THE UNIVERSITY OF TORONTO
UNDERGRADUATE MATHEMATICS COMPETITION

In Memory of Robert Barrington Leigh

Saturday, March 5, 2016

Time: $3\frac{1}{2}$ hours

`No aids or calculators permitted.`

The grading is designed to encourage only the stronger students to attempt more than five problems. Each solution is graded out of 10. If the sum of the scores for the solutions to the five best problems does not exceed 30, this sum will be the final grade. If the sum of these scores does exceed 30, then all solutions will be graded for credit.

1. Let a be a positive real number that is not an integer and let

$$n = \left\lfloor \frac{1}{a - \lfloor a \rfloor} \right\rfloor .$$

Prove that $\lfloor (n+1)a \rfloor - 1$ is divisible by $n+1$. [*Note:* $\lfloor x \rfloor$ denotes the largest integer that is not greater than x, so that $\lfloor x \rfloor \leq x < \lfloor x \rfloor + 1$.]

2. Determine all polynomial solutions $f(x)$ to the identity

$$f(x + y - xy) = f(x)f(y).$$

3. Let $n = \prod p^a$ be the prime factor decomposition of the positive integer n and define $s(n) = \sum ap$, the sum of all the primes involved in the decomposition counting repetitions. For each positive integer m exceeding 1, let $h(m)$ be the number of positive integers n for which $s(n) = m$.

(a) Prove that $\lim_{n\to\infty} s(n) = \infty$.

(b) Prove that $s(n)$ assumes every value exceeding 4 at least twice and that $\lim_{n\to\infty} h(n) = \infty$.

4. Let $p(x)$ be a monic polynomial of degree 3 with three distinct real roots. How many real roots does the polynomial $(p'(x))^2 - 2p(x)p''(x)$ have?

5. (a) Determine the largest positive integer n for which the following statement is NOT true:

There exists a finite set $\{a_1, a_2, \ldots, a_k\}$ $(k \geq 1)$ *of positive integers for which* $n < a_1 < a_2 < \cdots < a_k \leq 2n$ *and* $n \times a_1 \times a_2 \times \cdots \times a_k$ *is a perfect square.*

(b) Determine infinitely many integers n for which $n < a_1 < a_2 < \ldots < a_k$ and $n \times a_1 \times a_2 \times \cdots \times a_k$ is square implies that $a_k \geq 2n$.

(c) Let $n = m^2$. Is it possible to determine an integer m for which integers $a_1, a_2, \cdots, a_k$ can be chosen in the open interval $(m^2, (m+1)^2)$ for which the product $a_1 \times a_2 \times \cdots \times a_k$ is square?

Please turn over for more questions.

6. Suppose that f is a strictly increasing convex real-valued continuous function on $[0,1]$ for which $f(0)=0$ and $f(1)=1$ and $g(x)$ is a function that satisfies $g(f(x))=x$ for each $x \in [0,1]$. Prove that

$$\int_0^1 f(x)g(x)dx \leq \frac{1}{3}.$$

When does equality occur?

[*Note:* A function is *convex* if for any $t \in [0,1]$

$$f((1-t)x+ty) \leq (1-t)f(x)+tf(y)$$

whenever $x, y, (1-t)x+ty$ belong to the domain of f.]

7. Let m, n be integers for which $0 \leq m < n$ and let $p(x)$ be a polynomial of degree n over a field $\mathbf{F}$. What is the dimension over $\mathbf{F}$ of the vector space generated by the set of functions

$$\{1, x, x^2, \cdots, x^{n-m-1}, p(x), p(x+1), \cdots, p(x+m)\}?$$

8. Let S be a set of the positive integers that is closed under addition ($i.e.$, $x, y \in S \Rightarrow x+y \in S$) for which the set T of positive integers not contained in S is finite with $m \geq 1$ elements. Prove that the sum of the numbers in T is not greater than m^2 and determine all the sets S for which this sum is equal to m^2.

9. (a) Prove that every polyhedron has at least two faces with the same number of edges.

