THE UNIVERSITY OF TORONTO
UNDERGRADUATE MATHEMATICS COMPETITION

In Memory of Robert Barrington Leigh

March 12, 2017

Time: $3\frac{1}{2}$ hours

`No aids or calculators permitted.`

The grading is designed to encourage only the stronger students to attempt more than five problems. Each solution is graded out of 10. If the sum of the scores for the solutions to the five best problems does not exceed 30, this sum will be the final grade. If the sum of these scores does exceed 30, then all solutions will be graded for credit.

1. Determine the value of the infinite product

$$\prod_{n=2}^{\infty}\left(1-\frac{2}{1+n^3}\right).$$

2. Let S be a set of n points in the plane, no two pairs the same distance apart. Each point is joined by a straight line segment to the point that is nearest to it; no other segments are drawn. Prove:

   (a) No two segments have a point in common except possibly a point in S;

   (b) No point can be joined to more than five other points;

   (c) The set of segments contains no cycle. In other words, there is no set $\{A_1, A_2, \dots, A_k\}$ of points in S, with $k \geq 3$, such that A_k is joined to A_1 and A_i is joined to A_{i+1} for $1 \leq i \leq k-1$.

3. (a) Given six irrational real numbers, prove that there are always two subsets of three (not necessarily disjoint) such that the sum of any two numbers in each of the subsets is irrational.

   (b) Give an example of a set of six irrational numbers for which there are exactly two subsets of three numbers with all pair sums irrational.

4. 54 and 96 are two nonsquare positive integers whose product is a square; the squares 64 and 81 lie between them. Prove or disprove: *if m and n are two distinct nonsquare positive integers such that mn is a square, then there exists a square integer between them.*

5. Let $f(x)$ be a real continuous periodic function defined on the real numbers such that, for each positive integer n,

$$\frac{|f(1)|}{1}+\frac{|f(2)|}{2}+\cdots+\frac{|f(n)|}{n}\leq 1.$$

   Prove that there exists a real number r such that $f(r) = f(r+1) = 0$.

6. Let $n \geq 2$ and let M be a $n \times n$ matrix with n distinct eigenvalues (exactly) one of which is 0. Suppose that u is a nonzero row $n-$vector and v is a nonzero column $n-$v ector for which uM and Mv are zero $n-$vectors. Prove that $uv \neq O$ (*i.e.*, u is not orthogonal to v).

7. Let $p(z)$ be a polynomial of degree n, all of whose roots have absolute value 1. Prove that $|p'(1)| \geq \frac{n}{2}|p(1)|$.

8. Suppose that the real function $y = f(x)$ defined on $[0, \infty)$ satisfies the differential equation

$$y' = \frac{1}{x^2+y^2}$$

with the intial condition $f(0) > 0$. Prove that $\lim_{x\to\infty} f(x)$ exists and that this limit exceeds 1.

9. Let $f(x)$ be a real-valued continuous function defined on the closed interval $[0,1]$ for which

$$1 = \int_0^1 f(x)dx = \int_0^1 xf(x)dx.$$

Prove that

$$\int_0^1 (f(x))^2 dx \geq 4.$$

10. Prove that

$$0 < \int_{\pi/8}^{3\pi/8} \frac{\cos 2x dx}{1+\tan x} < \frac{\pi}{8\sqrt{2}} .$$

Solutions.

1. Determine the value of the infinite product

$$\prod_{n=2}^{\infty} \left(1 - \frac{2}{1+n^3}\right).$$

Solution 1.

$$\begin{aligned}
\prod_{k=2}^{n} \left(1 - \frac{2}{1+k^3}\right) &= \prod_{k=2}^{n} \left(\frac{k^3-1}{k^3+1}\right) \\
&= \prod_{k=2}^{n} \left(\frac{k-1}{k+1} \cdot \frac{k^2+k+1}{k^2-k+1}\right) \\
&= \left[\prod_{k=2}^{n} \frac{k-1}{k+1}\right] \cdot \left[\prod_{k=2}^{n} \frac{k^2+k+1}{(k-1)^2+(k-1)+1}\right] \\
&= \left[\frac{1\cdot 2}{n(n+1)}\right] \cdot \left[\frac{n^2+n+1}{3}\right] \\
&= \frac{2}{3} \cdot \frac{n^2+n+1}{n^2+n}.
\end{aligned}$$

Letting n tend to infinity, we find that the desired product is $2/3$.

