for solving a cubic in terms of radicals. The substitutions $x = y - (1/3)$ leads to the equation $y^3 + (2/3)y + (2/27) = 0$ which can be solved by using the formua $y = z - [2/(9z)]$. This yields a quadratic in z^3 which can be solved and substituted back into y then x.

There are a number of expressions for the roots. The real root is equal to

$$\frac{-1}{1+2^{1/3}} = -(1 - 2^{1/3} + 2^{2/3})/3,$$

which is approximately equal to -0.442493. One of the nonreal roots is

$$-1/(1+2^{1/3}\omega) = \frac{-(1-2^{1/3}\omega+2^{2/3}\omega)}{3} = (1+2^{1/3})(-2+2^{1/3}+2^{1/3}\sqrt{3}i)/6$$
$$= \frac{-2^{2/3}}{2^{2/3}-1+\sqrt{3}i} = \frac{1}{2^{-2/3}-1-2^{-2/3}\sqrt{3}i}.$$

2. Let n be a positive integer and S be a set of $2n-1$ integers, each equal to one of 0, 1, and 2. Each number occurs less than n times. Prove that there is a subset T of n of them whose sum is equal to n.

Solution 1. Since each number appears no more than $n-1$ times, each must appear at least once. Suppose that the number of 0's, 1's and 2's in S are respectively a, b, c. If we can find a suitable subset T, for which the number of 0's, 1's and 2's are respectively u, v, w, then

$$2n-1 = a+b+c, \qquad 1 \leq a,b,c \leq n-1,$$
$$n = u+v+w = v+2w,$$

so that $u=w$ and $0 \leq u \leq a$, $0 \leq v \leq b$, $0 \leq w \leq c$.

Suppose, first, that a and c are not less than $\lfloor n/2 \rfloor$. Let $n=2r$. Then we can take $(u,v,w)=(r,0,r)$.

Let $n=2r+1$. Then we can take $(u,v,w)=(r,1,r)$.

Suppose that $d = \min(a,c) < \lfloor n/2 \rfloor$. Then $a+c \leq d+(n-1)$, so that $b \geq (2n-1)-(n-1)-d = n-d$. In this case, we can take $(u,v,w)=(d,n-2d,d)$.

Solution 2. We can prove the result by induction. When $n=2$, the set must be $\{0,1,2\}$, and we can take $T=\{0,2\}$. Suppose that the result holds for $n=m$, and that we are given a set S of $2m+1$ numbers, each equal to 0, 1 or 2, with none of them appearing no more than m times. If 1 appears m times, then we can choose T to contain $m-1$ ones and one each of 0 and 2, to obtain $m+1$ numbers whose sum is $m+1$.

If S contains m each of 0 and 2, then we can form T by including $\lfloor (m+1)/2 \rfloor$ each of 0 and 2, and, when $m+1$ is odd, one 1, to make up the set T.

Suppose S contains fewer than m of either 0 or 2; then S must contain at least two 1s. Form S' be removing from S a 1 and either a 0 or 2 to ensure that there are no more than $m-1$ of either of these. Determine a subset T' of S of m numbers whose sum is m (by the induction hypothesis) and let $T = T' \cup \{1\}$.

Comment. This is a special case of the Erdös-Ginzbug-Ziv Theorem that provides that, for any positive integer n, any set of $2n-1$ integers contains a subset of n integers whose sum is a multiple of n.

3. Let a,b,c,d be the lengths of the sides of a quadrilateral. Suppose that the sum of any three sidelengths is an integer multiple of the fourth side. Prove that two of the sidelengths are equal.

Solution. Suppose that the sidelengths are unequal. Let d be the largest sidelength. Then $d < a+b+c < 3d$, whence $a+b+c=2d$. Let p be the perimeter of the quadrilateral; then $p=3d$. Since the no two sidelengths are equal, there are distinct positive integers u,v,w exceeding 2 for which $ua=b+c+d$, $vb=a+c+d$, $wc=a+b+d$, so that $p=(u+1)a=(v+1)b=(w+1)c$. Therefore

$$p = a+b+c+d \leq \frac{p}{3}+\frac{p}{4}+\frac{p}{5}+\frac{p}{6} = \left(\frac{57}{60}\right)p,$$

which is false. Hence two sidelengths must be equal.