   (b) Suppose that $k \geq 3$ and that all the faces in a polyhedron have at least k edges. Prove that there are k pairs of faces with the same number of edges (the pairs need not be disjoint).

10. Let X be a subset of the group G such that

$$\bigcap \{x^{-1}X : x \in X\}$$

contains an element a of finite order other than the identity. Prove that X is the union of cosets with respect to some nontrivial subgroup of G. [*Note:* for any set S and element g of G, $gS = \{gs : s \in S\}$.]

Solutions

1. Let a be a positive real number that is not an integer and let

$$n = \left\lfloor \frac{1}{a - \lfloor a \rfloor} \right\rfloor .$$

Prove that $\lfloor (n+1)a \rfloor - 1$ is divisible by $n+1$. [*Note:* $\lfloor x \rfloor$ denotes the largest integer that is not greater than x, so that $\lfloor x \rfloor \leq x < \lfloor x \rfloor + 1$.]

Solution. Since

$$n \leq \frac{1}{a - \lfloor a \rfloor} < n+1,$$
$$n(a - \lfloor a \rfloor) \leq 1 < (n+1)(a - \lfloor a \rfloor).$$

Therefore

$$1 + (n+1)\lfloor a \rfloor < (n+1)a \leq 1 + n\lfloor a \rfloor + a < 2 + (n+1)\lfloor a \rfloor,$$

so that $\lfloor (n+1)a \rfloor = 1 + (n+1)\lfloor a \rfloor$.

2. Determine all polynomial solutions $f(x)$ to the identity

$$f(x + y - xy) = f(x)f(y).$$

Solution 1. [M. Chow] It is straightforward to check that the only constant polynomials satisfying the equation are 0 and 1. Suppose that f is a nonconstant polynomial with a root r. Setting $y = r$, we find that $f(r + (1-r)x) = 0$ for each x. If $1 - r \neq 0$, then every number is a root of f, which is not possible. Hence $r = 1$ and $f(x) = c(1-x)^n$ for some nonzero constant c and positive integer n. Plugging this into the equation, we see that $c = c^2$ so that $c = 1$. Indeed, $f(x) = (1-x)^n$ does satisfy the equation.

Solution 2. The zero polynomial is a solution. Henceforth, let $f(x)$ be a nonzero polynomial solution. Setting $y = 1$ yields that $f(1) = f(x)f(1)$ for each x. If $f(1) \neq 0$, then $f(x) \equiv 1$ is a solution of the equation. This is the only nonzero constant polynomial satisfying the equation. Otherwise, we must have $f(1) = 0$, whereupon $f(x) = (1-x)g(x)$ for some polynomial $g(x)$. Plugging this into the equation yields that

$$(1 - x - y + xy)g(x + y - xy) = (1-x)(1-y)g(x)g(y)$$

so that

$$g(x + y - xy) = g(x)g(y).$$

This is the same as the original equation, so either $g(x) \equiv 1$ or $g(x)$ has a factor $(1-x)$ and so $f(x)$ is divisible by $(1-x)^2$. We can continue on in this vein. Since $(1-x)$ can appear as a factor of $f(x)$ with at most finite multiplicity, we conclude that $f(x) = (1-x)^n$ for some nonnegative integer n.

Solution 3. Let $g(x) = f(1-x)$. Then $g(x)$ satisfies the equation $g(xy) = g(x)g(y)$ for all x, y. This is satisfied by the constant polynomials 0 and 1. Suppose that $g(x) = \sum_{k=0}^{n} a_k x^k$ for some $n \geq 1$ with $a_n \neq 0$. Then

$$\sum_{k=0}^{n} (a_k y^k) x^k = \sum_{k=0}^{n} (a_k g(y)) x^k$$

for each x, y. For fixed y, this is a polynomial equation satisfied for all x, so that corresponding coefficients equate: $y^k = g(y)$ or $a_k = 0$ for $k = 0, \ldots, n$. Since $a_n \neq 0$, $g(y) = y^n$ identically, and $a_0 = a_1 = \cdots = a_{n-1} = 0$. Therefore $g(x) = x^n$ and $f(x) = (1-x)^n$. It is easily checked that these are actually solutions.

