PUTNAM PROBLEMS

CALCULUS, ANALYSIS

2018-A-5. Let $f : \mathbf{R} \to \mathbf{R}$ be an infinitely differentiable function satisfying $f(0) = 0$, $f(1) = 1$, and $f(x) \geq 0$ for all $x \in \mathbf{R}$. Show that there exist a positive integer n and a real number x such that $f^{(n)}(x) < 0$.

2018-B-5. Let $f = (f_1, f_2)$ be a function from $\mathbf{R}^2$ to $\mathbf{R}^2$ with continuous partial derivatives $\frac{\partial f_i}{\partial x_j}$ that are positive everywhere. Suppose that

$$\frac{\partial f_1}{\partial x_1}\frac{\partial f_2}{\partial x_2} - \frac{1}{4}\left(\frac{\partial f_1}{\partial x_2} + \frac{\partial f_2}{\partial x_1}\right)^2 > 0$$

everywhere. Prove that f is one-to-one.

2017-A-3. Let a and b be real numbers with $a < b$, and let f and g be continuous functions from $[a, b]$ to $(0, \infty)$ such that $\int_a^b f(x)\,dx = \int_a^b g(x)\,dx$ but $f \neq g$. For every positive integer n, define

$$I_n = \int_a^b \frac{(f(x))^{n+1}}{(g(x))^n}\,dx.$$

Show that $I_1, I_2, I_3, \ldots$ is an increasing sequence with $\lim_{n\to\infty} I_n = \infty$.

2017-B-3. Suppose that $f(x) = \sum_{i=0}^{\infty} c_i x^i$ is a power series for which each coefficient c_i is 0 or 1. Show that, if $f(2/3) = 3/2$, then $f(1/2)$ must be irrational.

2016-A-1. Find the smallest positive integer j such that for every polynomial $p(x)$ with integer coefficients and for every integer k, the integer

$$p^{(j)}(k) = \frac{d^j}{dx^j}p(x)|_{x=k}$$

(the j-th derivative of $p(x)$ at k) is divisible by 2016.

2016-A-2. Given a positive integer n, let $M(n)$ be the largest integer m such that

$$\binom{m}{n-1} > \binom{m-1}{n}.$$

Evaluate

$$\lim_{n\to\infty} \frac{M(n)}{n}.$$

2016-A-3. Suppose that f is a function from $\mathbf{R}$ to $\mathbf{R}$ such that

$$f(x) + f\left(1 - \frac{1}{x}\right) = \arctan x$$

for all real $x \neq 0$. (As usual, $y = \arctan x$ means $-\pi/2 < y < \pi/2$ and $\tan y = x$.) Find

$$\int_0^1 f(x)dx.$$

2016-A-6. Find the smallest constant C such that for every real polynomial $P(x)$ of degree 3 that has a root in the interval $[0,1]$,

$$\int_0^1 |P(x)|dx \le C \max_{x\in[0,1]} |P(x)|.$$

2016-B-5. Find all functions f from the interval $(1,\infty)$ to $(1,\infty)$ with the following property: if $x, y \in (1,\infty)$ and $x^2 \le y \le x^3$, then $(f(x))^2 \le f(y) \le (f(x))^3$.

2015-A-3. Compute

$$\log_2\left(\prod_{a=1}^{2015}\prod_{b=1}^{2015}(1+e^{2\pi iab/2015})\right).$$

Here i is the imaginary unit (that is, $i^2=-1$).

2015-A-4. For each real number x, let

$$f(x) = \sum_{n\in S_x} \frac{1}{2^n},$$

where S_x is the set of positive integers n for which $\lfloor nx \rfloor$ is even. What is the largest real number L such that $f(x) \ge L$ for all $x \in [0,1)$? (As usual, $\lfloor z \rfloor$ denotes the greatest integer less than or equal to z.)