4. Let $p(z)$ be a polynomial over the complex numbers with positive degree n. Prove that there are complex numbers c_0, c_1, ..., c_n, not all zero, for which $p(z)$ divides the polynomial

$$\sum_{k=0}^{n} c_k z^{k^2}.$$

Solution. For each integer $k \le n$, we can apply the division algorithm to obtain

$$z^{k^2} = q_k(z)p(z) + r_k(z),$$

where $r_k(z)$ is a remainder polynomial of degree $\le n-1$. The set $S_k = \{r_k(z) : 0 \le k \le n\}$ is a set of $n+1$ polynomials in the linear space P_{n-1} of polynomials of degree not exceeding $n-1$. Since P_{n-1} has dimension n, the set S_k is linearly dependent, so that there are constants c_k $(0 \le k \le n)$ not all zero for which $\sum_{k=0}^{n} c_k r_k(z) = 0$. Therefore

$$\sum_{k=0}^{n} c_k z^{k^2} = \left(\sum_{k=0}^{n} c_k q_k(z) \right) p(z),$$

and the desired result holds.

5. Let n be an integer, and let a, b, c be nonnegative reals for which $ab + bc + ca = 3$. Prove that

$$a^n + b^n + c^n \ge 3.$$

Solution 1. We first note that if (p, q, r) and (u, v, w) are both either increasing or either decreasing, then

$$3(pu + qv + rw) \ge (p + q + r)(u + v + w).$$

Taking the difference of the two sides yields

$$2(pu + qv + rw) - (pv + qu + pw + ru + qw + rv) = (p - q)(u - v) + (p - r)(u - w) + (q - r)(v - w) \ge 0.$$

Applying this to (a, b, c) in suitable order, we have for each positive integer n,

$$3(a^n + b^n + c^n) \ge (a + b + c)(a^{n-1} + b^{n-1} + c^{n-1}),$$

and

$$3(a^{-n} - b^{-n} - c^{-n}) \ge (a^{-1} + b^{-1} + c^{-1})(a^{-(n-1)} + b^{-(n-1)} + c^{-(n-1))}.$$

To obtain the result, we first prove that $a + b + c \ge 3$ and $a^{-1} + b^{-1} + c^{-1} \ge 3$.

Since

$$(a + b + c)^2 - 3(ab + bc + ca) = \frac{1}{2}((a - b)^2 + (a - c)^2 + (b - c)^2) \ge 0,$$

the result holds for $n = 1$. Since

$$a^{-1} + b^{-1} + c^{-1} = \frac{3}{abc}$$

and

$$(abc)^2 = a^2b^2c^2 \le \left(\frac{ab + bc + ca}{3} \right)^3 = 1,$$

by the arithmetic-geometric means inequality, the result holds for $n = -1$. We can now apply an induction argument.

Solution 2. The inequality is trivial for $n = 0$. Let n be a positive integer. Then

$$a^n + b^n + (n - 2) = a^n + b^n + 1 + \cdots + 1 \ge n(a^n b^n)^{1/n} = nab,$$

with a similar inequality for the other pairs of variables. Adding the three inequalities yields

$$2(a^n + b^n + c^n) + 3(n - 2) \ge n(ab + bc + ca) = 3n.$$

Hence $a^n + b^n + c^n \geq 3$.

