The plank problem

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Now that Ontario will have a Grade 12 course in calculus that will serve as a "fifth" and capstone course for the high school program, we need to take the injunction of Peter Taylor (of Queen's University) to heart that it be built on problems that will underline for the students the significance of the concepts and techniques. He has provided many examples of what would be of use, and I will not retail them here. Rather, I would like to point out that many of the traditional calculus problems can be pressed into service. Unfortunately, in many classrooms, these have been handled formulaically, and the students have learned to go through a chain of actions mechanically without understanding either the meaning or the significance of what they are involved in, or of appreciating the subtleties that can arise. I suspect that many teachers in the system see it as their task to provide their students with a "failsafe" set of instructions that will carry student through the standard fare that every calculus book seems bound to cover.

First, let me mention some of the desiderata in the Grade 12 course.

1. Syllabus. There has been tendency to overreact against the past tendency of simply listing topics. But the fact remains that university and college lecturers do expect that their students come knowing some mathematics; the ability of professors of both mathematics and science to cover the main ideas in their courses is sometimes severely compromised by the lack of student knowledge and facility. The syllabus should include basic results of polynomials, including the Factor Theorem, and standard trigonometric identities, including the sum-product conversion formulae. At the same time, topics on the syllabus cannot be covered in a vacuum, and should be accompanied by examples that provide context and exercises that provide challenge as well as practice. Part of the answer to the issue of crowding the syllabus is teaching students how to access for themselves some of the topics, and orchestrating the topics so that they are easy to understand and memorize. The example below involves quite a bit of basic knowledge in a setting where students can see the significance for themselves.

2. Algebraic facility. The facility of many students is poor and debilitating. But again the answer is not just to do things in isolation or to keep it at a low level. An algebraic expression is the carrier of information, both explicit and implicit, and algebraic manipulation is the process of coverting an expression to a form from which information can be more readily withdrawn. The ability to do this depends on a level of comfort, sufficient familiarity to foressee the results of a possible manupulation, and a sense of strategy that takes into account the nature of the information sought and the best way to access it. There is no substitute for practice, nor for a sequence of exercises of increasing complexity. This point of view of manipulation emphasizes the importance of context, and the necessity to provide tasks of sufficient interest and complexity that students can appreciate the challenge of coming to a conclusion. The current tendency in many quarters to test mechanical work only through multiple choice questions is precisely the wrong thing to do. The students are likely to see the tasks as meaningless and bothersome and they will not progress. The example below requires a great deal of algebraic footwork, but the student who is prepared to put in the mental effort to follow and master it will be rewarded with a sense of control over the mathematics.

3. Intuition, heuristics and formalism. If one takes a leisurely approach to calculus problems, then time can be alloted to "talking around" problems to see what might be induced about its solution, to decide on how firm a foundation one's conclusions might rest, and how mathematics can be brought in to solidify otherwise shaky conclusions.

4. Diligence. This is an overlooked characteristic, whose neglect is responsible for a lot of difficulty that students have in mathematics. Students need to realize how easy it can be to get the wrong slant on a piece of mathematics, or just to make a mechanical mistake. They need to develop a habit of mind that is attentive to detail and clarity and is continually checking their work. This can be both technical and conceptual. For example, technical calculations can be checked by such things as plugging in specific values, looking at the characteristics of expressions obtained (such as degree) and measuring it against expectations. Conceptual checking can involve trying to square what is coming out on the page against heuristic and intuitive considerations. Much of the ability to take this sort of care is borne of experience which in turn is the fruit of thoughtfully doing exercises that have been judiciously selected by the teacher.

5. Communication and group work. The example below gives a great deal of opportunity for communication. The problem can be embarked upon with the whole class and different possilble formulations arrived at. Subgroups can be responsible for tackling the problem in different ways, or, in the case of a more complicated solution, parts can be parcelled out to different students. The results can be critically compared as to their effectiveness and ease. Finally, students can be responsible for writing up the whole solution or in making oral presentations to their classmates.

I feel confident that students who can work in an aware and thoughtful way examples of this type will have nothing to fear from tertiary mathematics. In particular, the ability to handle inequalities (which involves algebraic skills and a clear mind) and to sketch graphs (which involves algebriac skill along with awareness of characteristics of functions and consideration of options) presage well for students coming to university mathematics. It should be pointed out that using technology to obtain graphs and sketching them by hand are two distinct processes, each with its own realm of effective use; handsketching can bring to light qualitative aspects that are either drowned in detail or easily missed in a computer version. In the example below, students should get in the habit of thinking about the graphs of all the functions introduced, both before and after they are constructed.