Solution 4. By substituting $y = 0$ and $y = 1$, we obtain that $f(x) = f(x)f(0)$ and $f(1) = f(x)f(1)$ for all x. This leads to the possibilities $f(x) \equiv 0$, $f(x) \equiv 1$, or $f(x)$ is a nonconstant polynomial with $f(0) = 1$ and $f(1) = 0$.

Letting $y = -x$ yields $f(x^2) = f(x)f(-x)$. Suppose that r is a root of $f(x)$. Then $f(r^{2^k}) = 0$ for all positive integers k. Since 0 is not a root and a polynomial has only finitely many roots, $r = 1$ or $r = -1$, and so

$$f(x) = (1-x)^b(1+x)^c,$$

for nonnegative integers b and c. Since $f(x^2) = f(x)f(-x)$, it follows that

$$(1-x)^b(1+x)^c = (1-x^2)^{b+c}.$$

Therefore $c = 0$ and $f(x) = (1-x)^b$. It can be checked that this satisfies the equation.

3. Let $n = \prod p^a$ be the prime factor decomposition of the positive integer n and define $s(n) = \sum ap$, the sum of all the primes involved in the decomposition counting repetitions. For each positive integer m exceeding 1, let $h(m)$ be the number of positive integers n for which $s(n) = m$.

   (a) Prove that $\lim_{n\to\infty} s(n) = \infty$.

   (b) Prove that $s(n)$ assumes every value exceeding 4 at least twice and that $\lim_{n\to\infty} h(n) = \infty$.

   Solution 1. (a) Since $\log x < x$ for all $x > 0$, we have that

$$\log n = \sum a \log p < \sum ap = s(n)$$

from which we deduce that $\lim_{n\to\infty} s(n) = \infty$.

(b) Every positive integer $m \geq 4$ can be written in the form $m = 2x + 3y$ where x and y are nonnegative integers. When m is even, we can let $(x, y) = ((m/2) - 3k, 2k)$ where $0 \leq k \leq \lfloor m/6 \rfloor$, so that there are at least $m/6$ possibilities. When m is odd, we can let $(x, y) = (\frac{1}{2}(m-3) - 3k, 1 + 2k)$ where $0 \leq k \leq \lfloor (m-3)/6 \rfloor$, so that there are at least $(m-3)/6$ possibilities. For each such representation, $s(2^x \cdot 3^y) = m$, so that $h(m) \geq (m-3)/6$ and $\lim_{m\to\infty} h(m) = \infty$.

This also establishes that $h(m) \geq 2$ for even $m \geq 12$ and odd $m \geq 15$. For lower values of m, we have $s(5) = s(6) = 5$, $s(8) = s(9) = 6$, $s(7) = s(12) = 7$, $s(15) = s(18) = 8$, $s(24) = s(27) = 9$, $s(32) = s(36) = 10$, $s(11) = s(45) = 11$ and $s(13) = s(108) = 13$.

Solution 2. (a) We prove that, if $2^u \leq n < 2^{u+1}$, then $u \leq s(n)$, from which the desired result will follow. This can be proved by induction. We first observe that, if p is a prime exceeding 2^v, then $s(p) = p > 2^v \geq 2v$. The assertion holds for $u = 1$; suppose it holds up to $u - 1$. Suppose that $n = 2k$. Then $2^{u-1} < k$, so that $s(n) = 2 + s(k) \geq 2 + (u-1) > u$. Now let n be odd. If n is prime, then $s(n) = n > 2^u > u$. Otherwise, let p be an odd prime divisor of n so that $n = pq$ with $q > 1$. Suppose that $2^t < p < 2^{t+1}$. Then $2^{u-t-1} < q$. By the induction hypothesis,

$$s(n) = s(p) + s(q) \geq 2t + (u - t - 1) = u + (t - 1) \geq u.$$

(b) We observe that if $n = 2^r$ and $n = 2^{r-3}3^2$ for $r \geq 3$, then $s(n) = 2r$ and that if $n = 2^{r-1}3$ and $n = 2^{r-2}5$ for $r \geq 2$, then $s(n) = 2r + 1$. Therefore every integer exceeding 4 is assumed by the function $s(n)$ at least twice

