**THE UNIVERSITY OF TORONTO**
**UNDERGRADUATE MATHEMATICS COMPETITION**

*In Memory of Robert Barrington Leigh*

*March 8, 2020*

*Time*: $3\frac{1}{2}$ hours

`No aids or calculators permitted.`

The grading is designed to encourage only the stronger students to attempt more than five problems. Each solution is graded out of 10. If the sum of the scores for the solutions to the five best problems does not exceed 30, this sum will be the final grade. If the sum of these scores does exceed 30, then all solutions will be graded for credit.

1. Determine all strictly increasing functions $f$ from $[0,1]$ onto $[0,1]$ for which either (1) $f(x)+g(x) \leq 2x$ for **all** $x \in [0,1]$, or (2) $f(x)+g(x) \geq 2x$ for **all** $x \in [0,1]$. Here, $g(x)$ is the composition inverse function $f^{-1}$ of $f$ satisfying $f(g(x)) = g(f(x)) = x$ for $x \in [0,1]$.

2. Determine all integers $m$ for which the following statement is FALSE: *there exists a nonnegative integer $r$ for which $m \equiv 2^r + 1 \pmod{2^{r+1}}$.*

3. Determine all finite sequences $(w_0, w_1, \ldots, w_n)$ of integers for which $n$ is a positive integer and for each $k$ $(0 \leq k \leq n)$, $w_k$ is the number of entries in the sequence equal to $k$.

4. Let $f$ be a real continuously differentiable function on $[0.1]$. Prove that

   (a) $\lim_{n\to\infty} n \int_0^1 x^n f(x^n)\, dx = \int_0^1 f(x)\, dx$;

   (b) $\lim_{n\to\infty} n \int_0^1 x^n f(x)\, dx = f(1)$.

5. (a) Let $A$ be a real $2 \times 2$ matrix for which $AA^{\mathbf{t}} = I$, where $\mathbf{t}$ denotes the transpose and $I$ is the identity matrix. Let $B$ be the matrix obtained from $A$ by replacing exactly one of its rows by its negative. Show that at least one of the matrices $A - I$ and $B - I$ must be singular.

   (b) Does the same result hold if $A$ is a real $n \times n$ matrix for $n \geq 3$?

6. Let $r$ be a positive integer. For $0 \leq k \leq 9r$, let $f_r(k)$ be the number of integers between 0 and $10^r - 1 = 9\ldots9$ inclusive the sum of whose digits is equal to $k$. Determine the maximum value of $f_4(k)$.

7. (a) Determine all polynomials $z^3 + az^2 + bz + c$ with complex coefficients $a, b, c$ whose roots are $a, b, c$ (with the same multiplicity) when at least one of its coefficients is 0 or 1.

   (b) Show that, if the roots of the polynomial $z^3 + az^2 + bz + c$ with complex coefficients $a, b, c$, all distinct from 0 and 1, are the same as the coefficients $a, b, c$ (with the same multiplicity), then $a$ is a root of an irreducible cubic polynomial with integer coefficients, and $b$ and $c$ can be expressed as polynomials in $a$.

   *Definition:* An irreducible polynomial is one that cannot be factored as a product of polynomials of lower degree with rational coefficients.

8. What is the minimum number of subgroups (including the trivial subgroups, the singleton identity and the whole group) that a non-commutative group can have?

9. Suppose that $a$ and $b$ are positive real numbers. What is the maximum value of $b/a$ for which there exist real numbers $x$ and $y$ for which $0 \leq x \leq b$, $0 \leq y \leq a$ and

$$a^2 + x^2 = b^2 + y^2 = (a-y)^2 + (b-x)^2.$$

10. Suppose that $x_0 = 0$, $x_1 = 1$, and

$$x_{n+1} = x_n\sqrt{x_{n-1}^2 + 1} + x_{n-1}\sqrt{x_n^2 + 1}$$

for $n \geq 2$. Determine $x_n$.

