THE UNIVERSITY OF TORONTO
UNDERGRADUATE MATHEMATICS COMPETITION

In Memory of Robert Barrington Leigh

March 10, 2018

Time: $3\frac{1}{2}$ hours

`No aids or calculators permitted.`

The grading is designed to encourage only the stronger students to attempt more than five problems. Each solution is graded out of 10. If the sum of the scores for the solutions to the five best problems does not exceed 30, this sum will be the final grade. If the sum of these scores does exceed 30, then all solutions will be graded for credit.

1. Prove or disprove: C is a $n \times n$ square matrix such that $C \neq O$, and $C^2 = O$ if and only if there are $n \times n$ matrices for which $AB = C$ and $BA = O$.

2. Let n be a positive integer. Determine the infimum (greatest lower bound) and supremum (least upper bound) of the set S_n of rationals r for which there exist positive integers a, b, c for which

$$r = \frac{a}{b}, \qquad \text{and} \qquad nr = \frac{a+c}{b+c}.$$

3. Determine a linear differential equation for y as a function of x that is satisfied by $y = x^3$ and $y = \cos x$, but not by a constant function.

4. Let p be a parabola with focus at F and directrix d. Suppose that P is an arbitrary point, except the vertex, on p and the tangent to p at P meets the directrix at the point Q. Prove that PQ bisects the angle between FQ and d.

   [*Note:* p is the locus of points P for which the length of PF is equal to the distance of P from d.]

5. Let n be a positive integer. Alf and Ben play the following game. Alf selects a number between 1 and n inclusive; Ben makes a guess as to what it is. If Ben guesses correctly, then he wins. Otherwise, the game continues into other rounds. At each round, Alf chooses a number that is either one more or one less than the number he chose on the previous round, and Ben makes a guess at to what it is. Prove that Ben has a winning strategy and determine the maximum number of rounds that that the game can proceed with this strategy.

6. Prove that

$$\int_0^1 \left[2x + (1+20x)^{1/30}(1+17x)^{1/51}\right] dx < e.$$

7. Let n be a positive integers and $u_1, u_2, \ldots, u_n$ be complex numbers. Determine necessary and sufficient conditions that the set

$$S = \{(u_1^k, u_2^k, \ldots, u_n^k) : 0 \leq k \leq n, k \neq n-1\}$$

   of n vectors in $\mathbf{C}$ be linearly independent. (Here $0^0 = 1$.)

8. Let $z_1, z_2, \ldots, z_n$ be complex numbers such that $z_1 + z_2 + \cdots + z_n = 0$. Prove that

$$2\sum_{k=1}^{n} |z_k|^2 \leq \left(\sum_{k=1}^{n} |z_k|\right)^2.$$

9. Let m be a positive integer and $f(x)$ be a continuous function on $[0, 1]$ for which

$$\int_0^1 f(t)t^k dt = 0$$

for $0 \leq k \leq m-1$. Prove that $f(x) = 0$ for at least m distinct values of x in $(0, 1)$.

10. Determine all polynomials $f(x)$ defined on the set of integers and taking integer values that satisfy the equation

$$f(x-1)f(x+1) = f(x)^2 - 2f(x) + 4$$

for each integer x and take at least one positive value.

END

Solutions.

1. Prove or disprove: C is a $n \times n$ square matrix such that $C \neq O$, and $C^2 = O$ if and only if there are $n \times n$ matrices for which $AB = C$ and $BA = O$.

Comment. Unfortunately, the statment of the problem gave rise to an ambiguity. It was intended that the logical equivalence apply only to nonzero matrics, but some students took $C \neq O$ as part of the conclusion from the second part of the equivalence.. In this case, $C = O$ yields a counterexample to the backwards implication.

Solution 1. The condition is sufficient, since $C^2 = ABAB = AOB = O$.

Consider the matrices as operating on an $n-$dimensional vector space. Let X be the nullspace of C spanned by $\langle x_1, x_2, \ldots, x_k \rangle$, and let Y be a complementary subspace spanned by $\langle x_{k+1}, \ldots, x_n \rangle$. We note that $k \neq 0, n$ and that $CY \subseteq X$. Let $A = C$ and let $Bx_i = O$ for $1 \leq i \leq k$ and $Bx_i = x_i$ for $k+1 \leq i \leq n$. Then

$$ABX = O = CX; \quad ABY = CY; \quad BAX = BCX = O; \quad BAY \subseteq BX = O.$$

Comment. Another possibility is to let A agree with C at x_i for $k+1 \leq i \leq n$ and with the identity on x_i for $1 \leq i \leq k$.