Solution 2. [Di Yang]

$$\begin{aligned}
\prod_{k=2}^{n} \left(1 - \frac{2}{1+k^3}\right) &= \prod_{k=1}^{n} \frac{(k+1)k(k-1)+(k-1)}{(k+1)k(k-1)+(k+1)} \\
&= \prod_{k=1}^{n} \frac{1+[k(k+1)]^{-1}}{1+[(k-1)k]^{-1}} \\
&= \prod_{k=2}^{n} (1+[k(k+1)]^{-1}) / \prod_{k=1}^{n-1} (1+[k(k+1]]^{-1})) \\
&= (1+[n(n+1)]^{-1})/(1+[1/2]) = \frac{2}{3}(1+[n(n+1)]^{-1}).
\end{aligned}$$

Letting n tend to infinity, we find that the desired product is $2/3$.

2. Let S be a set of n points in the plane, no two pairs the same distance apart. Each point is joined by a straight line segment to the point that is nearest to it; no other segments are drawn. Prove:

   (a) No two segments have a point in common except possibly a point in S;

   (b) No point can be joined to more than five other points;

   (c) The set of segments contains no cycle. In other words, there is no set $\{A_1, A_2, \ldots, A_k\}$ of points in S, with $k \geq 3$, such that A_k is joined to A_1 and A_i is joined to A_{i+1} for $1 \leq i \leq k-1$.

Solution. (a) We first observe that no segment $[X, Y]$ joining two points in S cannot contain a third point Z of S in its interior. (Otherwise, Z would be closer to X than Y, and closer to Y tan X.) Suppose that A is closest to B, and that C is closest to D, and that AB and CD intersect internally at P. Then $AB < CB$ and $CD < AD$. However

$$AB + CD = (AP + PB) + (CP + PD) = (AP + PD) + (CP + PB) > AD + CB$$

by the triangle inequality, yielding a contradiction.

(b) Suppose that PA and PB are connected. Suppose, if possible, that AP is the longest side of the triangle ABP. Then $AP > AB$ and $AP > BP$ so that P cannot be the closest point to A, nor A the closest point to P, so that we would not connect PA, yielding a contradiction. Therefore AB is the longest side of the triangle, so that $\angle APB$ exceeds $60°$. It follows that we cannot have more than five points connected to P.

(c) *Solution 1.* Suppose that there is a cycle of segments forming a polygon $A_1A_2 \ldots A_k$ and that A_kA_1 is the longest segment. Then A_kA_1 must exceed both A_1A_2 and $A_{k-1}A_k$, so that A_1 cannot be the closest point to A_k and vice versa. This contradicts that A_kA_1 is a segment.

(c) *Solution 2.* Suppose that there is a cycle involving the points $A_1, A_2, \ldots, A_k$. Wolog, we may suppose that A_1 is the nearest point to A_k. Since A_{k-1} is not the nearest point to A_k, A_k must be the nearest point to A_{k-1}. In a similar way, argue that A_{k-1} is the nearest point to A_{k-2} and so on along the cycle until we find that A_2 is the nearest point to A_1. Thus the lengths of the segments $[A_1, A_k]$, $[A_k, A_{k-1}]$, …, $[A_3, A_2]$, $[A_2, A_1]$ and $[A_1, A_k]$ constitute a strictly decreasing sequence, yielding a contradiction.

3. (a) Given six irrational real numbers, prove that there are always two subsets of three (not necessarily disjoint) such that the sum of any two numbers in each of the subsets is irrational.

   (b) Give an example of a set of six irrational numbers for which there are exactly two subsets of three numbers with all pair sums irrational.