$h(m)$ is the number of ways that m can be expressed as the sum of primes allowing repetitions. We show that $h(m+2) \geq h(m) + 1$ for $m \geq 8$. We can get a sum for $h(m+2)$ by appending a 2 to each sum for $h(m)$. In addition, each number m exceeding 7 can be written in the form $3a + 5b$. This yields the desired result.

Solution 3. (b) For each integer $m \geq 4$, let $f(m)$ be the number of primes that do not exceed $m - 2$. Since there are infinitely many primes, $\lim_{m\to\infty} f(m) = \infty$.

For a given integer $m \geq 4$, let p be a prime not exceeding $m - 2$ and let q be aprime dividing $m - p$. Then

$$s\left(p \times q^{(m-p)/q}\right) = p + q\left(\frac{m-p}{q}\right) = m.$$

When $m = p + q$, the sum of two primes, the number $p \times q^{(m-p)/q} = pq$ gets counted twice; otherwise $p \times q^{(m-p)/q}$ gets counted once. Thus there are at least $\lfloor \frac{1}{2} f(m) \rfloor$ numbers at which s takes the value m.

Observe that $s(5) = s(6) = 5$, $s(8) = s(9) = 6$, $s(7) = s(12) = 7$, $s(15) = s(16) = 8$ and $f(m) \geq 4$ for $m \geq 9$. Hence $h(m) \geq 2$ for $m \geq 5$. Since $h(m) \geq \lfloor \frac{1}{2} f(m) \rfloor$, it follows that $\lim_{m \to \infty} h(m) = \infty$.

4. Let $p(x)$ be a monic polynomial of degree 3 with three distinct real roots. How many real roots does the polynomial $(p'(x))^2 - 2p(x)p''(x)$ have?

Solution. Let $f(x) = (p'(x))^2 - 2p(x)p''(x)$. Then $f'(x) = -12p(x)$. Therefore, $f(x)$ has three extreme values at the three roots of $p(x)$, and the value of f at all of these extremes is positive. Since $f(x)$ has degree 4 with leading coefficient equal to $9 - 12 = -3$, the values of $f(x)$ are negative when $|x|$ is large. Therefore $f(x)$ has two real roots, one less than smallest root of $p(x)$, one larger than the greatest root of $p(x)$. Between the smallest and largest roots of $p(x)$, the values of $f(x)$ are positive.

[This problem appeared on the Swedish Mathematical Olympiad in 1988.]

5. (a) Determine the largest positive integer n for which the following statement is NOT true:

There exists a finite set $\{a_1, a_2, \ldots, a_k\}$ *(*$k \geq 1$*) of positive integers for which* $n < a_1 < a_2 < \cdots < a_k \leq 2n$ *and* $n \times a_1 \times a_2 \times \cdots \times a_k$ *is a perfect square.*

(b) Determine infinitely many integers for which, if $n < a_1 < a_2 < \cdots < a_k$ and $n \times a_1 \times a_2 \cdots < a_k$ is a perfect square requires that $a_k \geq 2n$.

(c) Let $n = m^2$. Is it possible to determine an integer m for which numbers $a_1, a_2, \cdots, a_k$ can be chosen in the open interval $(m^2, (m+1)^2)$ for which the product $a_1 \times a_2 \times \cdots \times a_k$ is square?