Solution 2. We can deal with the Jordan canonical for for C, which has 1×1 zero blocks and at least one 2×2 blocks of the form

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

down the main diagonal. Noting that

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

and

$$\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},$$

we can patch together blocks to form tbe matrics A and B with $A = C$.

2. Let n be a positive integer. Determine the infimum (greatest lower bound) and supremum (least upper bound) of the set S_n of rationals r for which there exist positive integers a, b, c for which

$$r = \frac{a}{b}, \qquad \text{and} \qquad nr = \frac{a+c}{b+c}.$$

Solution 1. We first deal with the case $n = 1$. Then

$$\frac{a}{b} = \frac{a+c}{b+c}$$

which is possible only if $a = b$. In this case, $S_1 = \{1\}$, and $\inf S_1 = \sup S_1 = 1$. Henceforth, let $n \geq 2$. Since

$$\frac{a+c}{b+c} - \frac{a}{b} = \frac{c(b-a)}{b(b+c)},$$

then $(a+c)/(b+c) > a/b$ if and only if $a/b < 1$. Also $(a+c)/(b+c) < 1$ when $a/b < 1$. It follows that $r < 1/n$ for all $r \in S_n$. Thus $S_n \subseteq (0, 1/n)$.

Observe that, if $c = ab$, then

$$\frac{a+c}{b+c} = \frac{a(b+1)}{b(a+1)} = n \cdot \frac{a}{b}$$

if and only if $b+1 = n(a+1)$ or $b = na + (n-1)$. If u is any positive integer, and we let

$$(a,b,c) = (u, nu + (n-1), nu^2 + (n-1)u),$$

then $r = u/(nu + (n-1)) \in S_n$. Since

$$\lim_{u \to \infty} \frac{u}{nu + (n-1)} = \frac{1}{n},$$

it follows that $\sup S_n = 1/n$.

On the other hand, if $v \geq 2$ and

$$(a,b,c) = (v-1, nv(v-1), (n-1)v),$$

then $r = 1/nv$ and

$$\frac{a+c}{b+c} = \frac{(v-1)+(n-1)v}{(nv^2 - nv) + (n-1)v} = \frac{nv-1}{nv^2 - v} = \frac{1}{v}.$$

Thus $1/nv \in S_n$ and $\lim_{v \to \infty} 1/nv = 0$, whence $\inf S_n = 0$.

Comment. To give insight on the choice $r = 1/nv$, consider the special case $n = 2$, Then the equation $2a/b = (a+c)/(b+c)$ is equivalent to

$$2c^2 = (c-a)(2c+b).$$

One solution of this is given by $a = c-1$ and $b = 2c^2 - 2c = 2c(c-1)$. This solution for $n = 2$ has its analogue for general n.

Solution 2. The case $n = 1$ can be disposed of as before. The condition $na/b = (a+c)/(b+d)$ leads to

$$c(b - an) = (n-1)ab,$$

from which $r < 1/n$. This also suggests that we try the triple

$$(a,b,c) = (a, an+1, (n-1)ab = n(n-1)a^2 + (n-1)a)$$

for the rational $r = a/(an+1)$. Since $\lim_{a \to \infty} r = 1/n$, it follows that $\sup S_n = 1/n$.

For the other bound, we follow Andrew Gomes, let d be a positive integer exceeding 2 and take

$$(a,b,c) = n^{d-2} + n^{d-3} + \cdots + 1, n^{2d-2} + n^{2d-3} + \cdots + n^d, n^{d-1})$$

to botain the rational $r = 1/n^d$. Since $\lim_{d \to \infty} r = 0$, it follows that $\inf S_n = 0$.

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3. Determine a linear differential equation for y as a function of x that is satisfied by $y = x^3$ and $y = \cos x$, but not by y equal to a constant function.

Comment. Most of the equations provided were obtained in an *ad hoc* way. We give the neatest ones. Then we provide two more formal solutions providing a systematic way of solving the the problem.