2015-B-1. Let f be a three times differentiable function (defined on $\mathbf{R}$ and real valued) such that f has at least five distinct real zeros. Prove that $f + 6f' + 12f'' + 8f'''$ has at least two distinct real zeros.

2014-A-1. Prove that every nonzero coefficient of the Taylor series of

$$(1-x+x^2)e^x$$

about $x=0$ is a rational number whose numerator (in lowest terms) is either 1 or a prime number.

2014-A-3. Let $a_0 = 5/2$ and $a_k = a_{k-1}^2 - 2$ for $k \ge 1$. Compute

$$\prod_{k=0}^{\infty}\left(1-\frac{1}{a_k}\right)$$

in closed form.

2014-B-2. Suppose that f is a function on the interval $[1,3]$ such that $-1 \le f(x) \le 1$ for all x and $\int_1^3 f(x)\,dx = 0$. How large can $\int_1^3 f(x)/x\,dx$ be?

2014-B-6. Let $f : [0,1] \to \mathbf{R}$ be a function for which there exists a constant $K > 0$ such that $|f(x)-f(y)| \le K|x-y|$ for all $x, y \in [0,1]$. Suppose also that, for each rational number $r \in [0,1]$, there exist integers a and b such that $f(r) = a + br$. Prove that there exist finitely many intervals $I_1, \ldots, I_n$ such that f is a linear function on each I_i and $[0,1] = \cup_{i=1}^n I_i$.

2013-A-3. Suppose that the real numbers $a_0, a_1, \ldots, a_n$ and x, with $0<x<1$, satisfy

$$\frac{a_0}{1-x} + \frac{a_1}{1-x^2} + \cdots + \frac{a_n}{1-x^{n+1}} = 0.$$

$0<y<1$ such that

$$a_0 + a_1 y + \cdots + a_n y^n = 0.$$

2013-B-2. Let $C = \cup_{N=1}^{\infty} C_N$, where C_N denotes the set of those 'cosine polynomials' of the form

$$f(x) = 1 + \sum_{n=1}^{N} a_n \cos(2\pi nx)$$

for which

(i) $f(x) \geq 0$ for all real x, and

(ii) $a_n = 0$ whenever n is a multiple of 3.

Determine the maximum value of $f(0)$ as f ranges through C, and prove that this maximum is attained.

2013-B-4, For any continuous real-valued function f defined on the interval $[0, 1]$, let

$$\mu(f) = \int_0^1 f(x)dx$$

$$\text{Var }(f) = \int +0^1(f(x) - \mu(f))^2 dx$$

$$M(f) = \max_{0 \leq x \leq 1} |f(x)|.$$

Show that if f and g are continuous real-valued functions defined on the interval $[0, 1]$, then

$$\text{Var}(fg) \leq 2\text{Var}(f)M(g)^2 + 2\text{Var}(g)M(f)^2.$$

2012-A-3. Let $f : [-1, 1] \to \mathbf{R}$ be a continuous function such that

(i) $f(x) = \frac{2-x^2}{2} f\left(\frac{x^2}{2-x^2}\right)$ for every $x \in [-1, 1]$;

(ii) $f(0) = 1$; and

(iii) $\lim_{x \to 1^-} \frac{f(x)}{\sqrt{1-x}}$ exists and is finite.

Prove that f is unique, and express $f(x)$ in closed form.

2012-A-6. Let $f(x, y)$ be a continuous, real-valued function on $\mathbf{R}^2$. Suppose that, for every rectangular region R of area 1, the double integral of $f(x, y)$ over R equals 0. Must $f(x, y)$ be identically zero?

2012-B-1. Let S be a class of functions from $[0, \infty)$ to $[0, \infty)$ that satisfies

(i) The functions $f_1(x) = e^x - 1$ and $f_2(x) = \ln(x + 1)$ are in S;

(ii) If $f(x)$ and $g(x)$ are in S, the functions $f(x) + g(x)$ and $f(g(x))$ are in S;

(iii) If $f(x)$ and $g(x)$ are in S and $f(x) \geq g(x)$ for $x \geq 0$, then the function $f(x) - g(x)$ is in S.