## SOLUTIONS

1. Determine all strictly increasing functions $f$ from $[0, 1]$ onto $[0, 1]$ for which either (1) $f(x) + g(x) \leq 2x$ for **all** $x \in [0, 1]$, or (2) $f(x) + g(x) \geq 2x$ for **all** $x \in [0, 1]$. Here, $g(x)$ is the composition inverse function $f^{-1}$ of $f$ satisfying $f(g(x)) = g(f(x)) = x$ for $x \in [0, 1]$.

*Solution 1.* Since $f(x)$ and $g(x)$ are one-one onto, both must be increasing and continuous. The function $h(x) = f(x) + g(x) - 2x$ has constant sign on $[0, 1]$ and satisfies

$$\int_0^1 h(x)dx = \int_0^1 (f(x) + g(x))dx - \int_0^1 2xdx = 1 - 1 = 0.$$

Hence $h(x) = 0$ and $f(x) + g(x) = 2x$ for all $x \in [0, 1]$.

If $f(x) \not\equiv x$, wolog, we may suppose that for some $x_0 \in (0, 1)$, $f(x_0) = x_0 + u$ with $u > 0$. Then $g(x_0) = x_0 - u$ and so $f(x_0 - u) = x_0$. For $n \geq 1$, let $x_n = x_{n-1} - u$. Suppose, as an induction hypothesis, that $n \geq 1$ and $f(x_{n-1}) = x_{n-1} + u < 1$. Then $g(x_{n-1}) = x_{n-1} - u > 0$ and

$$f(x_n) = f(x_{n-1} - u) = x_{n-1} = x_n + u < 1.$$

But this leads to a contradiction, since $x_n = x_0 - nu < 0$ for $n$ sufficiently large.

*Solution 2.* One such function is $f(x) \equiv x$. Suppose that $f$ is a different function for which $f(x)+g(x) \leq 2x$ for $x \in [0, 1]$. By interchanging the roles of $f$ and $g$ if necessary, we may suppose that there exists $x_0 \in (0, 1)$ for which $f(x_0) < x_0$. For $n \geq 0$, define $x_{n+1} = f(x_n)$, so that $g(x_{n+1}) = x_n$. Since $f$ is increasing, and $0 < x_1 < x_0$, it follows by induction that $0 < x_{n+1} < x_n$ for every positive integer $n$.

We have that $x_{n+1} + x_{n-1} = f(x_n) + g(x_n) \leq 2x_n$, so that $x_{n-1} - x_n \leq x_n - x_{n+1}$ for each positive integer $n$. Hence

$$x_0 > x_0 - x_{n+1} = (x_0 - x_1) + (x_1 - x_2) + \cdots + (x_n - x_{n+1}) \geq n(x_0 - x_1)$$

for every positive integer $n$. But this is impossible, so the function $f$ cannot exist.

Suppose that $f(x) + g(x) \geq 2x$. Define $F(x) = 1 - f(1 - x)$ and $G(x) = 1 - g(1 - x)$. Then $F$ and $G$ are strictly increasing functions from $[0, 1]$ onto $[0, 1]$,

$$F(G(x)) = 1 - f(1 - G(x)) = 1 - f(g(1 - x)) = 1 - (1 - x) = x = G(F(x)),$$

and

$$F(x) + G(x) = 2 - [f(1 - x) + g(1 - x)] \leq 2 - 2(1 - x) = 2x.$$

Hence, by the foregoing argument, $F(x) \equiv x$, and so $f(x) \equiv x$.

*Solution 3, by Samuel Li.* Suppose that $f(x) + g(x) \leq 2x$ for all $x$. Let

$$S = \{(u, v) : ux \leq f(x), g(x) \leq vx \quad \forall x \in [0, 1]\}.$$

The set contains $(0, 2)$ in particular. Let $(u, v) \in S$. Then

$$g(x) \geq ux \Longrightarrow f(x) \leq 2x - g(x) \leq (2 - u)x$$

with a similar relation for $f(x)$. Hence $(u, 2-u) \in S$. Also

$$x = f(g(x)) \leq vg(x) \Longrightarrow g(x) \geq \frac{1}{v}$$

with a similar relation for $g(x)$. Hence $(1/v, v) \in S$. Suppose, as an induction hypothesis, that

$$\left(\frac{n-1}{n}, \frac{n+1}{n}\right) \in S.$$

This is true for $n = 1$. Then

$$\left(\frac{n}{n+1}, 2 - \frac{n}{n+1}\right) = \left(\frac{n}{n+1}, \frac{n+2}{n+1}\right) \in S.$$

Since this is true for all $n$, we must have that $f(x) = g(x) = x$ for all $x$.

