PUTNAM PROBLEMS

DIFFERENTIAL EQUATIONS

First Order Equations

1. Linear $y' + p(x)y = q(x)$
   Muliply through by the integrating factor $\exp(\int p(x)dx)$ to obtain
$$(y\exp(\int p(x)dx))' = q(x)\exp(\int p(x)dx) \quad .$$

2. Separation of variables $y' = f(x)g(y)$
   Put in the form $dy/g(y) = f(x)dx$ and integrate both sides.

3. Homogeneous $y' = f(x, y)$ where $f(tx, ty) = f(x, y)$.
   Let $y = ux$, $y' = u'x + u$ to get $u'x + u = f(1, u)$.

4. Fractional linear
$$y' = \frac{ax + by}{cx + dy} \quad .$$
   Do as in case 3, or introduce an auxiliary variable t and convert to a system
$$\frac{dy}{dt} = ax + by \qquad \frac{dx}{dt} = cx + dy$$
   and try $x = c_1 e^{\lambda t}$, $y = c_2 e^{\lambda t}$. If this yields two distinct values for λ, the ratio $c_1 : c_2$ can be found.If there is only one value of λ, try $x = (c_1 + c_2 t)e^{\lambda t}$, $y = (c_3 + c_4 t)e^{\lambda t}$.

5. Modified fractional linear
$$y' = \frac{ax + by + c}{hx + ky + r} \quad .$$
   (i) If $ak - bh \neq 0$, choose p, q so that $ap + bq + c = 0$, $hp + kq + r = 0$ and make a change of variables: $X = x - p$, $Y = y - q$.
   (ii) If $ak - bh = 0$, set $u = ax + by + c$ to obtain a separation of variables equation in y and u.

6. General linear exact equation
$$P(x, y)dx + Q(x, y)dy = 0$$
   where
$$\Delta \equiv \frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x} = 0 \quad .$$
   The equation has a solution of the form $f(x, y) = c$, where $\partial f/\partial x = P$ and $\partial f/\partial y = Q$. We have
$$f(x, y) = \int_{x_0}^{x} P(u, x)du + \int_{y_0}^{y} Q(x_0, v)dv$$
   or
$$f(x, y) = \int_{y_0}^{y} Q(x, v)dv + \int_{x_0}^{x} P(u, y_0)du$$
   where (x_0, y_0) is any point.

7. General linear inexact equation
$$P(x, y)dx + Q(x, y)dy = 0$$

where

$$\Delta \equiv \frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x} \neq 0 \quad .$$

We need to find an "integrating factor" $h(x, y)$ to satisfy

$$h(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}) + P\frac{\partial h}{\partial y} - Q\frac{\partial h}{\partial x} = 0 \quad .$$

There is no general method for equations of this type, but one can try assuming that h is a function of x alone, y alone, xy, x/y or y/x. If h can be found, multiply the equation through by h and proceed as in Case 6.

8. Riccati equation $y' = f_0(x) + f_1(x)y + f_2(x)y^2$ where $f_2(x) \neq 0$.
   If a solution y_0 is known, set $y = y_0 + (1/u)$ to get a first order linear equation in u and x.

9. Bernoulli equation $y' + p(x)y = q(x)y^n$.
   If $n = 0$, use Case 1. If $n = 1$, use Case 2. If $n = 2$, consider Case 8. If $n \neq 0, 1$, set $u = y^{1-n}$ to get a first order equation in u and x.

Linear equations of nth order with constant coefficients

A linear equation of the nth degree with constant coefficients has the form

$$c_n y^{(n)} + c_{n-1}y^{(n-1)} + \cdots + c_2 y'' + c_1 y' + c_0 y = q(x)$$

where the c_i are constants and y is an unknown function of x. The general solution of such an equation is the sum of two parts:

the complementary function (general solution of the homogeneous equation formed by taking $q(x) = 0$);
a particular integral (any solution of the given equation).
The complementary function is the sum of terms of the form

$$(a_{r-1}x^{r-1} + a_{r-2}x^{r-2} + \cdots + a_2 t^2 + a_1 t + a_0)e^{\lambda x}$$

where λ is a root of multiplicity r of the auxiliary polynomial

$$c_n t^n + c_{n-1}t^{n-1} + \cdots + c_2 t^2 + c_1 t + c_0$$

and a_i are arbitrary constants.