Prove that if $f(x)$ and $g(x)$ are in S, then the function $f(x)g(x)$ is also in S.

2012-B-4. Suppose that $a_0 = 1$ and that $a_{n+1} = a_n + e^{-a_n}$ for $n = 0, 1, 2, \cdots$. Does $a_n - \log n$ have a finite limit as $n \to \infty$? (Here $\log n = \log_e n = \ln n$.)

2012-B-5. Prove that, for any two bounded functions $g_1, g_2 : \mathbf{R} \to [1, \infty)$, there exist functions $h_1, h_2 : \mathbf{R} \to \mathbf{R}$ such that for every $x \in \mathbf{R}$,

$$\sup_{s \in \mathbf{R}} (g_1(s)^x g_2(s)) = \max_{t \in \mathbf{R}}(xh_1(t) + h_2(t)).$$

2011-A-2. Let $\{a_1, a_2, \cdots\}$ and $\{b_1, b_2, \cdots\}$ be sequences of positive real numbers such that $a_1 = b_1 = 1$ and $b_n = b_{n-1}a_n - 2$ for $n = 2, 3, \cdots$. Assume that the sequence $\{b_n\}$ is bounded. Prove that

$$S = \sum_{n=1}^{\infty} \frac{1}{a_1 a_2 \cdots a_n}$$

converges, and evaluate S.

2011-A-3. Find a real number c and a positive number L for which

$$\lim_{r\to\infty} \frac{r^c \int_0^{\pi/2} x^r \sin x dx}{\int_0^{\pi/2} x^r \cos x dx} = L.$$

2011-A-5. Let $F : \mathbf{R}^2 \to \mathbf{R}$ and $g : \mathbf{R} \to \mathbf{R}$ be twice continuously differentiable functions with the following properties:

- $F(u, u) = 0$ for every $u \in \mathbf{R}$;
- for every $x \in \mathbf{R}$, $g(x) > 0$ and $x^2 g(x) \leq 1$;
- for every $(u, v) \in \mathbf{R}^2$, the vector $\nabla F(u, v)$ is either $\mathbf{0}$ or parallel to the vector $(g(u), -g(v))$.

Prove that there exists a constant C such that for every $n \geq 2$ and any $x_1, \cdots, x_{n+1} \in \mathbf{R}$, we have

$$\min\{|F(x_i, x_j)| : i \neq j\} \leq \frac{C}{n}.$$

2011-B-3. Let f and g be (real-valued) functions defined on an open interval containing 0, with g nonzero and continuous at 0; if fg and f/g are differentiable at 0, must f be differentiable at 0?

2011-B-5. Let $a_1, a_2, \cdots$ be real numbers. Suppose there is a constant A such that for all n,

$$\int_{-\infty}^{\infty} \left(\sum_{i=1}^{n} \frac{1}{1 + (x - a_i)^2} \right)^2 dx \leq An.$$

Prove that there is a constant $B > 0$ such that for all n,

$$\sum_{i,j=1}^{n} (1 + (a_i - a_j)^2) \geq Bn^2.$$

2010-A-2. Find all differentiable functions $f : \mathbf{R} \to \mathbf{R}$ such that

$$f'(x) = \frac{f(x+n) - f(x)}{n}$$

for all real numbers x and all positive integers n.

2010-A-3. Suppose that the function $h : \mathbf{R}^2 \to \mathbf{R}$ has continuous partial derivatives and satisfies the equation

$$h(x, y) = a\frac{\partial h}{\partial x}(x, y) + b\frac{\partial h}{\partial y}(x, y)$$

for some constants a, b. Prove that if there is a constant M such that $|h(x, y)| \leq M$ for all $(x, y) \in \mathbf{R}^2$, then h is identic ally zero.