One can ensure that the constant zero function is not a soution by making the equation nonhomogeneous.

$$(3x^2+6)y^{(5)} - (\sin x)y^{(3)} - \sin xy' + \sin x(3x^2+6) = 0.$$

$$x^3 y^{(7)} - 3x^2 y^{(6)} - 6xy^{(5)} + 6y^{(4)} + (\cos x)y''' + \sin xy'' + \cos xy' + \sin xy = 6\cos x + 6x\sin x + 3x^2\cos x + x^3 \sin x.$$

$$(x^3 - \cos x)y^{(4)} + (\cos x)y = x^3 \cos x.$$

$$(x^3 + 6x)y''' - (3x^2 + 6)y'' + (x^3 + 6x)y' - (3x^2 + 6)y = 0.$$

$$x^3 y^{(5)} + 6y^{4)} + x^3 y^{(3)} - 6y = 0.$$

Solution 1. We obtain a differential equation of the form

$$ay'' + by' + cy = 0,$$

where a, b, c are functions of x. Substituting x^3 and $\cos x$ for y gives us two algebraic linear equations for the functions a, b, c. Either by using Cramer's method or by computing the vector orthogonal to $(y'', y', y) = (6x, 3x^2, x^3)$ and $(y'', y', y) = (-\cos x, -\sin x, \cos x)$, we arrive at the equation

$$(3x^2 \cos x + x^3 \sin x)y'' - x \cos x(x^2 + 6)y' + x(3x \cos x - 6 \sin x)y = 0.$$

Solution 2. Let D denote the differential operator: $Dy = y'$. Let $Iy = y$. Then $y = \cos x$ is a solution of the differential equation $(D^2 + I)y = 0$. We need to determine a polynomial $p(D)$ of D for which

$$0 = p(D)(D^2 + I)x^3 = p(D)(6x + x^3).$$

There are two possibilities:

(a) If $z = 6x + x^3$, then $z' = 6 + 3x^2$ and $z'' = 6x$. By inspection, we note that $2z'' + xz - 3z = 0$ and can take $p(D) = 2D^2 + xD - 3I$. Then the equation

$$\begin{aligned} 0 &= p(D)(D^2 + I)y = (2D^2 + xD - 3I)(D^2 + I)y \\ &= (2D^4 + xD^3 - D^2 + xD - 3I)y \\ &= 2y^{(4)} + xy^{(3)} - y'' + xy' - 3y \end{aligned}$$

is satisfied by $y = \cos x$ and $y = x^3$.

(b) We can determine a linear equation that is satisfied by $y = x$ and $y = x^3$, and therefore by $y = 6x + x^3$. Trying the form $x^2 y'' + axy' + by = 0$ leads to the choice $p(D) = x^2 D^2 - 3xD + 3I$. Then the equation

$$\begin{aligned} 0 &= p(D)(D^2 + 1)y = (x^2 D^2 - 3xD + 3I)(D^2 + I)y \\ &= (x^2 D^4 - 3xD^3 + (x^2 + 3)D^2 - 3xD + 3I)y \\ &= x^2 y^{(4)} - 3xy^{(3)} + (x^2 + 3)y'' - 3xy' + 3y \end{aligned}$$

is satisfied by $y = \cos x$ and $y = x^3$.

Further comments. If we remove the restriction about the constant function, then there is a straightforward answer:

$$0 = D^4(D^2 + 1)y = (D^6 + D^4)y.$$

If we allow nonlinearity, then we have the equation

$$0 = (xy' - 3y)(y'' + y) = xy'y'' - 3yy'' + xyy' - 3y^2.$$

An approach similar to that in Solution 1 using y', y^2 and y leads to the possibility

$$(x^6 \cos x - x^3 \cos^2 x)y' - (x^3 \sin x + 3x^2 \cos x)y^2 + (3x^2 \cos^2 x + x^6 \sin x)y = 0.$$

4. Let p be a parabola with focus at F and directrix d. Suppose that P is an arbitrary point, except the vertex, on p and the tangent to p at P meets the directrix at the point Q. Prove that PQ bisects the angle between FQ and d.

   [*Note:* p is the locus of points P for which the length of PF is equal to the distance of P from d.]

Solution 1. Let the line parallel to the exis of the parabola passing through P meet the directrix at R.By the reflection property, $\angle QPR$, being equal to the vertically opposite ange at P is equation o $\angle QPF$. Since $PF = PR$, triangles QPF and QPR are congruent (SAS), and so $\angle FQP = \angle RQP$, as desired.

Solution 2. Since all parabolas are similar, we can assume that p is the locus of an equation of the form $y = kx^2$, where the focus is at $(0, 1)$ and the directrix has equation $y = -1$. The point $(1, k)$ is on p, so that we must have

$$(k-1)^2 + 1 = (k+1)^2$$

whence $k = 1/4$. Thus we consider the parabola with equation $4y = x^2$.