The case $f(x) + g(x) \geq 2x$ can be similarly handled.

2. Determine all integers $m$ for which the following statement is FALSE: *there exists a nonnegative integer $r$ for which $m \equiv 2^r + 1 \pmod{2^{r+1}}$.*

*Solution 1.* Since $2^r \not\equiv 2^{r+1}$ for any nonnegative value of $r$, 1 is not congruent to $2^r + 1 \pmod{2^{r+1}}$. Thus the statement is false for $m = 1$.

Let $m \geq 2$ and suppose that $2^r$ is the highest power of 2 that divides $m - 1$. Since $m - 1$ is not divisible by $2^{r+1}$, $m - 1 \equiv 2^r \pmod{2^{r+1}}$, so that $m \equiv 2^r + 1 \pmod{2^{r+1}}$ as desired.

*Solution 2.* As above, the statement is false for $m = 1$. However, the statement is true for every other integer. Let $k$ be any positive integer. If $m$ is even, in particular $m = 2^k$, then $m \equiv 2 = 2^0 + 1 \pmod 2$, so $r = 0$. If $m = 2^k + 1$, then $r = k$. This establishes the result for $2 \leq m \leq 5$, with $r \leq 1$ for $2 \leq m \leq 4$. Suppose that $k \geq 2$ and the result has been established for $m < 2^k$, with the corresponding $r$ in each case not exceeding $k - 1$.

Let $m = 2^k + j$ where $2 \leq j \leq 2^k - 1$. There exists a nonnegative intgeger $r \leq k - 1$ for which $j \equiv 2^r + 1 \pmod{2^{r+1}}$. Then

$$m = 2^{r+1}2^{k-r-1} + j \equiv 2^r + 1 \pmod{2^{r+1}}.$$

Therefore, by an induction argument, it follows that the only integer for which the statement is false is $m = 1$.

*Comments.* Observe that, when $m$ is even, then the corresponding value of $r$ is 0.

There cannot be two distinct values of $r$ for which the statement is true. Suppose, on the contrary, that $u \geq v + 1$ and

$$n \equiv 2^u + 1 \pmod{2^{u+1}} \qquad n \equiv 2^v + 1 \pmod{2^{v+1}}.$$

Then $2^u + 1 \equiv 2^v + 1 \pmod{2^{v+1}}$ so that $1 \equiv 2^v + 1 \pmod{2^{v+1}}$. However, this is false.

3. Determine all finite sequences $(w_0, w_1, \ldots, w_n)$ of integers for which $n$ is a positive integer and for each $k$ $(0 \leq k \leq n)$, $w_k$ is the number of entries in the sequence equal to $k$.

*Solution.* From the definition $w_k \geq 0$ for each $k$. Since $w_0 + w_1 + \cdots + w_k \leq n + 1$, the number of entries in the sequence, $w_k \leq n + 1$. It is clear that $w_0 \neq 0$ and $w_k \neq n + 1$ for each $k$.

Evaluating the sum of the entries in two ways, we find that

$$w_0 + w_1 + w_2 + \cdots + w_n = 0 \cdot 1 + 1 \cdot w_1 + 2 \cdot w_2 + \cdots + k \cdot w_k + \cdots + n \cdot w_n,$$

whence

$$w_0 = w_2 + 2w_3 + \cdots + (k-1)w_k \cdots + (n-1)w_n.$$

If $w_0 = 1$, then $w_2 = 1$ and $w_k = 0$ for $3 \le k \le n$. We must have $w_1 = 2$, $n = 3$, yielding the sequence $(1, 2, 1, 0)$.