2010-A-6. Let $f : [0, \infty) \to \mathbf{R}$ be a strictly decreasing continuous function such that $\lim_{x\to\infty} f(x) = 0$. Prove that

$$\int_0^\infty \frac{f(x+1) - f(x)}{f(x)} dx$$

diverges.

2010-B-5. Is there a strictly increasing fucntion $f : \mathbf{R} \to \mathbf{R}$ such that $f'(x) = f(f(x))$ for all x?

2009-A-6. Let $f; [0, 1]^2 \to \mathbf{R}$ be a continuous function on the closed unit square such that $\partial f/\partial x$ and $\partial f/\partial y$ exist and are continuous on the interior $(0, 1)^2$. Let $a = \int_0^1 f(0, y)dy$, $b = \int_0^1 f(1, y)dy$, $c = \int_0^1 f(x, 0)dx$, $d = \int_0^1 f(x, 1)dx$. Prove or disprove: There must be a point (x_0, y_0) in $(0, 1)^2$ such that

$$\frac{\partial f}{\partial x}(x_0, y_0) = b - a$$

and

$$\frac{\partial f}{\partial y}(x_0, y_0) = d - c \ .$$

2008-A-4. Define $f : \mathbf{R} \to \mathbf{R}$ by

$$f(x) = \begin{cases} x & \text{if } x \le e \\ x f(\ln x) & \text{if } x > e. \end{cases}$$

Does

$$\sum_{n=1}^\infty \frac{1}{f(n)}$$

converge?

2008-B-1. What is the maximum number of rational points that can lie on a circle in $\mathbf{R}^2$ whose centre is not a rational point? (A *rational point* is a point both of whose coordinates are rational numbers.)

2008-B-2. Let $F_0(x) = \ln x$. For $n \ge 0$ and $x > 0$, let $F_{n+1}(x) = \int_0^x F_n(t)dt$. Evaluate

$$\lim_{n\to\infty} \frac{n! F_n(1)}{\ln n} \ .$$

2008-B-5. Find all continuously differentiable functions $f : \mathbf{R} \to \mathbf{R}$ such that for every rational number q, the number $f(q)$ is rational and has the same denominator as q. (The denominator of a rational number q is the unique positive integer b such that $q = a/b$ for some integer a with $\gcd(a, b) = 1$.) (Note: gcd means greatest common divisor.)

2007-B-2. Suppose that $f : [0, 1] \longrightarrow \mathbf{R}$ has a continuous derivative and that $\int_0^1 f(x)dx = 0$. Prove that for every $\alpha \in (0, 1)$,

$$\left| \int_0^\alpha f(x)dx \right| \le \frac{1}{8} \max{}_{0\le x\le 1} |f'(x)| \ .$$

2006-A-1. Find the volume of the region of points (x, y, z) such that

$$(x^2 + y^2 + z^2 + 8)^2 \le 36(x^2 + y^2) \ .$$

2006-A-5. Let n be a positive odd integer and let θ be a real number such that θ/π is irrational. Set $a_k = \tan(\theta + k\pi/n)$, $k = 1, 2, \cdots, n$. Prove that

$$\frac{a_1 + a_2 + \cdots + a_n}{a_1 a_2 \cdots a_n}$$

is an integer and determine its value.

2006-B-2. Prove that, for every set $X = \{x_1, x_2, \cdots, x_n\}$ of n real numbers, there exists a non-empty subset S of X and an integer m such that

$$\left| m + \sum_{s \in S} s \right| \le \frac{1}{n+1} \ .$$

2006-B-5. For each continuous function $f : [0,1] \longrightarrow \mathbf{R}$, let $I(f) = \int_0^1 x^2 f(x) dx$ and $J(f) = \int_0^1 x (f(x))^2 dx$. Find the maximum value of $I(f) - J(f)$ over all such functions f.