Now let $n = -m$ be negative. By the arithmetic-geometric means inequality,

$$\left(\frac{1}{abc}\right)^{1/3} \leq \frac{1}{3}\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right) + \frac{1}{3}\left(\frac{ab + bc + ca}{abc}\right) = \frac{1}{abc},$$

whence $1/(abc) \geq 1$. Therefore

$$a^n + b^n + c^n = \frac{1}{a^m} + \frac{1}{b^m} + \frac{1}{c^m} \geq \frac{3}{(abc)^{m/3}} \geq 3.$$

6. Suppose that AB is a chord of a circle with centre O. The radius OY perpendicular to AB meets AB at Q. An arbitrary point P on the circumference of the circle distinct from A and B and on the same side of AB as O is chosen. PY intersects AB at X and PQ intersects the circle again at R. Prove that $XY > QR$.

Solution 1. Wolog, we will locate P and the same side of OY as B. Consider the reflection with axis OY that interchanges X and Z, P and S, and A and B. Since $PSRY$ is concyclic and SP and AB are parallel, $\angle YRP = \angle YSP = \angle YZQ$. Therefore $YRZQ$ is concyclic and (since $\angle ZQY = 90°$) YZ is a diameter of the circle containing its vertices, $XY = ZY > QR$.

Solution 2. Let $\alpha = \angle YRP = \angle YRQ$ and $\beta = \angle RYQ$. Let YQ produced meet the circle again at D. Since $DRYP$ is concyclic, $\alpha = \angle YRP = \angle YDP$. Since $\angle XQD = \angle XPD = 90°$, $DQXP$ is concyclic, so that $\angle YXQ = \angle QDP = \alpha$. Now $\sin\alpha = QY/XY$, and the Law of Sines applied to triangle RYQ yields

$$\frac{QY}{\sin\alpha} = \frac{QR}{\sin\beta},$$

so that $XY = QR \sin\beta > QR$.

Solution 3. Let RD intersect AB at S. By the Butterfly Theorem, $SQ = QX$. Since QY right bisects SX, $SY = XY$. Since $\angle YRS = \angle YQS = 90°$, Q, S, R, Y lies on a circle with diameter SY. Hence $QR < SY = XY$.

Note: A proof of the Butterfly Theorem was provided by Yao Kuang. Let T be the midpoint of RD and Z be the midpoint of PY, Triangles QRD and QYP are similar, as are triangles QRT and QZY. Hence $\angle QTS = \angle QZX$. Since

$$90° = \angle OQS = \angle OTS = \angle OZX = \angle OQX,$$

quadrilaterals $OTSQ$ and $OZXQ$ are concyclic. Therefore,

$$\angle QOS = \angle QTS = \angle QZX = \angle QOX,$$

from which it follows that $SQ = QX$.

7. Let n be a positive integer exceeding 1, and let $G = (V, E)$ be the graph with a set V of n vertices whose set S of $\binom{n}{2}$ edges consists of one edge joining each pair of vertices. Suppose that $f : V \longrightarrow \{0, 1\}$ is a function defined on V that takes only the values 0 and 1.

   Define $g : E \longrightarrow \{0, 1\}$ as follows: when the edge e connects the vertices x and y, then we select that value for g for which

   $$g(e) \equiv f(x) + f(y) \mod 2.$$

   For which values of n is it possible to find a function f for which the corresponding function g assumes each of the values 0 and 1 equally often?

Solution. Suppose that the subset A of V on which f assumes the value 0 has k elements and the subset B of V on which f assumes the value 1 has $n-k$ elements. Then $g(e)=0$ if and only if the endpoints of e belong to the same one of these two subsets and $g(e)=1$ if and only if the end points belong to different subsets. The number of edges of which g asumes the value 1 is $k(n-k)$. Therefore g assumes each of its values equally often if and only if $k(n-k)$ is half the total number $n(n-1)/2$ of edges. This gives the condition $n(n-1)=4k(n-k)$ which simplifies to

$$n=(n-2k)^2=[n-2(n-2k)]^2.$$

It is necessary that n be a perfect square.

[Alternatively, the number of edges on which g assumes the value 0 is $\binom{k}{2}+\binom{n-k}{2}=\frac{1}{2}[n^2-(2k+1)n+2k^2]$, and the condition is

$$n^2-(2k+1)n+2k^2=2kn-2k^2$$

which leads to the same result.]