The following problem can be introduced cold, or it can be introduced through a practical problem. Suppose that we have a couch or a long table that we have to transport along a corridor and carry around a corner. How can we determine that maximum length that can be accommodated in the turn? The couch has width, and while the table can be turned on its side, it does have legs sticking out that have to be accommodated. So there might be a preliminary discussion as to how we might model this situation mathematically, and, in particular, what are the consequences of ignoring the width of the couch or the legs of the table. One can perhaps get around to the plank problem, enunciated below, solve it and then look at what its solution tells us about the practical situation.

A second issue is the role of technology. Undoubtedly, it is possible to put a diagram of the corridor on the screen and turn segments of various lengths around the corner. This way of solving the problem as opposed to a paper-and-pencil solution should be discussed. One issue that should be raised is the “black box” effect of the computer: does the paper solution allow us to gain more insight as to what is going on? does it give us more flexibility to explore other approaches? An important point to be made is that as it is a calculus course, the focus should be on the techniques and insights of caluclus. A second point is that many of the students we are talking to may in due course be the ones that are responsible for designing the computer algorithms which so many find useful, and therefore they will need to be able to look more structurally and intimately at the underlying mathematics.

The Plank Problem: *Two corridors of width a and b meet at right angles at the corner of a building. A plank (assumed to be rigid and of zero thickness) is to be carried horizontally from one corridor to the other. What is the maximum length of the plank that can be accommodated.*

Comment. In many books, the corridors are given actual numerical widths, such as 8 and 27 feet. I have chosen to use the parameters a and b for two reasons. The first is that many students do not understand what parameters are, nor appreciate their role. Secondly, the use of parameters allows us to focus on the structure of the solution.

Step 1. Analyzing the situation and arriving at a mathematical formulation.

One can imagine many ways in which the plank is carried, even with the restriction that it be horizontal. As the plank becomes longer, it seems evident that it has to be carried ever more closely to the inner corner where the corridors intersect. So if we want to maximize the length of the plank, there is no loss of generality in assuming that as the plank turns from one corridor into another, it maintains contact with this inner corner. As it turns, the plank lies within a succession of line segments that pass through the inner corner and terminate on the opposite walls of the two corridors. The shortest of these segments measures the length of the longest plank that can be accommodated. Of course, we are supposing the plank to be rigid.

So we can reformulate the problem thus: *Two corridors of a building, of widths a and b intersect at right angles at a corner of a building. A line segment from the outer wall of one corridor passes to the outer wall of the second corridor and passes through the inner corner point where the corridors meet. What is the minimum length of such a line segment?*

Step 2: Casing the situation.

We can imagine the line segments turning about the inner corner. When it is flush with one of the inner walls, it is infinite in length. As it turns, the length decreases and then eventually increases until the segment is flush with the other inner wall. It stands to sense that somewhere, its length should be minimized? Where might this be? Is it possible that this minimum occurs when segments of equal length are cut off by the two corridors? Or, if one corridor is wider than the other, will the wider corridor cut off a larger segment than the narrower corridor? What does your intuition tell you?

As a check on the final answer, let us consider some special cases we might check it against. What happens if either a or b vanishes? When $a = 0$, we can begin to turn the plank only after it has cleared the corridor, so that its length cannot exceed b. When $a = b$, we can surmise by symmetry that the plank is most constrained when it makes an angle of 45° with each corridor; accordingly, we would not allow its length to exceed $2\sqrt{2}a$.

Comment. By making this dynamic visualization of the situation, one gets an idea of the dimensionality of the variation and how one might take one of the admissible positions of the segment. Also, by thinking intuitively about the problem, one begins to have some expectations as to what the solution might be. This will serve as a check on subsequent work.

Step 3: Selecting the variables and defining the function.

There seem to be essentially two ways to describe the situation. We observe that the position of the segment can be determined by a single variable, either the distance of the end of the segment along one of the corridors from the inner corner or the angle that the segment makes with one of the corridors. Corresponding to this *independent variable*, one determines the length of the segment; this will be the *dependent variable*, and will be the value of some function of the independent variable. Once the functions are defined, their domains should be carefully delineated and one should check that they are in accord with ones intuition about the variation.