If $w_0 = 2$, then either $w_2 = 2$, $w_k = 0$ for $3 \le k \le n$ or $w_3 = 1$ and $w_k = 0$ for $k = 2$ and $4 \le k \le n$. In the first case, $w_1 = 0$ and we obtain the sequence $(2, 0, 2, 0)$ or $w_1 = 1$ and we obtain the sequence $(2, 1, 2, 0, 0)$. We cannot have $w_1 \ge 2$. In the second case, some entry must be repeated three times and there must be at least one three. Since the only possible nonzero entries are $w_0, w_1, w_3$, this case is impossible.

Let $w_0 = r \ge 3$. Then $w_r \ge 1$ and $(r-1)w_r \le r$, so that $w_r \le 3/2 < 2$. Therefore, $w_r = 1$. Hence

$$1 = w_0 - (r-1)w_r = w_2 + 2w_3 + \cdots + (r-2)w_{r-1} + rw_{r+1} + \cdots + (n-1)w_n,$$

so that $w_2 = 1$ and $w_k = 0$ for $3 \le k \le n$ and $k \ne r$. Thus, $w_0 = r$, $w_2 = w_r = 1$ and $w_1 = 2$. We find that

$$n = (w_0 + w_1 + \cdots + w_n) - 1 = r + 3$$

and the finite sequence must be

$$(r, 2, 1, 0, \ldots, 0, 1, 0, 0, 0)$$

where there are $r - 3$ zeros followed by 1 in the $r$th position and three additional zeros.

4. Let $f$ be a real continuously differentiable function on $[0.1]$. Prove that

(a) $\lim_{n\to\infty} n \int_0^1 x^n f(x^n)\, dx = \int_0^1 f(x)\, dx$;

(b) $\lim_{n\to\infty} n \int_0^1 x^n f(x)\, dx = f(1)$.

(a) *Solution.* Let $M = \sup\{|f(x)| : 0 \le x \le 1\}$. Making the substitution $u = x^n$, we have that

$$n \int_0^1 x^n f(x^n)\, dx = \int_0^1 u^{1/n} f(u)\, du.$$

Therefore

$$\begin{aligned}\int_0^1 f(x)\, dx - n \int_0^1 x^n f(x^n)\, dx &= \int_0^1 f(u)(1 - u^{1/n})\, dx \\ &\le \int_0^1 |f(u)|(1 - u^{1/n})\, du \le M \int_0^1 (1 - u^{1/n})\, du = \frac{M}{n+1}.\end{aligned}$$

Letting $n$ tend to infinity yields the result.

(b) *Solution 1.* Let $K = \sup\{|f'(x)| : 0 \le x \le 1\}$. Then

$$|f(1) - f(x)| = \left| \int_x^1 f'(t) dt \right| \le K(1 - x),$$

for each $x \in [0, 1]$. Therefore

$$\begin{aligned}\left| f(1) - (n+1) \int_0^1 x^n f(x)\, dx \right| &= (n+1) \left| \int_0^1 x^n (f(1) - f(x))\, dx \right| \\ &\le (n+1) \int_0^1 x^n |f(1) - f(x)|\, dx \\ &\le (n+1)K \int_0^1 x^n (1-x)\, dx = \frac{K}{n+2}.\end{aligned}$$

Since

$$\left|(n+1)\int_0^1 x^n f(x)dx - n\int_0^1 x^n f(x)dx\right| = \left|\int_0^1 x^n f(x)dx\right| \leq M\int_0^1 x^n dx = \frac{M}{n+1},$$

the desired result follows.

(b) *Solution 2.*

$$(n+1)\int_0^1 x^n f(x)\,dx = \left[x^{n+1}f(x)\right]_0^1 - \int_0^1 x^{n+1}f'(x)\,dx.$$

The first term on the right is equal to $f(1)$ and the second is dominated in absolute value by $K\int_0^1 x^{n+1}\,dx = K(n+2)^{-1}$ with $K$ as above. The result follows.

5. (a) Let $A$ be a real $2\times 2$ matrix for which $AA^{\mathbf{t}} = I$, where $\mathbf{t}$ denotes the transpose and $I$ is the identity matrix. Let $B$ be the matrix obtained from $A$ by replacing exactly one of its rows by its negative. Show that at least one of the matrices $A - I$ and $B - I$ must be singular.