On the other hand, let $n=u^2$ and $2k=n-u=u(u-1))<n$. Let A be any set of $v=\frac{1}{2}u(u-1)$ vertices and B be its complement in v with $w=n-v=\frac{1}{2}u(u+1)$ vertices. Let $f(x)=0$ for $x\in A$ and $f(x)=1$ for $x\in B$. Then $2vw=u^2(u^2-1)=2\binom{n}{2}$ and the corresponding function assumes each of its values infinitely often.

[Alternatively,

$$\binom{v}{2}+\binom{w}{2}=\frac{1}{2}[(v^2+w^2)+(v+w)]=\frac{1}{4}[2(v+w)^2-2(v+w)-4vw]$$
$$\frac{1}{4}[2u^4-2u^2-u^2(u^2-1)]=\frac{u^2(u^2-1)}{4}=vw,$$

as desired.]

Note: This problem was contributed by Samer Seraj.

8. There exist polynomials p and q and a real number C for which

$$\int_0^x f^{2021}e^t\sin t\,dt=p(x)e^x\sin x+q(x)e^x\cos x+C.$$

Determine C.

Solution 1. Suppose that

$$I_n=\int_0^x t^ne^t\sin t\,dt,\qquad \text{and}\qquad J_n=\int_0^x t^ne^t\cos t\,dt.$$

Integrating by parts, we find that

$$I_0=\int_0^x e^t\sin t\,dt=\frac{e^x(\sin x-\cos x)}{2}+\frac{1}{2}$$

and

$$J_0=\int_0^x e^t\cos t\,dt=\frac{e^x(\sin x+\cos x)}{2}-\frac{1}{2},$$

and, for $n\geq 1$,

$$I_n=\frac{x^ne^x(\sin x-\cos x)}{2}-\frac{n}{2}I_{n-1}+\frac{n}{2}J_{n-1},$$

$$J_n = \frac{x^n e^x(\sin x + \cos x)}{2} - \frac{n}{2} I_{n-1} - \frac{n}{2} J_{n-1}.$$

Using an induction argument and noting that the integrals vanish when $x = 0$, we can determine the representations

$$I_n = p_n(x)e^x \sin x + q_n(x)e^x \cos x + C_n$$

and

$$J_n = r_n(x)e^x \sin x + s_n(x)e^x \cos x + D_n,$$

for some polynomials $p_n(x)$, $q_n(x)$, $r_n(x)$, $s_n(x)$ and constants C_n and D_n where $C_0 = -D_0 = 1/2$, and for $n \geq 1$,

$$C_n = -\frac{n}{2} C_{n-1} + \frac{n}{2} D_{n-1},$$

and

$$D_n = -\frac{n}{2} C_{n-1} - \frac{n}{2} D_{n-1}.$$

Adding these equations, we find that $D_n = -C_n - nC_{n-1}$, whence

$$\begin{aligned} C_{n+1} &= -\frac{n+1}{2} C_n + \frac{n+1}{2} D_n \\ &= -(n+1)C_n - \frac{(n+1)n}{2} C_{n-1}. \end{aligned}$$

Therefore

$$\begin{aligned} C_{n+1} &= -(n+1)C_n - \frac{(n+1)n}{2} C_{n-1} \\ &= -(n+1)\left[-nC_{n-1} - \frac{n(n-1)}{2} C_{n-2}\right] - \frac{(n+1)n}{2} C_{n-1} \\ &= \frac{(n+1)n}{2} C_{n-1} + \frac{(n+1)n(n-1)}{2} C_{n-2} \\ &= \frac{(n+1)n}{2}\left[-(n-1)C_{n-2} - \frac{(n-1)(n-2)}{2} C_{n-3}\right] + \frac{(n+1)n(n-1)}{2} C_{n-2} \\ &= -\frac{(n+1)n(n-1)(n-2)}{4} C_{n-3}. \end{aligned}$$

Therefore

$$C_{4k+1} = \frac{(-1)^k(4k+1)(4k)(4k-1)\cdots(4)(3)(2)}{2^{2k}} C_1 = \left(\frac{(-1)^k(4k+1)!}{2^{2k}}\right)\left(\frac{-1}{2}\right) = \frac{(-1)^{k+1}(4k+1)!}{2^{2k+1}}.$$

Setting $k = 505$ yields $C = C_{2021} = (2021!)/2^{1011}$.