Let x be the distance along the corridor with width a of the end of the segment from the inner corner. Note that $0 < x$. Let $f(x)$ be the length of the corresponding segment. Then

$$f(x) = b\left(1 + \frac{a^2}{x^2}\right)^{1/2} + (a^2 + x^2)^{1/2} = \left(\frac{b}{x} + 1\right)(a^2 + x^2)^{1/2}$$

for $0 < x$. There might be some discussion as to the best form of the function; what do we plan to do with it? Is it better to use the exponent rather than the square root sign to express the radical; why? The function will be differentiated, so we would like to make the process as painless as possible.

The practical situation demands that $f(x)$ should tend to infinity as x tends to either limit of its domain. Does this happen? This is clear when x tends to zero, and almost clear when x becomes large; just note that $(a^2 + x^2)^{1/2} > x$.

One might avoid the square root by noting that the function and its square will be minimized at the same time. So rather than deal with $f(x)$, consider rather

$$g(x) = (f(x))^2 = \left(\frac{b}{x} + 1\right)^2 (a^2 + x^2) = \frac{a^2b^2}{x^2} + \frac{2a^2b}{x} + (a^2 + b^2) + 2bx + x^2 \ .$$

There are many ways in which we might express $g(x)$ and the possibilities might be discussed with the students. I have decided simply to expand it out; is this the most convenient way to handle the differentiation?

The alternative approach takes as the independent variable θ, the angle that the segment makes with the inner wall of the corridor with width a. Then the length of the segment is given by

$$h(\theta) = a \csc\theta + b \sec\theta$$

for $0 < \theta < \pi/2$. One checks that as θ tends to either endpoint of its domain, the length of the segment tends to infinity. On the face of it, this choice of independent variable promises to be more tractable that the choice of x (provided, of course, the students are comfortable with trigonometry).

Step 4. The mathematical analysis

(a) Consider, first the function $f(x)$. We have that

$$f'(x) = -\frac{b}{x^2}(a^2 + x^2)^{1/2} + \left(\frac{b}{x} + 1\right)x(a^2 + x^2)^{-1/2} = \frac{(a^2 + x^2)^{-1/2}}{x^2}[x^3 - a^2 b] \ .$$

We need to check $f(x)$ for extreme values. Rather than just mechanically solving the equation $f(x) = 0$, let us consider why this is valid and also how we might check the nature of the extremum.

First the validity. Everywhere on its domain of definition, the function does have a derivative. We have also noted that it becomes ever larger as x tends to either 0 or infinity. Because of the continuity, it will assume a minimum value somewhere in the domain of definition. (In first year university, this will be shored up by the extreme value theorem for continuous functions on closed intervals; in Grade 12, this can be taken as a matter of intuition.) Any extremum inside the domain of definition will ential the vanishing of the derivative; the derivative vanishes exactly once (at $x = a^{2/3}b^{1/3}$). *Ergo*, where the derivative vanishes, a minimum must occur.

The context of the classroom might make the above approach risky to pursue (and if the teacher does decide to take this route, she has to proceed carefully and judiciously). If we want a more analytic test of the extremum, there are essentially three methods:

(i) Note that $f'(a^{2/3}b^{1/3}) = 0$ and check that $f(x) - f(a^{2/3}b^{1/3}$ is always nonnegative. Here, this approach looks like a nightmare, and an alternative should be investigated.

(ii) Check the sign of the first derivative on its domain. This is practically trivial: $f'(x) < 0$ for $x < a^{2/3}b^{1/3}$ and $f'(x) > 0$ for $x > a^{2/3}b^{1/3}$. So one readily deduces that f decreases to its extreme value and increases thereafter; thus we have a minimum.

(iii) Apply the second derivative test. Differenting $f'(x)$ and determining its sign looks as though it might be quite formidable. But there is another consideration. The second derivative test gives *local* information; it just tells us how the function behaves in some neighbourhood of the extreme value. The sign of the second derivative at the extremum, in and of itself, does not give information about global extreme values. Further analysis is required. This is an important point that can be made to Grade 12 students, one that I think can be perfectly well understood. It would go a long way to heading off a mechanical and sloppy approach to optimization in their later mathematical careers.