2006-B-6. Let k be an integer greater than 1. Suppose $a_0 > 0$, and define

$$a_{n+1} = a_n + \frac{1}{\sqrt[k]{a_n}}$$

for $n \ge 0$. Evaluate

$$\lim_{n \to \infty} \frac{a_n^{k+1}}{n^k} \ .$$

2005-A-5. Evaluate

$$\int_0^1 \frac{\ln(x+1)}{x^2+1} dx \ .$$

2005-B-3. Find all differentiable functions $f : (0, \infty) \longrightarrow (0, \infty)$ for which there is a positive real number a such that

$$f'\left(\frac{a}{x}\right) = \frac{x}{f(x)}$$

for all $x > 0$.

2004-A-6. Suppose that $f(x, y)$ is a continuous real-valued function on the unit square $0 \le x \le 1$, $0 \le y \le 1$. Show that

$$\int_0^1 \left(\int_0^1 f(x,y) dx \right)^2 dy + \int_0^1 \left(\int_0^1 f(x,y) dy \right)^2 dx \le \left(\int_0^1 \int_0^1 f(x,y) dx dy \right)^2 + \int_0^1 \int_0^1 [f(x,y)]^2 dx dy \ .$$

2004-B-3. Determine all real numbers $a > 0$ for which there exists a nonnegative continuous function $f(x)$ defined on $[0, a]$ with the property that the region

$$R = \{(x, y) : 0 \le x \le a, 0 \le y \le f(x)\}$$

has perimeter k units and area k square units for some real number k.

2004-B-5. Evaluate

$$\lim_{x \to 1-} \prod_{n=0}^{\infty} \left(\frac{1 + x^{n+1}}{1 + x^n} \right)^{x^n} \ .$$

2003-A-3. Find the minimum value of

$$|\sin x + \cos x + \tan x + \cot x + \sec x + \csc x|$$

for real numbers x.

2003-B-6. Let $f(x)$ be a continuous real-valued function defined on the interval $[0, 1]$. Show that

$$\int_0^1 \int_0^1 |f(x) + f(y)| dx dy \geq \int_0^1 |f(x)| dx \ .$$

2002-A-1. Let k be a positive integer. The nth derivative of $1/(x^k - 1)$ has the form $(P_n(x))/(x^k - 1)^{n+1}$ where $P_n(x)$ is a polynomial. Find $P_n(1)$.

2002-B-3. Show that, for all integers $n > 1$,

$$\frac{1}{2ne} < \frac{1}{e} - \left(1 - \frac{1}{n}\right)^n < \frac{1}{ne} \ .$$

2001-B-5. Let a and b be real numbers in the interval $(0, \frac{1}{2})$ and let g be a continuous ral-valued function such that $g(g(x)) = ag(x) + bx$ for all real x. Prove that $g(x) = cx$ for some constant c.

2000-A-4. Show that the improper integral

$$\lim_{B \to \infty} \int_0^B \sin(x) \sin(x^2) dx$$

converges.

2000-B-3. Let $f(t) = \sum_{j=1}^N a_j \sin(2\pi jt)$, where each a_j is real and $a_n \neq 0$. Let N_k denote the number of zeros (including multiplicities) of $d^k f/dt^k$. Prove that

$$N_0 \leq N_1 \leq N_2 \leq \cdots \qquad \text{and} \qquad \lim_{k \to \infty} N_k = 2N \ .$$

[Added note: Presumably one is to restrict t to the interval $[0, 1)$ when counting the zeros.]

2000-B-4. Let $f(x)$ be a continuous function such that $f(2x^2 - 1) = 2xf(x)$ for all x. Show that $f(x) = 0$ for $-1 \leq x \leq 1$.

1999-A-5. Prove that there is a constant C such that, if $p(x)$ is a polynomial of degree 1999, then

$$|p(0)| \leq C \int_{-1}^1 |p(x)| dx \ .$$

1999-B-4. Let f be a real function with a continuous third derivative such that $f(x)$, $f'(x)$, $f''(x)$, $f'''(x)$ are positive for all x. Suppose that $f'''(x) \leq f(x)$ for all x. Show that $f'(x) < 2f(x)$ for all x.