Solution 2. As in Solution 1, we obtain that $C_0 = -D_0 = -C_1 = 1/2$, $D_1 = 0$, and, for $n \geq 1$,

$$C_n = -\frac{n}{2} C_{n-1} + \frac{n}{2} D_{n-1}, \qquad D_n = -\frac{n}{2} C_{n-1} - \frac{n}{2} D_{n-1}.$$

$$\begin{pmatrix} C_n \\ D_n \end{pmatrix} = \frac{n}{2} A \begin{pmatrix} C_{n-1} \\ D_{n-1} \end{pmatrix},$$

where

$$A = \begin{pmatrix} -1 & 1 \\ -1 & -1 \end{pmatrix}.$$

Observe that
$$A^2 = \begin{pmatrix} 0 & -2 \\ 2 & 0 \end{pmatrix} \qquad \text{and} \qquad A^4 = -4I,$$
where I is the identity matrix. Therefore
$$\begin{pmatrix} C_{2021} \\ D_{2021} \end{pmatrix} = A^{2020} \begin{pmatrix} C_1 \\ D_1 \end{pmatrix} = -\frac{2021!}{4^{505}} I \begin{pmatrix} \frac{-1}{2} \\ 0 \end{pmatrix},$$
whence $C_{2021} = 2021!/2^{1011}$.

Solution 3. Observe that $\int_0^x t^n e^t \sin t\, dt = \Im \int_0^x t^n e^{wt}\, dt$ where $w = 1+i$. Let $\int_0^x t^n e^{wt}\, dt = f(x)e^{wx} + c_n$ where c_n is a complex constant and $f(x)$ is a complex polynomial. Then
$$x^n e^{wx} = f'(x)e^{wx} + wf(x)e^{wx}$$
whence $f'(x) + wf(x) = x^n$. The degree of $f(x)$ is n. Comparing the coefficients of the two sides, we find that
$$f(x) = \frac{1}{w}x^n - \frac{n}{w^2}x^{n-1} + \cdots + (-1)^i \frac{n(n-1)\cdots(n-i+1)}{w^{i+1}} + \cdots + \frac{(-1)^n n!}{w^{n+1}}.$$

From the integral equation, we must have
$$c_n = -f(0) = \frac{(-1)^{n+1} n!}{w^{n+1}}.$$
Note that $1/w = (1-i)/2$, whence $1/w^6 = i/8$ and $1/w^8 = 1/16$. We now restrict to $n = 2021 = 8 \times 252 + 5$. Then
$$C_{2021} = \Im \left(\frac{1}{w}\right)^{2022} (2021!) = \Im \left(\frac{1}{w}\right)^6 \times \left(\frac{1}{w}\right)^{8\times 252} \times (2021!) = \Im i \frac{2021!}{2^{1011}} = \frac{2021!}{2^{1011}},$$
and we obtain the same answer as before.

Note: This problem was contributed by Alfonso Gracia-Saz.

9. A middle school student presented the following procedure for trisecting an acute angle with straightedge and compasses:

Suppose the arms of the angle to be trisected meet at O. From a point A on one arm of the angle, drop a perpendicular to the other arm intersecting it at B. Construct an equilateral triangle ABC with O and C on opposite sides of AB. Then the angle AOB is trisected by the line OC.

(a) Provide an argument that this procedure does not work. Your solution will be graded on how elementary your argument is; in particular, it should involve mathematics accessible to a secondary student.

(b) Is there any acute angle for which the procedure works?