   (b) Does the same result hold is $A$ is a real $n \times n$ matrix when $n \geq 3$?

(a) *Solution 1.* The rows and columns of $A$ are unit vectors and the two rows are orthogonal, as are the two columns. We may assume that the first row of $A$ is $(\cos\theta, \sin\theta)$; then the second row is either $(-\sin\theta, \cos\theta)$ or $(\sin\theta, -\cos\theta)$. Let $A$ be

$$A = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix},$$

and $B$ be

$$B = \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix},$$

Observe that $AA^{\mathbf{t}} = BB^{\mathbf{t}} = I$; $A$ is a rotation and $B$ is a reflection with axis along $(\sin\theta, 1-\cos\theta)$. The second row of each is the negative of the second row of the other.

Wolog, we may suppose that we change the sign of the second row to obtain of $A$ to obtain $B$, and we can interchange the roles of $A$ and $B$. Thus, we just have to examine $A - I$ and $B - I$. We have

$$A - I = \begin{pmatrix} \cos\theta - 1 & \sin\theta \\ -\sin\theta & \cos\theta - 1 \end{pmatrix}$$

and

$$B - I = \begin{pmatrix} \cos\theta - 1 & \sin\theta \\ \sin\theta & -\cos\theta - 1 \end{pmatrix}.$$

Since the determinant of $B - I$ is equal to $1 - \cos^2\theta - \sin^2\theta = 0$, the result follows.

*Solution 2, by E. Hessami Pilehrood.* Let

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \qquad \text{and} \qquad B = \begin{pmatrix} a & b \\ -c & -d \end{pmatrix}.$$

Then $a^2 + b^2 = c^2 + d^2 = 1$ and $ac + bd = 0$. Since the transpose of $A$ has the same determinant as $A$ and the two matrices are inverses, the determinant $ad - bc$ of $A$ is equal to $\pm 1$.

Suppose that $ad - bc = 1$. Then

$$\begin{aligned}(a-d)^2 + (b+c)^2 &= (a^2 - 2ad + d^2) + (b^2 + 2bc + c^2) \\ &= (a^2 + b^2) + (c^2 + d^2) - 2(ad - bc) = 0.\end{aligned}$$

Hence $a = d$ and $b = -c$. The determinant of $B - I$ is equal to

$$-(a-1)(d+1) + bc = -(ad - bc) + (d - a) + 1 = 0.$$

Similarly, when $ad - bc = -1$, $(a+d)^2 + (b-c)^2 = 0$, so that $a = -d$, $b = c$. and the determinant of $A - I$ vanishes.

(b) *Solution.* The result holds for $n \geq 3$. We note that (1) $AA^{\mathbf{t}} = BB^{\mathbf{t}} = I$; (2) the eigenvalues of both $A$ and $B$ occur in complex conjugate pairs, and each pair has positive product; (3) If $\lambda$ is a real eigenvalue with eigenvector $x$, then

$$\lambda^2 x^{\mathbf{t}} x = (Ax)^t (Ax) = x^{\mathbf{t}} A^{\mathbf{t}} Ax = x^{\mathbf{t}} x,$$

with a similar computation for $B$, and so $\lambda = \pm 1$; (4) if $\lambda = 1$, then $x$ is an eigenvector of $A - I$ with eigenvalue 0 and $A - I$ is singular; (5) the sign of the determinant of either $A$ or $B$ is positive if $-1$ is an eigenvalue with even multiplicity and odd otherwise. In counting eigenvalues, we include multiplicity.

Suppose that $n$ is odd. Each matrix has oddly many real eigenvalues. If $\det A > 0$, then $-1$ occurs evenly often and so 1 is an eigenvalue. Since $A - I$ annihilates its eigenvector, then $A - I$ is singular. Otherwise, $\det B > 0$ and $B - I$ is singular.

Suppose that $n$ is even. Each matrix has evenly many real eigenvalues. If $\det A < 0$, then $-1$ occurs an odd number of times and so 1 is an eigenvalue and $A - I$ is singular. Otherwise, $\det B < 0$ and $B - I$ is singular.