A particular integral can be found when $q(x)$ can be written as the sum of terms of the type $h(x)e^{\rho x}$ where $h(x)$ is a polynomial and ρ is complex. This includes the cases $q(x) = h(x)e^{\alpha x}\sin\beta x$ and $q(x) = h(x)e^{\alpha x}\cos\beta x$, with α and β real. When the c_i and the coefficients of $h(x)$ are real, solve with $q(x) = h(x)e^{(\alpha+i\beta)}x$ and take the real or imaginary parts, respectively, of the solution obtained.

Operational calculus: Let $Du = u'$. The left side of the equation can be written $p(D)y = q(x)$ where $p(t) = c_n t^n + \cdots + c_1 t + c_0$. For any polynomial $p(t)$, we have the operational rules

$$p(D)e^{rx} = p(r)e^{rx} \qquad \text{and} \qquad p(D)(ue^{rx}) = e^{rx}p(D + r)u \quad .$$

The following examples illustrate how a particular integral can be obtained without having to deal with special cases or undetermined coefficients:

(i) $y'' + 2y' + 2y = 2e^{-x}\sin x$.
The solution of this equation is the imaginary part of the solution of the following equation

$$(D^2 + 2D + 2)y = 2e^{(-1+i)x} \quad .$$

We try for a particular integral of the form $y = ue^{(-1+i)x}$, where the exponent agrees with the exponent on the right side of the equation. Then, factoring the polynomial in D and substituting for y, we obtain:

$$(D+1-i)(D+1+i)(ue^{(-1+i)x}) = 2e^{(-1+i)x} \quad .$$

Bringing the exponent through and cancelling it, we get

$$D(D+2i)u = (D+2i)Du = 2 \qquad (*)$$

Differentiate:

$$(D+2i)D^2u = 0 \qquad (**) \ .$$

Any u for which (*) and (**) hold will yields a particular integral. Choose u such that $u'' = 0$. Then (**) holds. To satisfy (*), we need $2iDu = 2$ or $Du = -i$. Hence take $u(x) = -ix$. A particular integral of the complex equation is $-ix\exp((-1+i)x)$, and a particular integral of the original equation is

$$\text{Im } (-ixe^{(-1+i)x}) = -xe^{-x}\cos x \quad .$$

(ii) $y'' - 4y' + 4y = 8x^2e^{2x}\sin 2x$
A particular integral of this equation is the imaginary part of the particular integral of the equation

$$(D-2)^2y = 8x^2e^{(2+2i)x}.$$

Let $y = ue^{(2+2i)x}$. Then, bring the exponential through the operator as before and cancelling, we get

$$(D+2i)^2u = (D^2+4iD-4)u = 8x^2 \qquad (1)$$

$$(D^2+4iD-4)Du = 16x \qquad (2)$$

$$(D^2+4iD-4)D^2u = 16 \qquad (3)$$

$$(D^2+4iD-4)D^3u = 0 \qquad (4) \ .$$

Let $D^3u = 0$ to satisfy (4). Then (3) requires $-4D^2u = 16$ or $D^2u = -4$. Now, (2) requires $-16i - 4Du = 16x$ or $Du = -16x - 16i$. Finally, to make (1) hold, we need $u = -2x^2 - 4ix + 3$. The solution to (ii) is thus

$$y = \text{Im}(-2x^2 - 4ix + 3)e^{(2+2i)x} \quad .$$

Putnam questions

2010-A-3. Suppose that the function $h : \mathbf{R}^2 \to \mathbf{R}$ has continuous partial derivatives and satisfies the equation

$$h(x,y) = a\frac{\partial h}{\partial x}(x,y) + b\frac{\partial h}{\partial y}(x,y)$$

for some constants a, b. Prove that if there is a constant M such that $|h(x,y)| \leq M$ for all $(x,y) \in \mathbf{R}^2$, then h is identic ally zero.

2010-B-5. Is there a strictly increasing function $f : \mathbf{R} \to \mathbf{R}$ such that $f'(x) = f(f(x))$ for all x?

2009-A-2. Functions f, g, h are differentiable on some open interval around 0 and satisfy the equations and initial conditions

$$f' = 2f^2gh + \frac{1}{gh} \ , \qquad f(0) = 1 \ ,$$

$$g' = fg^2h + \frac{4}{fh} \ . \qquad g(0) = 1 \ ,$$

$$h' = 3fgh^2 + \frac{1}{fg}\ , \qquad h(0) = 1\ .$$

Find an explicit formula for $f(x)$ valid in some open interval around 0.

2009-B-5. Let $f : (1, \infty) \to \mathbf{R}$ be a differentiable function such that

$$f'(x) = \frac{x^2 - (f(x))^2}{x^2((f(x))^2 + 1)}$$

for all $x > 1$. Prove that $\lim_{x\to\infty} f(x) = \infty$.