Let $P \sim (2u, u^2)$; wolog, we may suppose by symmetry that $u > 0$. The slope of the tangent to u at P is equal to u, and its equation is $y = ux - u^2$. This intersects d at the point

$$Q \sim \left(\frac{u^2 - 1}{u}, -1 \right) .$$

Let θ be the (acute) angle between PQ and d. Then $\tan \theta = u$, so that

$$\tan 2\theta = \frac{2u}{1 - u^2} .$$

The tangent of the angle between FQ and d is equal to the slope of the line FQ, namely

$$\frac{2}{-(u^2 - 1)/u} = \frac{2u}{1 - u^2} .$$

Hence the angle between FQ and d is 2θ and the result follows.

5. Let n be a positive integer. Alf and Ben play the following game. Alf selects a number between 1 and n inclusive; Ben makes a guess as to what it is. If Ben guesses correctly, then he wins. Otherwise, the game continues into other rounds. At each round, Alf chooses a number that is either one more or one less than the number he chose on the previous round, and Ben makes a guess at to what it is. Prove that Ben has a winning strategy and determine the maximum number of rounds that that the game can proceed with thisstrategy.

Solution. Ben guesses in order the numbers $1, 2, 3, \ldots, n$. If Alf's first choice is an odd number, then if another round is needed, this choice must exceed 1. On succeeding rounds, Alf's choice and Ben's guess both alternate and so agree in parity. Since both Alf and Ben proceed by unit steps, their numbers must agree at some point. (The difference between Alf's and Ben's number is nonnegative to begin with, reduces by 2 or stays the same at each turn, and is nonpositive at the end.)

If Alf's first choice is an even number, then the numbers chosen by Alf and Ben will always have opposite parity and Ben will not succeed. However, when Ben reaches n without success, he will know that Alf chose an even number initially. If n is odd, then Alf's choice on move $n + 1$ will be odd, so Ben again guesses in order $1, 2, 3, \ldots, n$. If n is even, then Alf's guess on move $n + 1$ will be even and Ben guesses in order $2, 3, \ldots, n$. As in the first round, Ben must eventually guess Alf's number.

If n is odd and two complete run-throughs are needed, we may require $2n$ moves. If n is even, we may require $2n - 1$ moves.

Comment. It was intended, and so read by teh contestants, that Alf alwayas made his choice within the set $\{1, 2, \ldots, n\}$. However, if we allow Alf to pick a number exceeding n after the firsgt move, Ben can still succeed. To begin with, Ben's first guess is n and each subsequent guess reduces by 1 until he reaches 1. This will succeed if Alf's first choice has the same parity as n. If Ben has not succeeded at this stage, then Alf's first choice did not exceed $2n-1$ and cannot exceed $3n-2 = (2n-1)+(n-1)$ when Ben makes the guess 1. However, Ben now knows the parity of Alf's choices, and depending on this, can now start guessing at $3n-2$ or $3n-1$ and work his way down in unit steps.

6. Prove that
$$\int_0^1 \left[2x + (1+20x)^{1/30}(1+17x)^{1/51}\right] dx < e.$$

Solution 1. Recall that $\log(1+t) < t$ for $t > 0$. Hence
$$\frac{1}{30}\log(1+20x) + \frac{1}{51}\log(1+17x) < \frac{2}{3}x + \frac{1}{3}x = x.$$
The integrand is less than $2x + e^x$ when $0 < x \leq 1$ and so the integral is strictly less than
$$\int_0^1 (2x+e^x)dx = \left[x^2+e^x\right]_0^1 = 1 + (e-1) = e.$$

Solution 2. Observe that, for $0 \leq x \leq 1$,
$$(1+20x)^{1/30} < 27^{1/30} = 3^{1/10} < 3^{1/6}$$
and
$$(1+17x)^{1/51} < 27^{1/51} = 3^{1/17} < 3^{1/6},$$
so that the given integral is dominated by
$$\begin{aligned}\int_0^1 (2x + 3^{1/3})\, dx = 1 + 3^{1/3} &< 1 + \frac{3}{2} \\ &= 1 + 1 + \frac{1}{2!} < e.\end{aligned}$$

7. Let n be a positive integers and $u_1, u_2, \ldots, u_n$ be complex numbers. Determine necessary and sufficient conditions that the set
$$S = \{(u_1^k, u_2^k, \ldots, u_n^k) : 0 \leq k \leq n, k \neq n-1\}$$
of n vectors in $\mathbf{C}$ be linearly independent. (Here $0^0 = 1$.)