6. Let $r$ be a positive integer. For $0 \leq k \leq 9r$, let $f_r(k)$ be the number of integers between 0 and $10^r - 1 = 9 \ldots 9$ inclusive the sum of whose digits is equal to $k$. Determine the maximum value of $f_4(k)$.

   *Solution.* It is easily checked that $f_1(k) = 1$ for $0 \leq k \leq 9$ and

$$f_2(k) = \begin{cases} k+1 & \text{if } 0 \leq k \leq 9; \\ 19 - k & \text{if } 10 \leq k \leq 18. \end{cases}$$

Let $0 \leq n \leq 10^r - 1$. The sum of the digits of $n$ is $k$ if and only if the sum of the digits of $(10^r - 1) - n$ is $9r - k$. It follows that $f_r(k) = f_r(9r - k)$.

For $r = 2, 3$, if we write each number as an $r-$ digit number with first digit possibly 0 and categorize the numbers according to the first digit, we obtain

$$f_{r+1}(k) = f_r(k) + f_r(k-1) + f_r(k-2) + \cdots + f_r(k-9),$$

where we take $f_r(x) = 0$ when $x < 0$. Then, when $0 \leq k \leq 9$,

$$f_3(k) = (k+1) + k + \cdots + 1 = \frac{(k+1)(k+2)}{2}.$$

Then $f(9) = f(18) = 55$, $f(10) = f(17) = 63$, $f(11) = f(16) = 69$, $f(12) = f(15) = 73$ and $f(13) = f(14) = 75$.

Since $f_2(k)$ is increasing for $0 \leq k \leq 13$, so also is $f_3(k)$. Applying the foregoing formula for $r = 4$, we find that $f_4(13) = 480$, $f_4(14) = 540$, $f_4(15) = 592$, $f_4(16) = 633$, $f_4(17) = 660$ and $f_(18) = 670$. Since $f_4(k) = f(36 - k)$ for $k \geq 18$, we see that $f_4(k)$ assumes its maximum value of 670 when $k = 18$.

7. (a) Determine all polynomials $z^3 + az^2 + bz + c$ with complex coefficients $a, b, c$ whose roots are $a, b, c$ (with the same multiplicity) when at least one of its coefficients is 0 or 1.

   (b) Show that, if the roots of the polynomial $z^3 + az^2 + bz + c$ with complex coefficients $a, b, c$, all distinct from 0 and 1, are the same as the coefficients $a, b, c$ (with the same multiplicity), then $a$ is a root of an irreducible cubic polynomial with integer coefficients, and $b$ and $c$ can be expressed as polynomials in $a$.

*Note:* An irreducible polynomial is one that cannot be factored as a product of polynomials of lower degree with rational coefficients.

*Solution.* (a) If one of the coefficients/roots in 0, then $c$, the product of the roots, is 0. In this case, $z^2 + az + b$ is a quadratic polynomial whose roots are $a$ and $b$. Therefore $a + b = -a$ and $ab = b$, so that, either $(a, b) = (0, 0)$ or $(a, b) = (1, -2)$. We obtain the polynomials $z^3$ and

$$z^3 + z^2 - 2z = z(z - 1)(z + 2).$$

Suppose that $p(z) = z^3 + az^2 + bz + c$ with $c \neq 0$, roots $a, b, c$ and that $p(1) = 0$. Then

$$-a = a + b + c = p(1) - 1 = -1,$$

so that $a = 1$. Since $-c = abc = bc$, then $b = -1$. Thus

$$0 = p(b) = p(-1) = -1 + a - b + c = 1 + c.$$

Therefore

$$p(z) = z^3 + z^3 - z - 1 = (z^2 - 1)(z + 1) = (z - 1)(z + 1)^2.$$

(b) Suppose the roots of $z^3 + az^2 + bz + c$ are $a, b, c$ and that $p(0)p(1) \neq 0$. Then

$$a + b + c = -a, \qquad ab + bc + ca = b, \qquad abc = -c.$$

Since $c \neq 0$, $ab = -1$ and $b = -1/a$. Since $b + c = -2a$ and $a(b + c) + bc = b$, it follows that $c = (1/a) - 2a$ and