Solution 1. (a) We show that the proposed procedure fails when $\angle AOB = 60°$. Suppose that $AB = \sqrt{3}$. It is straightfoward to verify that $AO = 2$, $OB = 1$, $\angle OAC = 90°$, $\angle OBC = 150°$ and $OC = \sqrt{7}$ (either from the right triangle OAC or by applying the Law of Cosines to triangle OBC).

Suppose if possible that $\angle COB = 20°$ and $\angle AOC = 40°$. Then $\cos 40° = 2/\sqrt{7}$, so that
$$\sin 10° = \cos 80° = 2\cos^2 40° - 1 = \frac{1}{7}.$$

From the Law of Sines applied to triangle OBC, we have that
$$2 \sin 10° \cos 10° = \sin 20° = \sqrt{3} \sin 10°,$$

whence $\cos 10° = \sqrt{3}/2$. Therefore

$$\sin^2 10° + \cos^2 10° = \frac{1}{49} + \frac{3}{4} \neq 1,$$

which gives a contradiction. Therefore the angle AOB is not trisected, and this example disproves the generality of the method.

(b) The procedure works when $\angle AOB = 45°$. Let D be the foot of the perpendicular from C to OB produced. Then $OB = AB = BC$, so that triangle OBC is isosceles. Since $30° = \angle COD = \angle COB + \angle OCB = 2\angle COB$, then

$$15° = \angle COB = \frac{1}{3}\angle AOB.$$

Solution 2. (a) Use the same notation and measurements as in Solution 1. By the Law of Sines applied to triangle OBC, $\cos 10° = \sqrt{3}/2$. However, this contradicts $\cos x$ being one-one for acute angles and $\cos 30° = \sqrt{3}/2$.

(b) Let $\angle AOB = 45°$. We may suppose that $OB = AB = BC = 2$, so that $BD = \sqrt{3}$ and $CD = 1$. Then

$$\tan OCB = \frac{1}{2+\sqrt{3}} = 2 - \sqrt{3}.$$

From the formula $\tan 30° = (2\tan 15°)/(1-\tan^2 15°)$, it can be shown that $\tan 15° = 2-\sqrt{3}$. Alternatively, using the formula $\tan 3\theta = (3\tan\theta - \tan^3\theta)/(1-3\tan^2\theta)$, it can be checked that $\tan 3\angle BOC = 1$.]

Solution 3. (a) Let $\alpha = \angle AOC$, $\beta = \angle BOC$, so that $\angle OAB = 90° - (\alpha+\beta)$, $\angle OCA = 30° + \beta$ and $\angle OCB = 30° - \beta$. Let the length of AB, BC and CA be 1, and the length of OB be x. By the Law of Cosines applied to triangle OBC, the length of OC is equal to $\sqrt{1+x+x^2}$. From the Law of Sines applied to triangle AOC,

$$\frac{\sin\alpha}{\sin(30°+\beta)} = \frac{\sin\alpha}{\sin(150° - (\alpha+\beta))} = \frac{1}{\sqrt{1+x+x^2}}.$$

Similarly, from triangle OBC,

$$\frac{\sin\beta}{\sin(30°-\beta)} = \frac{1}{x}.$$

Hence

$$\frac{\sin\alpha}{\sin\beta} = \frac{x}{\sqrt{1+x+x^2}} \cdot \frac{\sin(30°+\beta)}{\sin(30°-\beta)}.$$

Now let x tend to ∞. Then α and β both tend to 0 Then $\sin\alpha/\sin\beta$ tends to 1. This is also the limit of the ratio α/β. Therefore, for a sufficiently small angle to be trisected, the ratio α/β cannot be 2 and the construction is invalid.