Solution. Suppose first that all the u_i are not distinct, that $u_r = u_s$ for distinct indices r and s. Then S is a set of n vectors in the $(n-1)-$dimensional subspace $\{(x_1, x_2, \ldots, x_n) : x_r = x_s\}$ and so must be linearly dependent.

Now suppose that all the u_i are distinct. If S is linearly dependent, then there are constants $a_0, a_1, \ldots, a_{n-2}, a_n$ for which
$$a_n(u_1^n, u_2^n, \ldots, u_n^n) + \sum_{k=0}^{n-2} a_k(u_1^k, u_2^k, \ldots, u_n^k) = 0.$$
Therefore $u_1, u_2, \ldots, u_n$ are all the roots of the polynomial
$$p(x) = a_n x^n + a_{n-2}x^{n-2} + \cdots + a_1 x + a_0.$$
Since the coefficient of x^{n-1} is 0, the sum $u_1 + u_2 + \cdots + u_n$ vanishes.

On the other hand, suppose that all the u_i are distinct and that $u_1 + u_2 + \cdots + u_n = 0$. Then

$$(x - u_1)(x - u_2) \cdot (x - u_n) = x^n + \sum_{k=0}^{n-2} b_k x^k$$

for some constants $b_0, b_1, \ldots, b_{n-2}$. Hence

$$(u_1^n, u_2^n, \ldots, u_n^n) + \sum_{k=0}^{n-2} b_k (u_1^k, u_2^k, \ldots, u_n^k) = 0,$$

and the set S is linearly dependent.

Hence the set S is linearly independent if and only if all the u_i are distinct and $u_1 + u_2 + \cdots + u_n \neq 0$.

Example.

$$(1, 16, 81, 256) - 15(1, 4, 9, 16) - 10(1, -2, -3, 4) + 24(1, 1, 1, 1) = (0, 0, 0, 0).$$

8. Let $z_1, z_2, \ldots, z_n$ be complex numbers such that $z_1 + z_2 + \cdots + z_n = 0$. Prove that

$$2 \sum_{k=1}^{n} |z_k|^2 \leq \left(\sum_{k=1}^{n} |z_k| \right)^2 .$$

Solution.

$$\begin{aligned} 0 = |z_1 + z_2 + \cdots + z_n|^2 &= (z_1 + z_2 + \cdots + z_n)(\overline{z_1} + \overline{z_2} + \cdots + \overline{z_n}) \\ &= \sum_{k=1}^{n} |z_k|^2 + 2 \sum_{1 \leq i,j \leq n} \text{Re } (z_i \overline{z_j}) \end{aligned}$$

whence

$$\begin{aligned} \sum_{k=1}^{n} |z_k|^2 &= -2 \sum_{1 \leq i,j \leq n} \text{Re } (z_i \overline{z_j}) \\ &\leq 2 \sum_{1 \leq i,j \leq n} |\text{Re } (z_i \overline{z_j})| \\ &\leq 2 \sum_{1 \leq i,j \leq n} |z_i \overline{z_j}| \\ &\leq 2 \sum_{1 \leq i,j \leq n} |z_i||z_j| \\ &= \left(\sum_{k=1}^{n} |z_k| \right)^2 - \sum_{k=1}^{n} |z_k|^2, \end{aligned}$$

so that

$$2 \sum_{k=1}^{n} |z_k|^2 \leq \left(\sum_{k=1}^{n} |z_k| \right)^2 .$$

9. Let m be a positive integer and $f(x)$ be a continuous function on $[0, 1]$ for which

$$\int_0^1 f(t) t^k dt = 0$$

for $0 \leq k \leq m-1$. Prove that $f(x) = 0$ for at least m distinct values of x in $(0, 1)$.

Solution. The condition implies that $\int_0^1 f(t)p(t)dt = 0$ whenever $p(t)$ is a polynomial of degree less than m. Suppose that $f(x)$ has exactly r zeros $a_1 < a_2 < \cdots < a_r$ in $(0, 1)$ where $r \leq m-1$. Let $b_1 < b_2 < \cdots < b_s$ be the largest subset of these for which f changes sign on an arbitrarily small neighbourhood of the zero. Note that $s \leq r \leq m-1$.