Solution. (a) There cannot be three numbers for which the sum of each pair is rational. (For, if a, b, c be three such numbers, then $2a = (a + b) + (a + c) - (b + c)$ would be rational, yielding a contradiction.) Suppose there are four numbers such that $a + b$, $a + c$ and $a + d$ are irrational. Since at least one of $b + c$, $b + d$ and $c + d$ is irrational, the the pair with the irrational sum can be taken with a to get the required triple.

Otherwise, there are at most two numbers for an irrational sum with a, so that there are three numbers u, v, w for which $a + u$, $a + v$, $a + w$ are all rational. But then, each of $u + v$, $u + w$ and $v + w$ must be irrational.

We now need to find a second triple. Suppose that a, b, c are three numbers for which each pair sum is irrational. Let the other three numbers be d, e, f. If each pair sum of these three is irrational, then we have our two triples. Otherwise, suppose wolog, then $d + e$ is rational. Then at least one of each of the following pairs must be irrational: $\{a + d, a + e\}$, $\{b + d, b + e\}$, $\{c + d, c + e\}$. So, one of the following triples must contain at least two irrationals: $\{a+d, b+d, c+d\}$, $\{a+e, b+e, c+e\}$. If, say, $a+d$ and $b+d$ are irrational, then $\{a, b, d\}$ has all its pairwise sums irrational. The result follows.

(b) An example is the set

$$\{1+\sqrt{2}, 2+\sqrt{2}, 3+\sqrt{2}, 1-\sqrt{2}, 2-\sqrt{2}, 3-\sqrt{2}\}.$$

4. 54 and 96 are two nonsquare positive integers whose product is a square; the squares 64 and 81 lie between them. Prove or disprove: *if m and n are two distinct nonsquare positive integers such that mn is a square, then there exists a square integer between them.*

Solution 1. The assertion is true. We establish the following lemma: *Let k be any nonsquare positive integer. Then for each positive integer r, there exists an integer $v \geq r+1$ for which $kr^2 < v^2 < k(r+1)^2$.*

The proof is by induction. For the base case $r = 1$, there is an integer u for which $\sqrt{k} < u < 2\sqrt{k}$ since $1 < \sqrt{k} = 2\sqrt{k} - \sqrt{k}$. Therefore $u \geq 2$ and $k < u^2 < 4k$.

Suppose that $r \geq 1$ and that $kr^2 < v^2 < k(r+1)^2$ with $v \geq r+1$. Then $v+1 \geq r+2$,

$$\frac{v+1}{v} = 1 + \frac{1}{v} \leq 1 + \frac{1}{r+1} = \frac{r+2}{r+1},$$

and

$$(v+1)^2 = \left(\frac{v+1}{v}\right)^2 v^2 < \left(\frac{r+2}{r+1}\right)^2 v^2 < k(r+2)^2.$$

If $k(r+1)^2 < (v+1)^2$, then the induction step is complete. Otherwise, we repeat the process with $v+1$ in place of v, and do so until eventually we get to some $s \geq 0$ for which

$$(v+s)^2 < k(r+1)^2 < (v+s+1)^2 < k(r+2)^2.$$

Now, let m and n be as in the problem, with $m < n$. We can write $m = ca^2$ and $n = db^2$ where c and d are respectively the largest squarefree divisors of m and n. The numbers are each distinct from 1 and a product of distinct primes. Since cd is square, they have exactly the same prime divisors, so that $c = d$. It follows that $a < b$.

We apply the lemma with $k = c$ and $r = a$ to obtain a square v for which

$$m = ca^2 < v^2 < c(a+1)^2 \leq cb^2 = n.$$

Solution 2. Let $m = x$ and $n = x + r$ be the two integers, with $r > 0$. There exists y such that $y^2 = x(x+r)$. Since the quadratic equation $x^2 + rx - y^2 = 0$ has integer roots, its discriminant is a square, so that $r^2 + 4y^2 = z^2$ for some integer z. Since $(r, 2y, z)$ is a pythagorean triple, there exist positive integers k, m, n such that $m > n$ and $(r, 2y, z) = (k(m^2 - n^2), 2k(mn), k(m^2 + n^2))$. Thus, the quadratic equation becomes

$$0 = x^2 + k(m^2 - n^2)x - k^2m^2n^2 = (x + km^2)(x - kn^2).$$

Therefore $x = kn^2$ and $x + r = km^2$.