$$-2a^2 + \left(-\frac{1}{a}\right)\left(\frac{1}{a} - 2a\right) = -\frac{1}{a},$$

whence

$$0 = 2a^4 - 2a^2 - a + 1 = (a - 1)(2a^3 + 2a^2 - 1).$$

The cubic factor cannot vanish for any integer or half-integer $a$, and so must be irreducible. It follows that

$$(a, b, c) = (a, -2a^2 - 2a, 2a^2).$$

Conversely, if $a$ satisfies $2a^3 + 2a^2 = 1$, then it can be checked that the polyonomial with these coefficients $a, b, c$ has roots $a, b, c$.

8. What is the minimum number of subgroups (including the trivial subgroups, the singleton identity and the whole group) that a non-commutative group can have?

*Solution.* The answer is *six*. The symmetric group $S_3$ has exactly six subgroups consisting of one subgroup of order 1, three cyclic subgroups of order 2, one cyclic subgroup of order 3 and the whole group of order 6. Another example is the quaternion subgroup of 8 elements, $\{1, i, j, k, -1, -i, -j, -k\}$ where $i^2 = j^2 = k^2 = -1$, $ij = k$, $jk = i$ and $ki = j$. This has one subgroup of order 1, one subgroup of order 2, three cyclic subgroups of order 4 and the whole group.

Let $G$ be a non-commutative group with two elements $a$ and $b$ for which $ab \neq ba$. Given any element $x$, the integer powers of $x$ constitute a cyclic group $\langle x \rangle$ of $g$. Since no pair of the elements $a, b, ab$ commute, no one of them can belong to the cyclic group generated by the others. Thus, we have five distinct subgroups $\langle e \rangle$, $\langle a \rangle$, $\langle b \rangle$, $\langle ab \rangle$ and $G$. If the order of $a$, $b$ or $ab$ is either composite or infinite, then its cyclic subgroup must have a proper subgroup distinct from each of the cyclic subgroups so far.

The remaining case is where the orders of $a$, $b$ and $ab$ are all prime. Let $A$, $B$, $C$ be the respective subgroups generated by these elements. Since any element outside their union would generate an additional subgroup, $G = A \cup B \cup C$. Also $\{e\} = A \cap B \cap C$. Let $u, v, w$ be elements (not the identity) in $A, B, C$ respectively. Since $uv$ cannot lie in $A$ nor $B$, it must belong to $C$ and so $uvw \in C$. Similarly, $uvw \in A$. Hence $uvw = e$. If $u'$ and $v'$ are any distinct elements of $A$ and $B$ respectively, then $u'v' = w^{-1} = uv$, whence $u^{-1}u' = vv'^{-1} = e \in A \cap B$. Thus $u = u'$ and $v = v'$, so that $A$ and $B$ are subgroups of order 2; likewise $C$ has order 2. This gives us the 4−group, which is commutative. So this case cannot occur and the result follows.

*Comment.* If $a^2 \neq e$, then $a^2b$ cannot commute with $ab$; otherwise $(a^2b)(ab) = (ab)(a^2b) \Rightarrow ab = ba$. Thus $a^2b$ would generate a subgroup distinct from that generated by $ab$. Also $a^2b$ cannot belong to the subgroups generated by either $a$ nor $b$. Once again, we are reduced to considering the case where every element not in the centre has order 2.

9. Suppose that $a$ and $b$ are positive real numbers. What is the maximum value of $b/a$ for which there exist real numbers $x$ and $y$ for which $0 \leq x \leq b$, $0 \leq y \leq a$ and

$$a^2 + x^2 = b^2 + y^2 = (a - y)^2 + (b - x)^2.$$

*Solution 1.* Let $ABCD$ be a rectangle with $|AB| = |CD| = a$ and $|AD| = |BC| = b$. Determine points $E$ and $F$ on the respective sides $BC$ and $CD$ for which $|BE| = x$ and $|DF| = y$. Then the given conditions signify that the triangle $AEF$ is equilateral. The question is quivalent to the maximum ratio of the sides $AD$ and $AB$ for which an equilateral triangle can be inscribed in the rectangle with one vertex at $A$.

