Meditations on the attainments of matriculating students

When university mathematicians are critical of the preparation of secondary students for university work, their comments are often interpreted as meaning that there should be more material in the syllabus. The issue is more subtle than this; I would like to report on some recent experiences that highlight what many of us are looking for in our first year students.

In December, 2006, I was at the University of Waterloo to join the marking team for the Canadian Open Mathematics Challenge. Questions B2 and B3 were both on geometry. I thought them excellent as they epitomized the sort of thing that every student graduating from Grade 12 mathematics should be capable of, if they are planning to use that mathematics in advanced study. Unfortunately, most of them did not do a good job on the questions, and many who did solve the problems were hardly fluent in managing even the most straightforward computations. The two questions were as follows:

B2. *The circle $x^2 + y^2 = 25$ intersects the $x-$axis at points $A(-5, 0)$ and $B(5, 0)$. The line $x = 11$ intersects the $x-$axis at point C. Point P moves along the line $x = 11$ above the $x-$axis and AP intersects the circle at Q.*

(a) Determine the coordinates of P when triangle AQB has maximum area. Justify your answer.

(b) Determine the coordinates of P when Q is the midpoint of AP. Justify your answer.

(c) Determine the coordinates of P when the area of triangle AQB is $\frac{1}{4}$ of the area of triangle APC. Justify your answer.

B3. *(a) The trapezoid $ABCD$ has parallel sides AB and DC of lengths 10 and 20, respectively. Also, the length of AD is 6 and the length of BC is 8. Determine the area of trapezoid $ABCD$.*

(b) $PQRS$ is a rectangle and T is the midpoint of RS. The inscribed circles of triangles PTS and RTQ each have radius 3. The inscribed circle of triangle QPT has radius 4. Determine the dimensions of rectangle $PQRS$.

Some of the solutions to problem **B2** (a) exemplified the adage that a little knowledge is a dangerous thing; the students saw the word "maximum" and looked for a derivative to take. Once students recognized that the area is maximized with the vertical height, and that the line AP would pass through $(0, 5)$, then it is simply a matter of identifying the intersection of the lines $y = x + 5$ and $x = 11$. Part (b) is a matter of recognizing that the abscissa of the midpoint of AP is 3; if they remember and can use the pythagorean triple $(3, 4, 5)$ to locate Q, then the answer is immediate. All that is needed is an internalization of proportionality and similar figures. A couple of students observed that BQ was the right bisector of AP, so that $|BP| = |AB| = 10$ and triangle BCP is a $6 - 8 - 10$ right triangle. Most of the successful solutions to (c) were painfully pedestrian. However, a couple noticed that triangles AQB and ACP were similar with factor $1/2$. This led quickly to $|AQ| = \frac{1}{2}|AC| = 8$. Since $|AB| = 10$, we find that $|BP| = 6$ and $|CP| = 2|BP| = 12$. There is no background here that cannot be reasonably expected on any secondary syllabus. The issues are less content and more mathematical fluency and structural insight.

Question **B3** is another question for which straightforward but tedious solutions exist, but become easy with the right perspective. The best solution given for (a) arose out of the realization that connecting A and B to the midpoint of CD partitioned the trapezoid into three congruent $6 - 8 - 10$ right triangles whose areas (by the base-height formula, if you look at the triangles sideways) are each 24. The best solution given for (b) relied on the insight that triangles TIU and VJP were similar with factor $3/4$, where I and J are the respective incentres of triangles PTS and PQT, ST is tangent to the smaller incircle at U and PQ is tangent to the larger incircle at V. If we take $|PV| = 4x$, then $|UT| = 3x$, whence $4x = 3 + 3x$ and $x = 3$. If $|PS| = 3 + 3y$, then an application of Pythagoras' theorem on triangle PTS yields the equation $(3x + 3y)^2 = (3x + 3)^2 + (3y + 3)^2$ which reduces to $xy = x + y + 1$ or $(x - 1)(y - 1) = 2$.

Let us look more closely at the ingredients in the solution for (b). There are three theoretical results needed. The first is to note that TP is a transveral of two parallels, so that the alternate angles STP and TPV are equal. The second is that the tangent rays from an external points to a circle are equal. The third is that the incentre of a triangle lies on the bisector of any of its angles. The last two can be convincingly established by appealing to the reflection of the configuration in the diameter of the circle passing through the external point. However, we need the insight that two crucial triangles are similar. Finally, for ease of dénouement, it helps to set the manipulations up in a way that delivers the result efficiently. While it might be unreasonable to expect every student to produce such a solution, exercises that they are regularly exposed to should be sufficiently rich that each student can on occasion produce solutions that indicate the assimilation of important mathematical values.