Observe that $k \neq 1$, so that $k \geq 2$. Also

$$\sqrt{x+r} - \sqrt{x} = (m-n)\sqrt{k} > 1,$$

so that there must be an integer c between $\sqrt{x}$ and $\sqrt{x+r}$. But then $x < c^2 < x + r$.

5. Let $f(x)$ be a real continuous periodic function defined on the real numbers such that, for each positive integer n,

$$\frac{|f(1)|}{1} + \frac{|f(2)|}{2} + \cdots + \frac{|f(n)|}{n} \leq 1.$$

Prove that there exists a real number r such that $f(r) = f(r+1) = 0$.

Solution. Suppose that the function has period $p > 0$. Then it is bounded by some number $M > 0$ on the closed interval $[0, p]$ and therefore bounded on $\mathbf{R}$ by M. Let $g(x) = |f(x)| + |f(x+1)|$. If the statement of the problem is false, then $g(x)$ never vanishes, and therefore assumes a positive minimum m on $[0, p]$ and therefore on $\mathbf{R}$. Therefore,

$$\sum_{k=1}^{n} \frac{g(k)}{k} \geq m \sum_{k=1}^{n} \frac{1}{k}.$$

Therefore the sum of the left becomes arbitrarily large as n grows.

On the other hand,

$$\begin{aligned}\sum_{k=1}^{n} \frac{g(k)}{k} &= \sum_{k=1}^{n} \frac{|f(k)|}{k} + \sum_{k=1}^{n} \frac{|f(k+1)|}{k} \\ &\leq \sum_{k=1}^{n} \frac{|f(k)|}{k} + \sum_{k=1}^{n} \frac{|f(k+1)|}{k+1} + \sum_{k=1}^{n} \frac{|f(k+1)|}{k(k+1} \\ &\leq 2 + M \sum_{k=1}^{n} \left(\frac{1}{k} - \frac{1}{k+1} \right) < 2 + M.\end{aligned}$$

Ths contradicts the previous assertion and so the conclusion of the problem must hold.

6. Let $n \geq 2$ and let M be a $n \times n$ matrix with n distinct eigenvalues (exactly) one of which is 0. Suppose that u is a nonzero row $n-$vector and v is a nonzero column $n-$v ector for which uM and Mv are zero $n-$vectors. Prove that $uv \neq O$ (*i.e.*, u is not orthogonal to v).

Solution 1. Let the eigenvalues be $\lambda_1 = 0, \lambda_2, \ldots, \lambda_n$. We can pick a basis $\{v_1 = v, v_2, \ldots, v_n\}$ for $\mathbf{F}^n$ (where $\mathbf{F}$ is the underlying scalar field) of eigenvectors for which $Mv_i = \lambda_i v_i$. Then, when $i \geq 2$,

$$\lambda_i u v_i = u(\lambda_i v_i) = u(Mv_i) = (uM)v_i = O.$$

Since u is nonzero, it cannot be orthogonal to every basis element, and so $uv = uv_1$ cannot vanish.

Solution 2. Since the eigenvalues are distinct, there exists an invertible $n \times n$ matrix A such that $A^{-1}MA$ is a diagonal matrix. Since the diagonal matrix is symmetric, the row eigenvectors and the transposes of the column eigenvectors for each eigenvalue. Let $p = uA$ and $q = A^{-1}v$; the former is a row $n-$vector and the latter a column $n-$vector. We have that

$$p(A^{-1}MA) = uMA = O \qquad \text{and} \qquad (A^{-1}MA)q = A^{-1}Mv = 0,$$

so that p is a row and q a column eigenvalue corresponding to the eigenvalue 0 of the diagonal matrix. Since the eigenspace is one-dimensional, the transpose of q is a nontrivial multiple of p, so that $pq \neq O$.

Therefore

$$uv = (pA^{-1})(Aq) = pq \neq O.$$

7. Let $p(z)$ be a polynomial of degree n, all of whose roots have absolute value 1. Prove that $|p'(1)| \geq \frac{n}{2}|p(1)|$.

