PUTNAM PROBLEMS

SEQUENCES, SERIES AND RECURRENCES

Notes

1. $x_{n+1} = ax_n$ has the general solution $x_n = x_1 a^{n-1}$.
2. $x_{n+1} = x_n + b$ has the general solution $x_n = x_1 + (n-1)b$.
3. $x_{n+1} = ax_n + b$ (with $a \neq 1$) can be rewritten $x_{n+1} + k = a(x_n + k)$ where $(a-1)k = b$ and so reduces to the recurrence 1.
4. $x_{n+1} = ax_n + bx_{n-1}$ has different general solution depending on the discriminant of the characteristic polynomial $t^2 - at - b$.
(a) If $a^2 - 4b \neq 0$ and the distinct roots of the characteristic polynomial are r_1 and r_2, then the general solution of the recurrence is
$$x_n = c_1 r_1^n + c_2 r_2^n$$
where the constants c_1 and c_2 are chosen so that
$$x_1 = c_1 r_1 + c_2 r_2 \qquad \text{and} \qquad x_2 = c_1 r_1^2 + c_2 r_2^2 \quad .$$
(b) If $a^2 - 4b = 0$ and r is the double root of the characteristic polynomial, then
$$x_n = (c_1 n + c_0) r^n$$
where c_1 and c_0 are chosen so that
$$x_1 = (c_1 + c_0) r \qquad \text{and} \qquad x_2 = (2c_1 + c_0) r^2 \quad .$$
5. $x_{n+1} = (1-s)x_n + sx_{n-1} + r$ can be rewritten $x_{n+1} - x_n = -s(x_n - x_{n-1}) + r$ and solved by a previous method for $x_{n+1} - x_n$.
6. $x_{n+1} = ax_n + bx_{n-1} + c$ where $a + b \neq 1$ can be rewritten $(x_{n+1} + k) = a(x_n + k) + b(x_{n-1} + k)$ where $(a + b - 1)k = c$ and solved for $x_n + k$.
7. The general homogeneous linear recursion has the form
$$x_{n+k} = a_{k-1} x_{n+k-1} + \cdots + a_1 x_{n+1} + a_0 \quad .$$
Its characteristic polynomials is
$$t^k - a_{k-1} t^{k-1} - \cdots - a_1 t - a_0 \quad .$$
Let r be a root of this polynomial of multiplicity m; then the nth term of the recurrence is a linear combination of terms of the type
$$(c_{m-1} r^{m-1} + \cdots + c_1 r + c_0) r^n \quad .$$

Putnam questions

2018-B-4. Given a real number a, we define a sequence by $x_0 = 1$, $x_1 = x_2 = a$, and $x_{n+1} = 2x_n x_{n-1} - x_{n-2}$ for $n \geq 2$. Prove that, if $x_n = 0$ for some n, then the sequence is periodic.

2017-A-3. Let a and b be real numbers with $a < b$, and let f and g be continuous functions from $[a, b]$ to $(0, \infty)$ such that $\int_a^b f(x)\,dx = \int_a^b g(x)\,dx$ but $f \neq g$. For every positive integer n, define
$$I_n = \int_a^b \frac{(f(x))^{n+1}}{(g(x))^n}\,dx.$$

Show that $I_1, I_2, I_3, \ldots$ is an increasing sequence with $\lim_{n\to\infty} I_n = \infty$.

2017-B-3. Suppose that $f(x) = \sum_{i=0}^{\infty} c_i x^i$ is a power series for which each coefficient c_i is 0 or 1. Show that, if $f(2/3) = 3/2$, then $f(1/2)$ must be irrational.

2017-B-4. Evaluate the sum

$$\sum_{k=0}^{\infty}\left(3\cdot\frac{\ln(4k+2)}{4k+2} - \frac{\ln(4k+3)}{4k+3)} - \frac{\ln(4k+4)}{4k+4} - \frac{\ln(4k+5)}{4k+5}\right)$$

$$= 3\cdot\frac{\ln 2}{2} - \frac{\ln 3}{3} - \frac{\ln 4}{4} - \frac{\ln 5}{5} + 3\cdot\frac{\ln 6}{6} - \frac{\ln 7}{7} - \frac{\ln 8}{8}$$
$$-\frac{\ln 9}{9} + 3\cdot\frac{\ln 10}{10} - \cdots.$$

(As usual, $\ln x$ denotes the natural logarithm of x.