Comment. We can get a fuller story on when the procedure is valid. Let $t = \tan\angle AOB$ and $s = \tan\angle COD$, where D is as defined above. Suppose that $OB = 2$, so that $AB = BC = 2t$, $BD = t\sqrt{3}$, and $CD = t$. Then $s = t/(2+t\sqrt{3})$. We have that

$$\begin{aligned}
\tan 3\angle COB &= \frac{3s-s^3}{1-3s^2} = \frac{3t(2+t\sqrt{3})^2 - t^3}{(2+t\sqrt{3})^3 - 3t^2(2+t\sqrt{3})} \\
&= \frac{3t(4+4t\sqrt{3}+3t^2) - t^3}{(8+12t\sqrt{3}+18t^2+3t^3\sqrt{3}) - (6t^2+3t^3\sqrt{3})} \\
&= \frac{8t^3+12t^2\sqrt{3}+12t}{12t^2+12t\sqrt{3}+8} = t\left[\frac{8t^2+12t\sqrt{3}+12}{12t^2+12t\sqrt{3}+8}\right]
\end{aligned}$$

It is readily checked that $\tan 3\angle COB = t = \tan\angle AOB$ if and only if $t = 1$ and $\angle AOB = 45°$.

10. Determine all bilateral sequences $\{x_n : n \in \mathbf{Z}\}$ whose entries are nonzero integers, that satisfy the recursion

$$x_{n+1} = \frac{x_n + x_{n-1} + 1}{x_{n-2}}$$

for each integer n.

Solution. Suppose that a, b, c are three successive terms. The following terms then turn out to repeat with a period of length 8:

$$\Big(a, \quad b, \quad c, \quad \frac{b+c+1}{a}. \quad \frac{a+b+c+1+ac}{ab},$$

$$\frac{(a+b+1)(b+c+1)}{abc}, \quad \frac{a+b+c+1+ac}{bc}, \quad \frac{a+b+1}{c}\Big).$$

Thus we need only arrange that eight successive terms are integers. In fact, some sequences may have periods of length 1, 2 or 4. Note that we can read this sequence in the opposite direction and it will still satisfy the rule of formation.

Let us investigate when all the entries are integers; this requires that only the numbers in each period are integers. To begin with, we suppose that all the entries are positive and that a is a maximum entry. Then $0 < b + c + 1 \le 2a + 1$. It follows from this that the fourth term of the period, $(b + c + 1)/a$, being an integer, does not exceed 2. Hence, either $b + c = 2a - 1$ or $b + c = a - 1$.

In the first instance, the only possibilities are $(b, c) = (a, a - 1)$ or $(b, c) = (a - 1, a)$. The fifth term is respectively $1 + (2/a)$ or $1 + (4/(a - 1))$ so that a is one of 2, 3, 5. Checking these out leads to sequences with one of these periods: $(5, 4, 5, 2, 2, 1, 2, 2)$ and $(3, 2, 3, 2, 3, 2, 3, 2)$

Otherwise $b + c = a - 1$, and the period is

$$(a, b, a - b - 1, 1, \frac{a+1}{b} - 1, \frac{a+b+1}{b(a-b-1)} = \frac{1}{b}\left(1 + \frac{2(b+1)}{a-(b+1)}\right), \ldots).$$

Since the sixth term is an integer, we must have $ab - b^2 - b \le a + b + 1$, or

$$a \le \frac{(b+1)^2}{b-1} = b + 3 + \frac{4}{b-1}.$$

When $b = 2$, a must be odd and $a - 3$ must divide 6. Hence $a = 5$ or $a = 9$ and we obtain the periods: $(5, 2, 2, 1, 2, 2, 5, 4)$ and $(9, 2, 6, 1, 4, 1, 6, 2)$.

When $b = 3$, $a - 2$ must be a multiple of 3 and $a - 4$ must divide 8. Hence $a = 5, 8$ and we obtain the periods: $(5, 3, 1, 1, 1, 3, 5, 9)$ and $(8, 3, 4, 1, 2, 1, 4, 3)$.

When $b = 4$, a must be either 7 or 15, but both these fail on the sixth term.

When $b = 5$, then $a = 9$ and we obtain $(9, 5, 3, 1, 1, 1, 3, 5)$.