(a) *Solution 1.* Define $x_k = 2k^2$ and note that $x_k < x_{k+1} < 2x_k$ for $k \geq 3$. Let $n \geq 18$, and select k so that $x_k \leq n < x_{k+1}$. Then, since $n < x_{k+1} < 2x_k \leq 2n$, it follows that the product of n, x_{k+1} and $2n$ is a square and the statement holds. If $10 \leq n \leq 17$, then the statement holds since $n \times 18 \times 2n$ is a square. If $n = 5, 6, 7$, then $n \times 8 \times 2n$ is square and the statement holds. Since $8 \times 10 \times 12 \times 15$ and 9×16 are square, the statement holds for 8 and 9. However, it can be checked that it does not hold for $n = 1, 2, 3, 4$, so 4 is the largest number for which the statement fails.

Solution 2. Define the sequence $x_1 = 18 = 2 \times 3^2$, $x_2 = 32 = 2^5$, $x_3 = 50 = 2 \times 5^2$, and $x_m = 4 \times x_{m-3}$ for $m \geq 4$. Then each x_m is the product of an odd power of 2 and a square. Furthermore, note that $x_1 < x_2 < 2x_1$, $x_2 < x_3 < 2x_2$, $x_3 < x_4 < 2x_4$, so that $x_m < x_{m+1} < 2x_m$ for each $m \geq 1$. Suppose that $10 \leq n \leq 17$. Then $n \times 18 \times 2n$ is square. Let $n \geq 18$, and suppose that m is the largest integer for which $a_m \leq n$. Then $n < a_{m+1} < 2a_m \leq 2n$, and $n \times a_{m+1} \times 2n$ is a square. Thus, the statement is true whenever $n \geq 10$.

For integers less than 10, we check the sets with the smallest maximum number that will yield square products: $2 \times 3 \times 6 = 6^2$, $3 \times 6 \times 8 = 12^2$, $4 \times 9 = 6^2$, $5 \times 8 \times 10 = 20^2$, $6 \times 8 \times 12 = 24^2$, $7 \times 8 \times 14 = 28^2$, $8 \times 10 \times 12 \times 15 = 120^2$ and $9 \times 16 = 12^2$. (Wolog, we can assume that none of the factors is square unless n itself is square. With 2 and 8, we require a factor that is divisible by at most an odd power of 2; with 3, we require a factor that is divisible by at most an odd power of 3.) When $1 \leq n \leq 4$, we need a factor that exceeds $2n$, and the statement fails for only these two values.

Solution. (b) If p is prime, we need a factor also divisible by p, and so at least equal to $2p$.

(c) The interval $(5^2, 6^2)$ contains the integers $27, 28, 30, 32, 35$ whose product is $(2^4 \times 3^3 \times 5 \times 7)^2 = (7!)^2$; the interval $(7^2, 8^2)$ contains the integers $50, 56, 63$ whose product is $(2^2 \times 3 \times 5 \times 7)^2$; the interval $(8^2, 9^2)$ contains the integers $66, 70, 75, 77, 80$ whose product is $(2^3 \times 3 \times 5^2 \times 7 \times 11)^2$ and the integers $65, 72, 78, 80$ whose product is $(2^4 \times 3 \times 5 \times 13)^2$.

6. Suppose that f is a strictly increasing convex real-valued continuous function on $[0, 1]$ for which $f(0) = 0$

and $f(1) = 1$ and $g(x)$ is a function that satisfies $g(f(x)) = x$ for each $x \in [0,1]$. Prove that

$$\int_0^1 f(x)g(x)dx \le \frac{1}{3}.$$

When does equality occur?