Suppose that $AB$ is fixed and let $\theta = \angle BAE$. For the inscription to be possible, $0 \leq \theta \leq 30°$. The side length $s$ of the triangle $AEF$ is equal to $a \sec \theta$ and $|AD| = s \cos(30° - \theta) = a \sec \theta \cos(30° - \theta)$. Since $\sec \theta$ and $\cos(30° - \theta)$ are both increasing, we see that $|AD| \leq a \sec 30° = 2a/\sqrt{3}$. Therefore the maximum value of $\o b/a$ is $2\sqrt{3}/3$.

*Solution 2, by Richard Chow.* The conditions imply that the points $O \sim (0,0)$, $A \sim (a, x)$ and $B \sim (y, b)$ constitute an equilateral triangle in the first quadrant of the Cartesian plane with $A$ and $B$ located on the circumference of a circle of radius $r$ centred at the origin. Let $A$ start on the $x-$axis and rotate the triangle counterclockwise until $B$ arrives on the $y-$axis. Then $a$ decreases and $b$ increases, so that $b/a$ increases. The maximum value of $b/a$ occurs when $A \sim (r\sqrt{3}/2, r/2)$ and $B \sim (0, r)$. Thus the desired maximum is $2/\sqrt{3} = 2\sqrt{3}/3$.

10. Suppose that $x_0 = 0$, $x_1 = 1$, and

$$x_{n+1} = x_n \sqrt{x_{n-1}^2 + 1} + x_{n-1} \sqrt{x_n^2 + 1}$$

for $n \geq 2$. Determine $x_n$.

*Solution 1.* Let $x_n = \sinh u_n$ and $v_n = e^{u_n}$ for $n \geq 0$. Then $(u_0, v_0) = (0, 1)$ and $v_1$, being the positive root of the equation $\frac{1}{2}(t - t^{-1}) = 1$, is equal to $1 + \sqrt{2}$. We have, for each $n \geq 2$, that

$$\sinh u_{n+1} = x_{n+1} = \sinh u_n \cosh u_{n-1} + \sinh u_{n-1} \cosh u_n = \sinh(u_n + u_{n-1}),$$

whence $u_{n+1} = u_n + u_{n-1}$ amd $v_{n+1} = v_n v_{n-1}$.

Thus $v_2 = v_1 v_0 = v_1$, $v_3 = v_2 v_1 = v_1^2$, $v_4 = v_3 v_2 = v_1^3$, and, by induction,

$$v_n = v_1^{f_n} = (1 + \sqrt{2})^{f_n},$$

for $n \geq 0$, where $f_n$ is the Fibonacci sequence givn by $f_0 = 0$, $f_1 = 1$ and $f_{n+1} = f_n + f_{n-1}$. Note that

$$v_n^{-1} = (-1)^{f_n}(1 - \sqrt{2})^{f_n},$$

so that $v_n^{-1} = (1-\sqrt{2})^{f_n}$ when $n \equiv 0 \pmod 3$ and $v_n^{-1} = -(1-\sqrt{2})^{f_n}$ otherwise.

Therefore

$$x^n = \frac{1}{2}(v_n - v_{n-1})$$

for each nonnegative integer $n$.

$$\{x_n\} = \{0, 1, 1, 2\sqrt{2}, 7, 41, 204\sqrt{2}, 47321, \ldots\}.$$

*Solution 2.* Let $\sinh c = 1$. Then $x_0 = \sinh 0 = \sinh cf_0$ and $x_1 = \sinh cf_1$. Suppose as an induction hypothesis for $n \geq 1$, $0 \leq k \leq n$, $x_k = \sinh cf_k$. Then

$$x_{n+1} = \sinh cf_n \cosh cf_{n-1} + \sinh cf_{n-1} \cosh cf_n = \sinh c(f_n + f_{n-1}) = \sinh cf_{n+1}.$$

Hence, $x_n = \sinh cf_n$ for each nonnegative integer $n$.