Another question that was interesting is this regard was Problem **B1** on the 2006 Putnam competition:

B1. *Show that the curve* $x^3 + 3xy + y^3 = 1$ *contains only one set of three distinct points* A, B, *and* C, *which are the vertices of an equilateral triangle, and find its area.*

Of course, we will not know until November, 2007 how the candidates fared with this question. However, while it was given to tertiary students, it is definitely not out of line for secondary students to be asked to attempt it. The expression on the left side of the equation should ring some bells, evoking the expansion of $(x + y)^3$. At this point, a rather sophisticated bit of algebraic insight comes in handy: if we make the restriction that $x + y = 1$, we can put the factor $(x + y)$ with $3xy$ to transform the equation into the form $(x + y)^3 = 1$. Thus, $x + y = 1$ is consistent with the equation, so that the line is part of the locus of the equation. This also means that $x + y - 1$ should be a factor of $x^3 + 3xy + y^3 - 1$. Indeed,

$$x^3 + 3xy + y^3 - 1 = (x + y - 1)(x^2 + y^2 - xy + x + y + 1) \ .$$

Setting the second factor equal to zero apparently gives the equation of a conic section, which, because of the symmetry in x and y should have an axis along the line $x = y$. It is quickly found that the points $(\frac{1}{2}, \frac{1}{2})$ and $(-1, -1)$ lie on the locus. This suggest that the expressions $(x + 1)$ and $(y + 1)$ and $(x - y)$ might possibly bear on the second factor. In fact,

$$2(x^2 + y^2 - xy + x + y + 1) = (x - y)^2 + (x + 1)^2 + (y + 1)^2 \ .$$

The locus of the original equation is thus a straight line and a single point off the line, and it is now straightforward to answer the question.

Is **B1** an appropriate question for a test or entrance examination? No. But it, and a diet of other questions of similar ilk, should certainly be given on homework assignments and used in group work. Such exercises promote the powers of recognition, analysis and exploitation of basic facts that constitute an indispensible part of student progress in even the most mundane of university mathematics courses.

Another example is a problem that was brought to the COMC marking from one of the teachers, who found it as a bonus question on a grade 11 test. She told me that the students had not studied logarithms, so that the solution could only use the laws of exponents. *Given that* $60^a = 3$ *and* $60^b = 5$, *determine*

$$12^{\frac{1-a-b}{2(1-b)}} \ .$$

You might want to try it before proceeding. The solution does not appear to be obvious, and it flummoxed me at first. However, a reasonably direct approach delivers the goods. From the second condition, one finds that $5 = 12^{b/(1-b)}$. Write the first equation as $2^2 \cdot 5^a \cdot 12^a = 12$ so that $2^2 = 12^{1-a} \cdot 5^{-a}$; substitute for 5 in terms of 12 and roll to a successful conclusion. While one might wonder about this as a test question, even for a bonus, it is a useful challenge to post on a bulletin board in the classroom. Its artificiality should not detract from the challenge of finding the simple answer that the proposer had in mind. Indeed, indeed, one might pause to pay homage to the creator of this question.

While it might be wished that this or that piece of mathematics were better known, most thoughtful observors see the problem of student preparation more with the use that students make of what is already on the curriculum and the analysis and connections that they might or might not be able to make. None of the foregoing problems discussed above involves material not already on nor easily incorporated into the existing syllabus. But a student who approaches them with a cookbook of procedures and formulae is unable to make much progress; some insight into the underlying structure along with a strategic approach and knitting together of knowledge from different sources is needed.

My comments should be seen as supplementing rather than countering the recommendations of Peter Taylor of Queen's University. In his address at the recent CMS Toronto meeting, where he received the Adrien Pouliot Award, he presented two nice examples of the sort of classroom investigations he promotes. In the first instance, students had to measure the fall in air pressure of an inflated tire after it had been suddenly punctured and find a function that modelled it. In the second, students experimented to find the relative frequency $f(x)$ of success in throwing a beanbag into a box located a distance x meters away, and then decide on the distance that would maximize a "reward function" $x^2 f(x)$.

The first of these leads students to an understanding of the situations the give rise to exponential decay and how such situations can be described and anlyzed mathematically. The second leads to an appreciation of the properties that a function $f(x)$ can be expected to have and how one balances a situation with contradictory aspects, here the interaction of a function x^2 that increases with x against one $f(x)$ that decreases. Unlike the contest problems, the intial focus is not on the mathematical description but on the intuitive grasp of a situation which can then be mathematized appropriately.

All discussions of student preparation lead back to the characteristics of their teachers

- how much mathematics they know, how they know it, the experiences they have as practising mathematicians, their confidence is handling whatever students may bring to an unprogrammed situation and their ability to interpret student responses and take up advantages and disadvantages of different approaches to a situation.