2016-B-1. Let $x_0, x_1, x_2, \ldots$ be the sequence such that $x_0 = 1$ and for $n \geq 0$,

$$x_{n+1} = \ln(e^{x_n} - x_n)$$

(as usual, the function ln is the natural logarithm). Show that the infinite series

$$x_0 + x_1 + x_2 + \cdots$$

converges and find its sum.

2016-B-6. Evaluate

$$\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k}\sum_{n=0}^{\infty}\frac{1}{k2^n+1}.$$

2015-A-2. Let $a_0 = 1$, $a_1 = 2$, and $a_n = 4a_{n-1} - a_{n-2}$ for $n \geq 2$. Find an odd prime factor of a_{2015}.

2015-B-5. Let P_n be the number of permutations π of $\{1, 2, \ldots, n\}$ such that

$$|i-j| = 1 \qquad \text{implies} \qquad |\pi(i) - \pi(j)| \leq 2$$

for all i, j in $\{1, 2, \ldots, n\}$. Show that for $n \geq 2$, the quantity

$$P_{n+5} - P_{n+4} - P_{n+3} + P_n$$

does not depend on n, and find its value.

2014-A-1. Prove that every nonzero coefficient of the Taylor series of

$$(1 - x + x^2)e^x$$

about $x = 0$ is a rational number whose numerator (in lowest terms) is either 1 or a prime number.

2014-A-3. Let $a_0 = 5/2$ and $a_k = a_{k-1}^2 - 2$ for $k \geq 1$. Compute

$$\prod_{k=0}^{\infty}\left(1 - \frac{1}{a_k}\right)$$

in closed form.

2013-B-1. For positive integers n, let the number $c(n)$ be determined by the rules $c(1) = 1$, $c(2n) = c(n)$, and $c(2n+1) = (-1)^n c(n)$. Find the value of

$$\sum_{n=1}^{2013} c(n)c(n+2).$$

2010-B-1. Is there an infinite sequence of real numbers $a_1, a_2, a_3, \dots$ such that $a_1^m + a_2^m + a_3^m + \cdots = m$ for every positive integer m?

2009-B-6. Prove that for every positive integer n, there is a sequence $a_0, a_1, \dots, a_{2009}$ with $a_0 = 0$ and $a_{2009} = n$ such that each term after a_0 is either an earlier term plus 2^k for some nonegative integer k or of the form b mod c for some earlier positive terms b and c. (Here b mod c denotes the remainder when b is divided by c, so $0 \le (b \text{mod} c) < c$.)

2007-B-3. Let $x_0 = 1$ and for $n \ge 0$, let $x_{n+1} = 3x_n + \lfloor x_n\sqrt{5} \rfloor$. In particular, $x_1 = 5$, $x_2 = 26$, $x_3 = 136$, $x_4 = 712$. Find a closed-form expression for x_{2007}. ($\lfloor a \rfloor$ means the largest integer $\le a$,)

2006-A-3. Let $1, 2, 3, \cdots, 2005, 2006, 2007, 2009, 2012, 2016, \cdots$ be a sequence defined by $x_k = k$ for $k = 1, 2, \cdots, 2006$ and $x_{k+1} = x_k + x_{k-2005}$ for $k \ge 2006$. Show that the sequence has 2005 consecutive terms each divisible by 2006.

2006-B-6. Let k be an integer greater than 1. Suppose $a_k > 0$, and define

$$a_{n+1} = a_n + \frac{1}{\sqrt[k]{a_n}}$$

for $n \ge 0$. Evaluate

$$\lim_{n\to\infty} \frac{a_n^{k+1}}{n^k}.$$

2004-A-3. Define a sequence $\{u_n\}_{n=0}^{\infty}$ by $u_0 = u_1 = u_2 = 1$, and thereafter by the condition that

$$\det \begin{pmatrix} u_n & u_{n+1} \\ u_{n+2} & u_{n+3} \end{pmatrix} = n!$$

for all $n \ge 0$. Show that u_n is an integer for all n. (By convention, $0! = 1$.)