1998-A-3. Let f be a real function on the real line with continuous third derivative. Prove that there exists a point a such that

$$f(a) \cdot f'(a) \cdot f''(a) \cdot f'''(a) \geq 0 \ .$$

1997-A-3. Evaluate

$$\int_0^{\infty}\left(x-\frac{x^3}{2}+\frac{x^5}{2\cdots4}-\frac{x^7}{2\cdot4\cdot6}+\cdots\right)\left(1+\frac{x^2}{2^2}+\frac{x^4}{2^2\cdot4^2}+\frac{x^6}{2^2\cdot4^2\cdot6^2}+\cdots\right)dx\ .$$

1996-A-6. Let $c\geq 0$ be a constant. Give a complete description, with proof, of the set of all continuous functions $f:\mathbf{R}\to\mathbf{R}$ such that $f(x)=f(x^2+c)$ for all $x\in\mathbf{R}$. [Note: $\mathbf{R}$ is the set of real numbers.]

1995-A-2. For what pairs (a,b) of positive real numbers does the improper integral

$$\int_b^{\infty}\left(\sqrt{\sqrt{x+a}-\sqrt{x}}-\sqrt{\sqrt{x}-\sqrt{x-b}}\right)dx$$

converge?

1994-A-2. Let A be the area of the region in the first quadrant bounded by the line $y=\frac{1}{2}x$, the $x-$axis, and the ellipse $\frac{1}{9}x^2+y^2=1$. Find the positive number m such that A is equal to the area of the region in the first quadrant bounded by the line $y=mx$, the $y-$axis, and the ellipse $\frac{1}{9}x^2+y^2=1$.

1994-B-3. Find the set of all real numbers k with the following property:

For any positive, differentiable function f that satisfies $f'(x)>f(x)$ for all x, there is some number N such that $f(x)>e^{kx}$ for all $x>N$.

1994-B-5. For any real number α, define the function f_α by $f_\alpha(x)=\lfloor\alpha x\rfloor$. Let n be a positive integer. Show that there exists an α such that for $1\leq k\leq n$,

$$f_\alpha^k(n^2)=n^2-k=f_{\alpha^k}(n^2)\ .$$

($\lfloor x\rfloor$ denotes the greatest integer $\leq x$, and $f_\alpha^k=f_\alpha\circ\cdots\circ f_\alpha$ is the $k-$fold composition of f_α.)

1993-A-1. The horizontal line $y=c$ intersects the curve $y=2x-3x^3$ in the first quadrant as in the figure. Find c so that the areas of the two shaded regions are equal.

1993-A-5. Show that

$$\int_{-100}^{-10}\left(\frac{x^2-x}{x^3-3x+1}\right)^2dx+\int_{\frac{1}{100}}^{\frac{1}{11}}\left(\frac{x^2-x}{x^3-3x+1}\right)^2dx+\int_{\frac{101}{100}}^{\frac{11}{10}}\left(\frac{x^2-x}{x^3-3x+1}\right)^2dx$$

is a rational number.

1993-B-4. The function $K(x,y)$ is positive and continuous for $0\leq x\leq 1$, $0\leq y\leq 1$, and the functions $f(x)$ and $g(x)$ are positive and continuous for $0\leq x\leq 1$. Suppose that for all x, $0\leq x\leq 1$,

$$\int_0^1 f(y)K(x,y)dy=g(x)\qquad\text{and}\qquad\int_0^1 g(y)K(x,y)dy=f(x)\ .$$

Show that $f(x) = g(x)$ for $0 \le x \le 1$.