2005-B-3. Find all differentiable functions $f : (0, \infty) \longrightarrow (0, \infty)$ for which there is a positive real number a such that

$$f'\left(\frac{a}{x}\right) = \frac{x}{f(x)}$$

for all $x > 0$.

1997-B-2. Let f be a twice differentiable real-valued function satisfying

$$f(x) + f''(x) = -xg(x)f'(x)\ ,$$

where $g(x) \geq 0$ for all real x. Prove that $|f(x)|$ is bounded.

1995-A-5. Let $x_1, x_2, \cdots, x_n$ be differentiable (real-valued) functions of a single variable t which satisfy

$$\frac{dx_1}{dt} = a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n$$

$$\frac{dx_2}{dt} = a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n$$

$$\cdots$$

$$\frac{dx_n}{dt} = a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n$$

for some constants $a_{ij} \geq 0$. Suppose that for all i, $x_i(t) \to 0$ as $t \to \infty$. Are the functions $x_1, x_2, \cdots, x_n$ necessarily linearly dependent?

1989-B-3. Let f be a function on $[0, \infty)$, differentiable and satisfying

$$f'(x) = -3f(x) + 6f(2x)$$

for $x > 0$. Assume that $|f(x)| \leq e^{-\sqrt{x}}$ for $x \geq 0$ (so that $f(x)$ tends rapidly to 0 as x increases). For n a nonnegative integer, define

$$\mu_n = \int_0^\infty x^n f(x)dx$$

(sometimes called the nth moment of f).

a. Express μ_n in terms of μ_0.

b. Prove that the sequence $\{\mu_n \frac{3^n}{n!}\}$ always converges, and that this limit is 0 only if $\mu_0 = 0$.

1988-A-2. A not uncommon calculus mistake is to believe that the product rule for derivatives says that $(fg)' = f'g'$. If $f(x) = e^{x^2}$, determine, with proof, whether there exists an open interval (a, b) and a nonzero function g defined on (a, b) such that this wrong product rule is true for x in (a, b).

1979-B-4. (a) Find a solution that is not identically zero, of the homogeneous linear differential equation

$$(3x^2 + x - 1)y'' - (9x^2 + 9x - 2)y' + (18x + 3)y = 0.$$

Intelligent guessing of the form of a solution may be helpful.

(b) Let $y = f(x)$ be the solution of the *non-homogeneous* differential equation

$$(3x^2 + x - 1)y'' - (9x^2 + 9x - 2)y' + (18x + 3)y = 6(6x + 1)$$

that has $f(0) = 1$ and $(f(-1) - 2)(f(1) - 6) = 1$. Find integers a, b, c such that $)f(-2) - a)(f(2) - b) = c$.

1975-A-5. On some interval I of the real line, let $y_1(x)$ and $y_2(x)$ be linearly independent solutions of the differential equation $y'' = f(x)y$, where $f(x)$ is a continuous real-valued function. Suppose that $y_1(x) > 0$ and $y_2(x) > 0$ on I. Show that there exists a positive constant c such that, on I, the function $z(x) = c\sqrt{y_1(x)y_2(x)}$ satisfies the equation

$$z'' + \frac{1}{z^3} = f(x)z.$$

State clearly the manner in which c depends on $y_1(x)$ and $y_2(x)$.

1973-A-5. A particle moves in $3-$space according to the equations:

$$\frac{dx}{dt} = yz, \qquad \frac{dy}{dt} = zx, \qquad \frac{dz}{dt} = xy.$$

(Here $x(t)$, $y(t)$, $z(t)$ are real-valued functions of the real variable t.) Show that:

(a) If two of $x(0)$, $y(0)$, $z(0)$ equal zero, then the particle never moves.

(b) If $x(0) = y(0) = 1$, $z(0) = 0$, then the solution is

$$x = \sec t, \qquad y = \sec t, \qquad z = \tan t,$$

whereas if $x(0) = y(0) = 1$, $z(0) = -1$, then

$$x = \frac{1}{t+1} \qquad y = \frac{1}{t+1}, \qquad z = \frac{-1}{t+1}.$$

(c) If at least two of the values $x(0)$, $y(0)$, $z(0)$ are different from zero, then either the particle moves to infinity at some finite time in the future, or it came from infinity at some finite time in the past. (A point (x, y, z) in $3-$space "moves to infinity" if its distance from the origin approaches infinity.)