(b) Consider the function $g(x)$. We find that

$$\begin{aligned} g'(x) &= -2a^2b^2x^{-3} - 2a^2bx^{-2} + 2b + 2x = 2b[-a^2bx^{-3} + 1] + 2[-a^2bx^{-2} + x] \\ &= 2bx^{-3}[x^3 - a^2b] + 2x^{-2}[x^3 - a^2b] = 2x^{-3}(x^3 - a^2b)(x + b) \ . \end{aligned}$$

I have recast the function in a form that will not be obvious to most students. But dealing with such functions and getting them into a form from which information can be extracted is precisely the sort of challenge that Grade 12 students should be faced with. It would help for students to work together on this task, compare notes and decide which is the best form to work with. They may well improve on what is above. However, the form given above allows us to read off exactly where the derivative vanishes and what its sign is elsewhere. So we come to a conclusion consistent with (a).

(c) Consider the function $h(\theta)$. We find that

$$h'(\theta) = -a \csc\theta \cot\theta + b \sec\theta \tan\theta = \csc\theta \cot\theta(-a + b\tan^3\theta) \ .$$

By checking out the sign of the first derivative, we can conclude that the function $h(\theta)$ assumes its minimum value when $\tan\theta = a^{1/3}b^{-1/3}$.

This will be challenging for two reasons, the first being the use of uncommon trigonometric functions. One could cast h in terms of the more standard sines and cosines, at the price of yielding a more complicated expression. Or one could roll up one's sleeves and decide to get used to working with the less familiar functions. It is not a big deal to determine their derivatives from the laws of differentiation and the derivatives of sine and cosine.

However, teachers will complain that differentiating of trigonometric functions is not on the syllabus. It is at this point that a strategic decision has to be made as to whether you are teaching calculus or just covering the syllabus; i.e. is there a pedagogical benefit from students' familiarity with trigonometric functions and their understanding of the concepts of calculus to make the digression worth while?

If one reduces the problem of differentiation to that of differentiating the sine function (or, if you like, the cosine function), we find that the cancellation "trick" used for polynomials and rational functions is not available to us here, and we have to focus more carefully on what the limit of the differential quotient means and how it is calculated. We have to consider the expression

$$\frac{\sin(x+h)-\sin x}{h} \ .$$

The first impulse might be to expand out $\sin(x+h)$, but this increases the mess, rather than alleviates it. If the students have seen the sum to product conversion formulae, then the use of this can be tried, and we are reduced to finding out the limit as u tends to 0 for $\sin u/u$. This can be managed with a Grade 12 class.

Step 4. Answering the question.

We were asked for the maximum length of the plank, or, equivalently, the minimum length of the segment, so we should conclude the solution with a statement to that effect. If the students have been working in groups or have been assigned to look at different approaches, they should make sure that everybody has the same answer: the maximum length of the plank is $(a^{2/3}+b^{2/3})^{3/2}$.

Step 5. Mopping up

Now it is time to do a reality check on the situation. Suppose the corridors were of equal width; what would we expect the answer to be? Does our solution bear this out? We wondered whether the maximum length of the plan occurred when the inner corner bisected the minimum segment. So let us draw a picture and see what the situation is. When this occurs, the length of the segment turns out to be $2(a^2+b^2)^{1/2}$. Is this equal to $(a^{2/3}+b^{2/3})^{3/2}$? It certainly should be larger. Let us check that out; to avoid troublesome computations, rather compare the squares:

$$\begin{aligned}4(a^2+b^2)-(a^{2/3}+b^{2/3})^3 &= 3(a^2-a^{4/3}b^{2/3}-a^{2/3}b^{4/3}+b^2)\\ &3(a^{4/3}-b^{4/3})(a^{2/3}-b^{2/3}) = 3(a^{2/3}+b^{2/3})(a^{2/3}-b^{2/3})^2 \geq 0 \ ,\end{aligned}$$

with equality if and only if $a=b$. Notice that the presence of the fractional exponents might make the computations hard to deal with; perhaps a change of variable might make things more transparent - why not make $a=r^3$ and $b=s^3$?

Another conjecture that might be entertained is that the segment is minimized when it makes the same angle with both corridors, *i.e.*, 45°. This possibility is easily checked with the function $h(\theta)$. How would we check whether this is so when the functions $f(x)$ and $g(x)$ are used?

In conclusion, while on the face of it, while much of the criticism of school preparation for students matriculating to university might seem to bear on knowledge and technological facility, this is only part of the story. The real issue is whether, given their knowledge and facility, the students behave *in a mathematical way*. If this is to happen, we must not just focus on a syllabus of topics, but introduce into the classroom tasks that impose imperatives on the students to learn and master certain things and that provide contexts that inform the usefulness of these.