1992-A-2. Define $C(\alpha)$ to be the coefficient of x^{1992} in the power series expansion about $x = 0$ of $(1+x)^\alpha$. Evaluate

$$\int_0^1 C(-y-1)\left(\frac{1}{y+1} + \frac{1}{y+2} + \frac{1}{y+3} + \cdots + \frac{1}{y+1992}\right)dy \quad .$$

1992-A-4. Let f be an infinitely differentiable real-valued function defined on the real numbers. If

$$f(\frac{1}{n}) = \frac{n^2}{n^2+1}, \qquad n = 1, 2, 3, \cdots,$$

compute the values of the derivatives $f^{(k)}(0)$, $k = 1, 2, 3, \cdots$.

1992-B-3. For any pair (x, y) of real numbers, a sequence $(a_n(x,y))_{n\ge 0}$ is defined as follows:

$$a_0(x,y) = x$$

$$a_{n+1}(x,y) = \frac{(a_n(x,y))^2 + y^2}{2} \quad , \qquad \text{for all} \quad n \ge 0 \ .$$

Find the area of the region

$$\{(x,y) | (a_n(x,y))_{n\ge 0} \ \text{ converges}\}$$

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1992-B-4. Let $p(x)$ be a nonzero polynomial of degree less than 1992 having no nonconstant factor in common with $x^3 - x$. Let

$$\frac{d^{1992}}{dx^{1992}}\left(\frac{p(x}{x^3-x}\right) = \frac{f(x)}{g(x)}$$

for polynomials $f(x)$ and $g(x)$. Find the smallest possible degree of $f(x)$.

1990-A-2. Is $\sqrt{2}$ the limit of a sequence of numbers of the form $\sqrt[3]{n} - \sqrt[3]{m}$ $(n, m = 0, 1, 2, \ldots)$?

1990-A-4. Consider a paper punch that can be centered at any point of the plane and that, when operated, removes from the plane precisely those points whose distance from the center is irrational. How many punches are needed to remove every point?

1990-B-1. Find all real-valued continuously differentiable functions f on the real line such that for all x

$$(f(x))^2 = \int_0^x ((f(t))^2 + (f'(t))^2)dt + 1990.$$

1989-A-2. Evaluate

$$\int_0^a \int_0^b e^{\max(b^2x^2, a^2y^2} dydx,$$

where a and b are positive.

1989-B-3. Let f be a function on $[0, \infty)$, differentiable and satisfying

$$f'(x) = -3f(x) + 6f(2x)$$

for $x > 0$. Assume that $|f(x)| \leq e^{-\sqrt{x}}$ for $x \geq 0$ (so that $f(x)$ tends rapidly to 0 as x increases). For n a non-negative integer, define

$$\mu_n = \int_0^\infty x^n f(x)dx$$

(sometimes called the nth moment of f).

(a) Express μ_n in terms of μ_0.

(b) Prove that the sequence $\{\mu_n(3^n/n!)\}$ always converges, and that its limit is 0 only if $\mu_0 = 0$.

1988-A-2. A not uncommon calculus mistake is to believe that the product rule for derivatives says that $(fg)' = f'g'$. If $f(x) = e^{x^2}$, determine, with proof, whether there exists an open interval (a, b) and a nonzero function g defined on (a, b) such that this wrong product rule is true for x in (a, b).

1988-A-3. Determine, with proof, the set of real numbers x for which

$$\sum_{n=1}^\infty \left(\frac{1}{n} \csc \frac{1}{n} - 1 \right)^x$$

converges.

1988-A-5. Prove that there exists a *unique* function f from the set $\mathbf{R}^+$ of positive real numbers to $\mathbf{R}^+$ such that $f(f(x)) = 6x - f(x)$ and $f(x) > 0$ for all $x > 0$.

1988-B-4. Prove that, if $\sum_1^\infty a_n$ is a convergent series of positive real numbers, then so is $\sum_1^\infty (a_n)^{n/n+1}$.

1971-A-6. Let c be a real number such that n^c is an integer for every positive integer n. Show that c is a non-negative integer.