Solution. Equality occurs when $f(x) = g(x) = x$, so we suppose henceforth that $f(x)$ is not identically equal to x. Since $f(x)$ is continuous, its image is precisely the interval $[0,1]$. Since $f(g(f(x)) = f(x)$ for each $x \in [0,1]$, $f(g(x)) = x$ on the interval, so that f and g are composition inverses. The function g is continuous, increasing and concave. There exists $w \in (0,1)$ such that $f(w) \neq w$. Since $w = (1-w)\cdot 0 + w \cdot 1$, $f(w) < w$. Suppose $0 < u < w < v < 1$. Then $wf(u) \le (w-u)f(0) + uf(w) < wu$ and so $f(u) < u$. Since

$$v = \left(\frac{1-v}{1-w}\right) w + \left(\frac{v-w}{1-w}\right),$$

we have that

$$f(v) \le \left(\frac{1-v}{1-w}\right) f(w) + \left(\frac{v-w}{1-w}\right) f(1) < \left(\frac{1-v}{1-w}\right) w + \left(\frac{v-w}{1-w}\right) 1 = v.$$

Thus $f(x) < x$ on $(0,1)$. Similarly, because of the concavity of $g(x)$, $g(x) > x$ on $(0,1)$. Suppose that $0 < u < v \le 1$. Then for some positive t less than 1, $u = tv = (1-t)0 + tv$, so that $f(u) \le (1-t)f(0) + tf(v) = tf(v)$. Dividing by $u = tv$, we find that

$$\frac{f(u)}{u} < \frac{f(v)}{v}.$$

Let $u = x$ and $v = g(x)$, we find that

$$\left(\frac{f(x)}{x} \cdot \frac{g(x)}{f(g(x))}\right) < 1$$

whence $f(x)g(x) < x^2$ for $x \in (0,1)$. It follows that

$$\int_0^1 f(x)g(x)dx < \int_0^1 x^2 dx = \frac{1}{3}.$$

Equality occurs if and only if $f(x) = x$.

Comment. The result can be verified in the case that $f(x) = x^n$ and $g(x) = x^{1/n}$.

7. Let m, n be integers for which $0 \le m < n$ and let $p(x)$ be a polynomial of degree n over a field $\mathbf{F}$. What is the dimension over $\mathbf{F}$ of the vector space generated by the set of functions

$$\{1, x, x^2, \cdots, x^{n-m-1}, p(x), p(x+1), \cdots, p(x+m)\}?$$

Solution. For a set S of vectors, let $\langle S \rangle$ denote the linear space generated by S. In the case $m = 0$, we note that the dimension of

$$\langle 1, x, x^2, \cdots, x^{n-1}, p(x) \rangle = \langle 1, x, x^2, \cdots, x^{n-1}, x^n \rangle$$

equals $n+1$, as does the dimension of

$$\langle 1, x, x^2, \cdots, x^{n-2}, p(x), p(x+1) \rangle = \langle 1, x, x^2, \cdots, x^{n-2}, p(x) - p(x+1), p(x) \rangle,$$

since the degrees of the last two polynomials in the latter generating set are $n-1$ and n.

For any polynomial $q(x) = c_k x^k + c_{k-1}x^{k-1} + \cdots + c_1 x + c_0$ of degree k, define

$$\Delta q(x) = q(x+1) - q(x)$$

and

$$\Delta^j q(x) = \Delta(\Delta^{j-1} q(x))$$

for $j \geq 2$. Since $\Delta q(x) = kc_k x^{k-1} + \cdots$, the degree of $\Delta q(x)$ is equal to $k-1$. More generally, the degree of $\Delta^j q(x)$ is $k - j$ for $1 \leq j \leq k$.

Observe that

$$\begin{aligned}
\langle 1, x, x^2, \cdots, p(x), & p(x+1), p(x+2), \cdots, p(x+m)\rangle \\
&= \langle 1, x, x^2, \cdots, x^{n-m-1}, p(x), \Delta p(x), \Delta p(x+1), \cdots, \Delta p(x+m-1)\rangle \\
&= \langle 1, x, x^2, \cdots, x^{n-m-1}, p(x), \Delta p(x), \Delta^2 p(x), \cdots, \Delta^2 p(x+m-2)\rangle \\
&= \langle 1, x, x^2, \cdots, x^{n-m-1}, p(x), \Delta p(x), \Delta^2 p(x), \cdots, \Delta^m p(x)\rangle \\
&= \langle 1, x, x^2, \cdots, x^{n-m-1}, \Delta^m p(x), \cdots, \Delta p(x), p(x)\rangle \\
&= \langle 1, x, x^2, \cdots, x^{n-m-1}, x^{n-m}, \cdots, x^{n-1}, x^n\rangle.
\end{aligned}$$

Hence the space of the problem has dimension $n + 1$.