2002-A-5. Define a sequence by $a_0 = 1$, together with the rules $a_{2n+1} = a_n$ and $a_{2n+2} = a_n + a_{n+1}$ for each integer $n \ge 0$. Prove that every positive rational number appears in the set

$$\left\{ \frac{a_{n-1}}{a_n} : n \ge 1 \right\} = \left\{ \frac{1}{1}, \frac{1}{2}, \frac{2}{1}, \frac{1}{3}, \frac{3}{2}, \cdots \right\}.$$

2001-B-3. For any positive integer n let $\langle n \rangle$ denote the closest integer to $\sqrt{n}$. Evaluate

$$\sum_{n=1}^{\infty} \frac{2^{\langle n \rangle} + 2^{-\langle n \rangle}}{2^n}.$$

2001-B-6. Assume that $\{a_n\}_{n\ge 1}$ is an increasing sequence of positive real numbers such that $\lim a_n/n = 0$. Must there exist infinitely many positive integers n such that

$$a_{n-1} + a_{n+i} < 2a_n$$

for $i = 1, 2, \cdots, n-1$?

2000-A-1. Let A be a positive real number. What are the possible values of $\sum_{j=0}^{\infty} x_j^2$, given that $x_0, x_1, x_2, \cdots$ are positive numbers for which $\sum_{j=0}^{\infty} x_j = A$?

2000-A-6. Let $f(x)$ be a polynomial with integer coefficients. Define a sequence $a_0, a_1, \cdots$ of integers such that $a_0 = 0$ and $a_{n+1} = f(a_n)$ for all $n \geq 0$. Prove that if there exists a positive integer m for which $a_m = 0$, then either $a_1 = 0$ or $a_2 = 0$.

1999-A-3. Consider the power series expansion

$$\frac{1}{1 - 2x - x^2} = \sum_{n=0}^{\infty} a_n x^n \ .$$

Prove that, for each integer $n \geq 0$, there is an integer m such that

$$a_n^2 + a_{n+1}^2 = a_m \ .$$

1999-A-4. Sum the series

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{m^2 n}{3^m (n 3^m + m 3^n)} \ .$$

1999-A-6. The sequence $\{a_n\}_{n \geq 1}$ is defined by $a_1 = 1$, $a_2 = 2$, $a_3 = 24$, and for $n \geq 4$,

$$a_n = \frac{6a_{n-1}^2 a_{n-3} - 8a_{n-1}a_{n-2}^2}{a_{n-2}a_{n-3}} \ .$$

Show that, for all n, a_n is an integer multiple of n.

1999-B-3. Let $A = \{(x,y) : 0 \leq x, y \leq 1\}$. For $(x,y) \in A$, let

$$S(x,y) = \sum_{\frac{1}{2} \leq \frac{m}{n} \leq 2} x^m y^n \ ,$$

where the sum ranges over all pairs (m,n) of positive integers satisfying the indicated inequalities. Evaluate

$$\lim \{(1 - xy^2)(1 - x^2 y) S(x,y) : (x,y) \longrightarrow (1,1), (x,y) \in A\} \ .$$

1998-A-4. Let $A_1 = 0$ and $A_2 = 1$. For $n > 2$, the number A_n is defined by concatenating the decimal expansions of A_{n-1} and A_{n-2} from left to right. For example, $A_3 = A_2 A_1 = 10$, $A_4 = A_3 A_2 = 101$, $A_5 = A_4 A_3 = 10110$, and so forth. Determine all n such that 11 divides A_n.

1998-B-4. Find necessaary and sufficient conditions on positive integers m and n so that

$$\sum_{i=0}^{mn-1} (-1)^{\lfloor i/m \rfloor + \lfloor i/n \rfloor} = 0 \ .$$

1997-A-6. For a positive integer n and any real number c, define x_k recursively by $x_0 = 0$, $x_1 = 1$, and for $k \geq 0$,

$$x_{k+2} = \frac{c x_{k+1} - (n-k) x_k}{k+1} \ .$$

Fix n and then take c to be the largest value for which $x_{n+1} = 0$. Find x_k in terms of n and k, $1 \le k \le n$.

1994-A-1. Suppose that a sequence $a_1, a_2, a_3, \cdots$ satisfies $0 < a_n \le a_{2n} + a_{2n+1}$ for all $n \ge 1$. Prove that the series $\sum_{n=1}^{\infty} a_n$ diverges.

1994-A-5. Let $(r_n)_{n\ge 0}$ be a sequence of positive real numbers such that $\lim_{n\to\infty} r_n = 0$. Let S be the set of numbers representable as a sum

$$r_{i_1} + r_{i_2} + \cdots + r_{i_{1994}}$$

with $i_1 < i_2 < \cdots < i_{1994}$. Show that every nonempty interval (a, b) contains a nonempty subinterval (c, d) that does not intersect S.