We may suppose that $f(x)$ is positive on the open interval $(0, b_1)$. Let $p(t) = (-1)^s(x-b_1)(x-b_2)\dots(x-b_s)$. Then $f(t)$ and $p(t)$ have the same sign on each of the intervals $(0, b_1)$, (b_i, b_{i+1}) $(1 \leq i \leq s-1)$ and $(b_s, 1)$, so that $f(x)p(x) \geq 0$ on $[0, 1]$. But then $f(x)p(x) \equiv 0$, yielding a contradiction.

10. Determine all polynomials $f(x)$ defined on the set of integers and taking integer values that satisfy the equation

$$f(x-1)f(x+1) = f(x)^2 - 2f(x) + 4$$

for each integer x and takes at least one positive value.

Solution. We make some preliminary observations:

(1) The equation can be rewritten as

$$f(x-1)f(x+1) = (f(x) - 1)^2 + 3.$$

(2) If $f(x)$ satisfies the equation, then so also so $g(x) = f(-x)$ and $g(x) = f(x+c)$ for any contstant c.

(3) $f(x)$ never assumes the value 0, since the product of its values at either consecutive even or consecutive odd integers is always at least 3.

(4) $f(x)$ always assumes positive values. From (1), we see that it has the same sign on all even integers, and the same sign on all odd integers. If these signs were opposite, then there would be a zero between each pair of integers. But this is impossible for a polynomial with at most finitely many zeros.

(5) Since $f(x)$ is integer valued, it assumes its minimum value m. Because of (2), we may assume that it assumes is mimimum value at 0 and that $f(-1) \leq f(1)$.

(6) $m = 1$ or $m = 2$. We have that

$$m^2 \leq f(-1)f(1) = m^2 - 2m + 4,$$

when $f(0) = m \leq 2$.

Suppose that $f(0) = 2$. Then $f(-1)f(1) = 4$. Since $f(-1)$ and $f(1)$ are both not less than 2, $f(-1) = f(1) = 2$. It follows that $f(x) = 2$ for all integers x.

Suppose that $f(0) = 1$. Then $f(-1) = 1$ and $f(1) = 3$. We continue using the given equation to determine that $f(2) = 7$; $f(3) = 13$; $f(4) = 21$. Noting that the second differences are constant, we conjecture that $f(x)$ is given by a quadratic polynomial: $f(n) = n^2 + n + 1$.

This can be established by induction: it holds for $0 \leq n \leq 4$. Assume it holds for $0 \leq n \leq m$. Then

$$(m^2 - m + 1)f(m+1) = m^4 + 2m^3 + m^2 + 3 = (m^2 - m + 1)(m^2 + 3m + 3),$$

whence

$$f(m+1) = m^2 + 3m + 3 = (m+1)^2 + (m+1) + 1.$$

The result follows by induction and we deduce that $f(x) = x^2 + x = 1$.

Taking account of observation 1, we find that all the required solutions are $f(x) \equiv 2$ and

$$f(x) = (x+a)^2 \pm (x+a) + 1 = x^2 + (2a \pm 1)x + (a^2 \pm a + 1),$$

where a is any integer.

Comment. There may be nonpolynomial solutions that take both positive and negative value. One observation that we can make is that if $f(n)$ is odd for any integer n, then $f(n-1)$ and $f(n+1)$ must both be odd. Hence we can conclude that $f(n)$ is always odd or is always even.

An interesting case is $f(0) = -2$. In this case, we may assume that $f(x) = 2u(x)$ and deduce that $u(x)$ satisfies the equation

$$u(x-1)u(x+1) = u(x)^2 - u(x) + 1.$$

Assuming $|u(-1)| \leq |u(1)|$, and supposing $u(x)$ to be negative, we find that $u(-1) = -1$ and $u(1) = -3$. Consulting a table of values, we conjecture that

$$u(x+2) = 5u(x+1) - u(x) + 1/$$

Indeed, assume that u satisfies the recurion and, and for a particular value n of x,the equation. Then

$$\begin{aligned} u(n)u(n+2) &= u(n)[5u(n+1) - u(n) + 1] \\ &= 5u(n)u(n+1) - [u(n)^2 - u(n)] \\ &= 5u(n)u(n+1) - [u(n-1)u(n+1) - 1] \\ &= u(n+1)[5u(n) - u(n-1)] + 1 \\ &= u(n+1)[u(n+1) - 1] + 1 \\ &= u(n+1)^2 - u(n+1) + 1. \end{aligned}$$