Solution 1. If $p(1) = 0$, the result is trivial. Henceforth, suppose that $p(1) \neq 0$. We have that

$$\frac{p'(z)}{p(z)} = \sum \frac{1}{z - w},$$

where the sum, with n terms, is over the roots w of $p(z)$, each term repeated according to the multiplicity of the roots.

We note that, when $|w| = 1$, then $w\bar{w} = 1$, so that

$$\frac{1+w}{1-w} = -\frac{1+\bar{w}}{1-\bar{w}},$$

so that the real part of $(1+w)/(1-w)$ is equal to 0. Since

$$\frac{2}{1-w} = 1 + \frac{1+w}{1-w},$$

the real part of $1/(1-w) = 1/2$.

It follows that the real part of

$$\frac{p'(1)}{p(1)} = \sum \frac{1}{1-w}$$

is equal to $n/2$. Since the absolute value of a complex number is not less than the absolute value of its real part, the result follows.

Solution 2. [Michael Chow; Gal Gross] We may suppose that p is monic. Let $p(z) = \prod_{k=1}^{n}(z - a_k)$ and $q(z) = \prod_{k=1}^{n}(z - \overline{a_k}) = \overline{p(\overline{z})}$, where $|a_k| = 1$ for each k. Let

$$u(z) = p'(z)/p(z) = \sum_{k=1}^{n} \frac{1}{z - a_k}$$

and

$$v(z) = q'(z)/q(z) = \overline{u(\overline{z})} = \sum_{k=1}^{n} \frac{1}{z - \overline{a_k}}.$$

Then

$$2\text{Re } u(1) = u(1) + \overline{u(1)} = u(1) + v(1) = \sum_{k=1}^{n} \left(\frac{1}{1-a_k} + \frac{1}{1-\overline{a_k}} \right)$$
$$= \sum_{k=1}^{n} \frac{2 - a_k - \overline{a_k}}{1 - a_k - \overline{a_k} + a_k\overline{a_k}} = n.$$

The result follows, as in Solution 1.

Comment. If $p(z) = (z+1)^n$, then equality occurs.

8. Suppose that the real function $y = f(x)$ defined on $[0, \infty)$ satisfies the differential equation

$$y' = \frac{1}{x^2 + y^2}$$

and that $f(0) = u > 0$. Prove that $\lim_{x\to\infty} f(x)$ exists and that this limit exceeds 1.

Solution. Since its derivative is positive, the function $f(x)$ increases on $[0, \infty)$, and so it suffices to show that it is bounded.

Suppose first that $u \geq 1$. Then

$$f'(x) \leq \frac{1}{x^2 + 1}$$

so that

$$f(x) = u + \int_0^x \frac{dt}{t^2 + f^2(t)} \leq u + \int_0^x \frac{dt}{t^2 + 1} = u + \arctan x < u + \frac{\pi}{2}.$$

On the other hand, if $0 < u < 1$ and $f(x) \leq 1$ on a closed interval $[0, a]$, then

$$f(a) = u + \int_0^a \frac{dt}{t^2 + f^2(t)} \geq u + \int_0^a \frac{dt}{t^2+1} = u + \arctan a.$$

It follows that $\arctan a \leq f(a) - u < 1$ so that $a < \pi/4$. Therefore $f(\pi/4) > 1$ and there is a number $c \in (0, \pi/4)$ such that $f(c) = 1$.

Therefore

$$f'(x) \leq \begin{cases} \frac{1}{u^2}, & \text{for } 0 \leq x \leq c; \\ \frac{1}{x^2+1}, & \text{for } c < x. \end{cases}$$

Thus

$$\begin{aligned} f(x) &\leq u + \frac{c}{u^2} + \int_c^x \frac{dt}{t^2+1} < u + \frac{c}{u^2} + \arctan x \\ &< u + \frac{c}{u^2} + \frac{\pi}{2} < 1 + \frac{\pi}{4u^2} + \frac{\pi}{2}. \end{aligned}$$

Since $f(\pi/4) > 1$, $f(x)$ in increasing and bounded, it follows that $\lim_{x\to\infty} f(x)$ exists and exceeds 1.