8. Let S be a set of the positive integers that is closed under addition ($i.e.$, $x, y \in S \Rightarrow x + y \in S$) for which the set T of positive integers not contained in S is finite with $m \geq 1$ elements. Prove that the sum of the numbers in T is not greater than m^2 and determine all the sets S for which this sum is equal to m^2.

Solution. We prove the result by induction on m. Since S does not contain all the positive integers, it cannot contain 1. Hence $1 \in T$ and the result holds for $m = 1$. We first show that the largest element in T cannot exceed $2m - 1$. Let $n \geq 2m$ and consider the pairs $(1, n-1), (2, n-2), (3, n-3), \ldots, (m, n-m)$. There are m pairs, and if $n \in T$, each pair would have to contain at least one element of T. But this along with n would give at least $m + 1$ elements of T, and we have a contradiction.

Suppose that u is the largest element of T and that the result holds for $m - 1$. Then the set $S \cup \{u\}$ is closed under addition, and by the induction hypothesis $\sum\{x : x \in T\backslash\{u\}\} \leq (m-1)^2$. Therefore

$$\sum\{x : x \in T\} \leq (m-1)^2 + (2m-1) = m^2.$$

Equality requires that $u = 2m - 1$. Backtracking for the previous values of m, we find that all the odd numbers up to $2m - 1$ must belong to t for equality. The equality holds only for the sets

$$S = \{2, 4, 6, \ldots, 2m-2, 2m, 2m+1, 2m+2, 2m+3, \ldots\}.$$

9. (a) Prove that every polyhedron has at least two faces with the same number of edges.

   (b) Suppose that $k \geq 3$ and that all the faces in a polyhedron have at least k edges. Prove that there are k pairs of faces with the same number of edges (the pairs need not be disjoint).

Solution. (a) Let the number of faces of the polyhedron be F. Suppose that the result is false. For the indices $i = 1, 2, \ldots, F - 1$, arrange the faces in order so that the number of edges E_{i+1} in the $(i+1)$th face exceeds the number of edges E_i in the ith face: $E_{i-1} - E_i \geq 1$ for $i = 1, 2, \ldots, F - 1$. Then, noting that $E_1 \geq 3$, we have that

$$E_F - 3 \geq E_F - E_1 = \sum_{i=1}^{F-1}(E_{i+1} - E_i) \geq F - 1$$

so that $E_F \geq F + 2$. This says that the number of faces adjacent to the face with the most edges is more than the number of faces of the polyhedron, which is clearly false.

(b) Follow the argument of (a) to obtain

$$E_F - k \geq (E_2 - E_1) + (E_3 - E_2) + \cdots + (E_F - E_{F-1}).$$

The sum on the right has $F - 1$ terms, but does not exceed $E_F - k \leq (F - 1) - k$. Therefore there must be at least k values of i for which $E_{i+1} - E_i$ vanishes.

10. Let X be a subset of the group G such that

$$\cap\{x^{-1}X : x \in X\}$$

contains an element a of finite order other than the identity. Prove that X is the union of cosets with respect to some nontrivial subgroup of G.

Solution. Since $xa \in X$ for each $x \in X$, $Xa \subseteq X$. Hence, if $a^n = e$, the identity element,

$$X \supseteq Xa \supseteq Xa^2 \supseteq \ldots \supseteq Xa^n = X,$$

so that $Xa = X$.

Let H be the cyclic subgroup generated by a. Then for each element $h = a^i$ of H, $Xh = X$. Thus $xH \subseteq X$ for each $x \in X$. On the other hand, each element of X is of the form yh for some $y \in H$, and so belongs to yH. Therefore

$$X = \cup\{xH : x \in X\}.$$