If $b \ge 6$, then $4/(b - 1) < 1$ and so $a \le b + 3$. Hence $a - 3 \le b \le a$. Since $b + c = a - 1$ and $c \ge 1$, we must have $b = a - 2$ or $b = a - 1$. We are led to the periodic sequences:

$$(a, a-2, 1, 1, \frac{3}{a-2}, \ldots) \quad \text{and} \quad (a, a-3, 2, 1, \frac{4}{a-3}).$$

Since $a - 2 \ge 6$, $a \ge 8$ and the fifth term in not an integer. Thus, this case is not possible.

Consider periods that have at least one negative number. If the sequence has three negative terms in a row, the next term must be positive. Thus, the period must have at least one positive entry. Therefore, wolog, we assume that $a \geq 1 > -1 \geq b$.

First suppose that $c > 0$; we can also suppose that $c \leq a$, since reversing the sequence gives a valid sequence. Let $b + c + 1 > 0$. Since $b + c + 1$ is divisible by a, then

$$a \leq b + c + 1 = c + (b + 1) \leq c \leq a,$$

from which $a = c$ and $b = -1$. This leads to the period

$$(a, -1, a, 1, -(a + 2), -1, -(a + 2), 1)$$

for $a \neq 0, -2$.

The other possibility is that

$$a \geq c \geq 1 > -1 \geq b + c + 1 > b.$$

Since a divides $b + c + 1$, then $b + c + 1 = -ka$ where k is a positive integer. Therefore $b = -ka - c - 1$ and we are led to the period

$$(a, -ka - c - 1, c, -k, -\frac{c + 1 - k}{c + 1 + ka} \dots).$$

The denominator of the fraction in the fifth entry is positive and exceeds the numerator. so if the fifth entry is an integer, $k - c - 1$ must be positive and at least as large as $c + 1 + ka$. But this would imply that $k - c - 1 \geq c + 1 + ka$ or $0 \geq 2(c + 1) + k(a - 1)$, an impossibility.

If $a > 0 > -1 \geq b, c$, then $b + c + 1 < 0$, and the fourth term is negative. But then the fifth term in the period must be positive. Noting that we cannot have two negative terms immediately preceded and followd by positive terms, we have these possibilities for the signs of the terms in the period:

$$(+, -, -, -, +, +, -, +)$$

$$(+, -, -, -, +, -, +, ?)$$

$$(+, -, -, -, +, -, -, -)$$

Since the first and second of these involve the subsequence $+, -, +$, which can be placed at the front, these cases have been considered. Only the third possibility remains to be considered.

Suppose that the period is $(p, -q, -r, -s, t, -u, -v, -w)$ with all of p, q, r, s, t, u, v, w positive. Then $p - q + 1 = rw$ and $-q - r + 1 = -ps$, whence $p + r = rw + ps$ or $p(s - 1) + r(w - 1) = 0$. Hence $w = s = 1$. Also $p - w + 1 = qv$ and $-v - w + 1 = -pu$, whence $p(u - 1) + v(q - 1) = 0$. Hence $u = q = 1$. It follows that $p = r = v$ and we are led to

$$(p, -1, -p, -1, p, -1, -p, -1).$$

In summary, we have the following possible periods all of whose entries are integers:

$$(9, 5, 3, 1, 1, 1, 3, 5)$$

$$(9, 2, 6, 1, 4, 1, 6, 2)$$

$$(8, 3, 4, 1, 2, 1, 4, 3)$$

$$(5, 4, 5, 2, 2, 1, 2, 2)$$

$$(3, 2, 3, 2, 3, 2, 3, 2)$$

$$(a, -1, a, 1, -(a + 2), -1, -(a + 2), 1) \quad (a \neq 0, -2)$$

$$(p, -1, -p, -1, p, -1, -p, -1) \quad (p \neq 0).$$

Note that all of the sequences obtained with these periods are symmetrical, *i.e.*, they are the same when they are read backwards.