9. Let $f(x)$ be a real-valued continuous function defined on the closed interval $[0, 1]$ for which

$$1 = \int_0^1 f(x)dx = \int_0^1 xf(x)dx.$$

Prove that

$$\int_0^1 (f(x))^2 dx \geq 4.$$

Solution 1. Using the method of undetermined coefficients, we can find a linear polynomial $g(x) = 2(3x-1)$ for which $\int_0^1 g(x)dx = \int_0^1 xg(x)dx = 1$. Then

$$\int_0^1 (g(x))^2 dx = 2\left(3\int_0^1 xg(x)dx - \int_0^1 g(x)dx\right) = 2 \times 2 = 4.$$

Also

$$\begin{aligned} 0 \leq \int_0^1 (f(x) - g(x))^2 dx &= \int_0^1 (f(x))^2 dx - 2\int_0^1 f(x)g(x)dx + \int_0^1 (g(x))^2 dx \\ &= \int_0^1 (f(x))^2 dx - 12\int_0^1 xf(x)dx + 4\int_0^1 f(x)dx + \int_0^1 (g(x))^2 dx \\ &= \int_0^1 (f(x))^w dx - 12 + 4 + 4 = \int_0^1 (f(x))^2 dx - 4. \end{aligned}$$

from which the desired result follows.

Comment. The result can be generalized. If $1 = \int_0^1 x^k f(x)dx$ for $k = 0, 1, 2$, then we find that the condition is satisfied by $g(x) = 3(10x^2 - 8x + 1)$ where $\int_0^1 (g(x))^2 dx = 9$. Looking at $\int_0^1 (f(x) - g(x))^2 dx$, we can follow a similar argument to show that $\int_0^1 (f(x))^2 dx \geq 9$.

If we impose the condition for $0 \leq k \leq 3$ and take $g(x) = 4(35x^3 - 45x^2 + 15x - 1)$, we find that $\int_0^1 (f(x))^2 dx \geq 16$.

Solution 2. [Michael Chow] Let a, b be arbitrary real numbers. Then

$$0 \leq \int_0^1 (f(x) + ax + b)^2 dx \leq \int_0^1 [(f(x))^2 + 2axf(x) + 2b(f(x)) + a^2x^2 + 2abx + b^2]dx,$$

whence

$$\int_0^1 (f(x))^2 dx \geq -\left[2a\int_0^2 xf(x)dx + 2b\int_0^1 f(x)dx + a^2\int_0^1 x^2dx + 2ab\int_0^1 xdx + b^2\int_0^1 dx\right]$$
$$= -\frac{1}{3}(6a + 6b + a^2 + 3ab + 3b^2) \equiv g(a,b).$$

Since $\partial g/\partial x = \partial g/\partial y = 0$ if and only if $(a,b) = (-6,2)$, g has a critical value at this point. Thus

$$\int_0^1 (f(x))^2 dx \geq g(-6,2) = -\frac{1}{3}(-12) = 4,$$

as desired.

Solution 3. [Shuyang Shen] Recall the Cauchy-Schwarz inequality:

$$\int_a^b (u(x))^2 dx \cdot \int_a^b (v(x))^2 dx \geq \left(\int_a^b u(x)v(x)dx\right)^2 .$$

Applying this to $[a,b] = [0,1]$ and $(u(x), v(x)) = (f(x) + kx, 1)$, where k is any real, yields that

$$\int_0^1 (f(x) + kx)^2 dx \geq \left(\int_0^1 (f(x) + kx)dx\right)^2 .$$

Therefore

$$\int_0^1 (f(x))^2 dx + 2k\int_0^1 xf(x)dx + \frac{1}{3}k^2 \geq \left(1 + \frac{k}{2}\right)^2 ,$$

so that

$$\int_0^1 (f(x))^2 dx \geq 1 + k + \frac{k^2}{4} - 2k - \frac{k^2}{3}$$
$$= \frac{1}{12}[12 - 12k - k^2] = 4 - \frac{(k-6)^2}{12}.$$

Letting $k = 6$ yields the desired result.