1993-A-2. Let $(x_n)_{n\ge 0}$ be a sequence of nonzero real numbers such that

$$x_n^2 - x_{n-1}x_{n+1} = 1$$

for $n = 1, 2, 3, \cdots$. Prove that there exists a real number a such that $x_{n+1} = ax_n - x_{n-1}$ for all $n \ge 1$.

1993-A-6. The infinite sequence of $2's$ and $3's$

$$2, 3, 3, 2, 3, 3, 3, 2, 3, 3, 3, 2, 3, 3, 2, 3, 3, 3, 2, 3, 3, 3, 2, 3, 3, 3, 2, 3, 3, 2, 3, 3, 3, 2, \cdots$$

has the property that, if one forms a second sequence that records the number of $3's$ between successive $2's$, the result is identical to the given sequence. Show that there exists a real number r such that, for any n, the nth term of the sequence is 2 if and only if $n = 1 + \lfloor rm \rfloor$ for some nonnegative integer m. (Note: $\lfloor x \rfloor$ denotes the largest integer less than or equal to x.

1992-A-1. Prove that $f(n) = 1 - n$ is the only integer-valued function defined on the integers that satisfies the following conditions
(i) $f(f(n)) = n$, for all integers n;
(ii) $f(f(n + 2) + 2) = n$ for all integers n;
(iii) $f(0) = 1$.

1992-A-5. For each positive integer n, let

$$a_n = \begin{cases} 0, & \text{if the number of 1's in the binary representation of } n \text{ is even,} \\ 1, & \text{if the number of 1's in the binary representation of } n \text{ is odd.} \end{cases}$$

Show that there do not exist integers k and m such that

$$a_{k+j} = a_{k+m+j} = a_{k+2m+j}$$

for $0 \le j \le m - 1$.

1991-B-1. For each integer $n \ge 0$, let $S(n) = n - m^2$, where m is the greatest integer with $m^2 \le n$. Define a sequence $(a_k)_{k=0}^{\infty}$ by $a_0 = A$ and $a_{k+1} = a_k + S(a_k)$ for $k \ge 0$. For what positive integers A is this sequence eventually constant?

1990-A-1. Let

$$T_0 = 2, T_1 = 3, T_2 = 6,$$

and for $n \ge 3$,

$$T_n = (n + 4)T_{n-1} - 4nT_{n-2} + (4n - 8)T_{n-3} \quad .$$

The first few terms are

$$2, 3, 6, 14, 40, 152, 784, 5168, 40576, 363392.$$

Find, with proof, a formula for T_n of the form $T_n = A_n + B_n$, where (A_n) and (B_n) are well-known sequences.

1988-B-4. Prove that if $\sum_{n=1}^{\infty} a_n$ is a convergent series of positive real numbers, then so is

$$\sum_{n=1}^{\infty} (a_n)^{n/(n+1)} \ .$$

1985-A-4. Define a sequence $\{a_i\}$ by $a_1 = 3$ and $a_{i+1} = 3^{a_i}$ for $i \geq 1$. Which integers between 00 and 99 inclusive occur as the last two digits in the decimal expansion of infinitely many a_i?

1981-B-1. Find

$$\lim_{n\to\infty} \left[\frac{1}{n^5} \sum_{h=1}^{n} \sum_{k=1}^{n} (5h^4 - 18h^2k^2 + 5k^4) \right].$$

1979-A-3. Let x_1, x_2, x_3, $\cdots$ be a sequence of nonzero real numbers satisfying

$$x_n = \frac{x_{n-2}x_{n-1}}{2x_{n-2} - x_{n-2}} \qquad \text{for} \quad n = 3, 4, 5, \cdots \quad .$$

Establish necessary and sufficient conditions on x_1 and x_2 for x_n to be an integer for infinitely many values of n.

1976-B-5. Evaluate

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} (x-k)^n.$$

1975-B-6. Show that, if $s_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$, then
(a) $n(n+1)^{1/n} < n + s_n$ for $n > 1$, and
(b) $(n-1)n^{-1/(n-1)} < n - s_n$ for $n > 2$.

1962-I-5. Evaluate in closed form

$$\sum_{k=1}^{n} \binom{n}{k} k^2.$$

1962-II-1. Let $x^{(n)} = x(x-1)\cdots(x-n+1)$ for n a positive integer and let $x^{(0)} = 1$. Prove that

$$(x+y)^{(n)} = \sum_{k=0}^{n} \binom{n}{k} x^{(k)} y^{(n-k)}.$$