1971-B-5. Show that the graphs in the $x-y$ plane of all solutions of the system of differential equations

$$x'' + y' + 6x = 0, \qquad y'' - x' + 6y = 0 \qquad (' = d/dt)$$

which satisfy $x'(0) = y'(0) = 0$ are hypocycloids, and find the radius of the fixed circle and the two possible values of the radius of the rolling circles for each solution. (A hypocycloid is the path described by a fixed point on the circumference of a circle which rolls on the inside of a given fixed circle.)

1969-A-5. Let $u(t)$ be a continuous function in the system of differential equations

$$\frac{dx}{dt} = -2y + u(t), \qquad \frac{dy}{dt} = -2x + u(t).$$

Show that, regardless of the choice of $u(t)$, the solution of the system which satisfies $x = x_0$, $y = y_0$ at $t = 0$ will never pass through $(0, 0)$ unless $x_0 = y_0$. When $x_0 = y_0$, show that, for any positive value t_0 of t, it is possible to choose $u(t)$ so the solution is at $(0, 0)$ when $t = t_0$.

1967-A-4. Show that if $\lambda > \frac{1}{2}$ there does not exist a real-valued function u such that for all x in the closed interval $0 \le x \le 1$,

$$u(x) = 1 + \lambda \int_x^1 u(y)u(y - x)dy.$$

1966-B-6. Show that all solutions of the differential equation $y'' + e^x y = 0$ remain bounded as $x \longrightarrow \infty$.

1960-II-7. Let $g(t)$ and $h(t)$ be real, continuous functions for $t \ge 0$. Show that any function $v(t)$ satisfying the differential inequality

$$\frac{dv}{dt} + g(t)v \ge h(t), \qquad v(0) = c,$$

satisfies the further inequality $v(t) \ge u(t)$, where

$$\frac{du}{dt} + g(t)u = h(t), \qquad u(0) = c.$$

From this, conclude that for sufficiently small $t > 0$, the solution of

$$\frac{dv}{dt} + g(t)v = v^2, \qquad v(0) = c_1,$$

may be written

$$v = \max \left[c_1 \exp\left(-\int_0^t (g(s) - 2w(s))ds \right) - \int_0^t \exp\left(-\int_0^t (g(s_1) - 2w(s_1))ds_1 \right) w^2(s)ds \right],$$

where the maximum is over all continuous functions $w(t)$ defined over the $t-$interval $[0, t_0]$.

1960-II-3. The motion of the particles of a fluid in the plane is specified by the following components of velocity

$$\frac{dx}{dt} = y + 2x(1 - x^2 - y^2), \qquad \frac{dy}{dt} = -x.$$

Sketch the shape of the trajectories near the origin. Discuss what happens to an individual particle as $t \longrightarrow +\infty$ and justify your conclusion.

1958-I-3. Under the assumption that the following set of relations has a unique solution for $u(t)$, determine it.

$$\frac{d(u(t))}{dt} = u(t) + \int_0^1 u(s)ds, \qquad u(0) = 1.$$

1958(Feb.)-II-6. A projectile moves in a resisting medium. The resisting force is a function of the velocity and is directed along the velocity vector. The equation $x = f(t)$ gives the horizontal distance in terms of the time t. Show that the vertical distance y is given by

$$y = -gf(t) \int \frac{dt}{f'(t)} + g \int \frac{f(t)dt}{f'(t)} + Af(t) + B,$$

where A and B are constants and g is the acceleration due to gravity.

1957-II-6. The curve $y = f(x)$ passes through the origin with a slope of 1. It satisfies the differential equation

$$(x^2 + 9)y'' + (x^2 + 4)y = 0.$$

Show that it crosses the $x-$axis between $x = (3/2)\pi$ and $x = (\sqrt{63/53})\pi$.

1955-I-7. Consider the function f defined by the differential equation

$$f''(x) = (x^2 + ax)f(x)$$

and the initial conditions $f(0) = 1$, $f'(0) = 0$. Prove that the roots of f are bounded above but unbounded below.

1954-I-3. Prove that the family of integral curves of the differential equation

$$\frac{dy}{dx} + p(x)y = q(x), \qquad p(x)q(x) \neq 0$$

is cut by the line $x = k$, the tangents at the points of intersection are concurrent.

1953-II-3. Solve the equations

$$\frac{dy}{dx} = z(y + z)^n, \qquad \frac{dz}{dx} = y(y + z)^n,$$

given the initial conditions $y = 1$ and $z = 0$ when $x = 0$.

1952-II-2. Find the surface generated by the solutions of

$$\frac{dx}{yz} = \frac{dy}{zx} = \frac{dz}{xy}$$

which intersects the circle $y^2 + z^2 = 1$, $x = 0$.