10. Prove that

$$0 < \int_{\pi/8}^{3\pi/8} \frac{\cos 2x dx}{1 + \tan x} < \frac{\pi}{8\sqrt{2}} .$$

Solution 1.

$$\int_{\pi/8}^{3\pi/8} \frac{\cos 2x dx}{1 + \tan x} = \int_{\pi/8}^{3\pi/8} \frac{\cos^2 x - \sin^2 x}{(\cos x + \sin x)/\cos x} dx$$
$$= \int_{\pi/8}^{3\pi/8} (\cos x - \sin x)\cos x dx = \int_{\pi/8}^{3\pi/8} (\cos^2 x - \sin x \cos x)dx$$
$$= \frac{1}{2}\int_{\pi/8}^{3\pi/8} (1 + \cos 2x - \sin 2x)dx = \frac{1}{2}\left[x + \frac{\sin 2x + \cos 2x}{2}\right]_{\pi/8}^{3\pi/8}$$
$$= \frac{1}{4}\left[2x + \sqrt{2}\sin(2x + \frac{\pi}{4})\right]_{\pi/8}^{3\pi/8} = \frac{1}{8}[\pi - 2\sqrt{2}].$$

$$\left(\frac{\pi}{8\sqrt{2}}\right) - \left(\frac{1}{8}[\pi - 2\sqrt{2}]\right) = \frac{1}{8\sqrt{2}}[(1-\sqrt{2})\pi + 4] > \frac{1}{8\sqrt{2}}[-\pi + 4] > 0.$$

The result follows.

Solution 2.

$$\begin{aligned}\int_{\pi/8}^{3\pi/8} \frac{\cos 2x}{1+\tan x} dx &= \int_{\pi/8}^{\pi/4} \frac{\cos 2x}{1+\tan x} dx + \int_{\pi/4}^{\frac{3\pi}{8}} \frac{\cos 2x}{1+\tan x} dx \\ &= \int_{\pi/8}^{\pi/4} \frac{\cos 2x}{1+\tan x} dx + \int_{\pi/4}^{\pi/8} \frac{\cos(\pi - 2y)}{1+\tan(\frac{\pi}{2} - y)} (-dy) \\ &= \int_{\pi/8}^{\pi/4} \frac{\cos 2x}{1+\tan x} dx + \int_{\pi/8}^{\pi/4} \frac{\cos 2y}{1+\cot y} dy \\ &= \int_{\pi/8}^{\pi/4} \frac{\cos 2x(1-\tan x)}{1+\tan x} dx > 0 \ .\end{aligned}$$

Since $\cos 2x$ decreases on $[\pi/8, 3\pi/8]$,

$$\int_{\pi/8}^{3\pi/8} \frac{\cos 2x}{1+\tan x} dx \le \cos(\pi/4) \int_{\pi/8}^{3\pi/8} \frac{dx}{1+\tan x} \ .$$

Using the substitution $y = \frac{\pi}{2} - x$, we find that

$$\int_{\pi/8}^{3\pi/8} \frac{dx}{1+\tan x} = \int_{\pi/8}^{3\pi/8} \frac{\tan x dx}{1+\tan x} = \frac{1}{2} \int_{\pi/8}^{3\pi/8} \frac{(1+\tan x)dx}{1+\tan x} = \frac{\pi}{8} \ ,$$

and the result follows.

Comment. An alternative approach to obtaining the integral of $\sin x \cos x$ from $\pi/8$ to $3\pi/8$ is as follows:

$$\begin{aligned}\int_{\pi/8}^{3\pi/8} \sin x \cos x dx &= \left[\frac{1}{2}\sin^2 x\right]_{\pi/8}^{3\pi/8} = \frac{1}{2}\left(\sin^2 \frac{3\pi}{8} - \sin^2 \frac{\pi}{8}\right) \\ &= \frac{1}{2}\left(\cos^2 \frac{\pi}{8} - \sin^2 \frac{\pi}{8}\right) = \frac{1}{2}\cos\frac{\pi}{4} = \frac{\sqrt{2}}{4}.\end{aligned}$$