Department of Education, Ontario

Annual Examinations, 1961

GRADE XIII

PROBLEMS

(To be taken only by candidates writing for certain University Scholarships involving Mathematics)

Note 1: Ten questions constitute a full paper.

Note 2: A supply of squared paper and a book of mathematical tables my be obtained from the Presiding Officer.

1. (a) What condition must be satisfied by the constant a, b, c for the quadratic function

$$f(x, y) = x^2 - y^2 + 2ax + 2by + c$$

to be the product of two linear functions?

(b) Indicate the nature of the graph of $f(x, y) = 0$ in the general case, and also when the condition referred to in part (a) is satisfied.

2. To what extent are x, y, z which satisfy the equations

$$x + y + z = 2$$

$$xy + yz + zx = 1$$

determined by the condition that they be real?

3. If s_n denotes the sum of the first n natural numbers, find the sum of the infinte series

$$\frac{s_1}{1} + \frac{s_2}{2} + \frac{s_3}{4} + \frac{s_4}{8} + \cdots \ .$$

4. Prove that

$$\frac{m}{(m+1)(2m+1)} < \sum_{r=m+1}^{2m} \frac{1}{r^2} < \frac{1}{2m} \ ,$$

for all integers n, where

$$\sum_{r=m+1}^{2m} \frac{1}{r^2} = \frac{1}{(m+1)^2} + \frac{1}{(m+2)^2} + \cdots + \frac{1}{(2m)^2} \ .$$

5. (a) In the triangle with vertices $(a, 0)$, $(-a, 0)$, $(0, b)$, prove that (i) the medians concur in a point G (the median), (ii) the right-bisectors of the sides concur in a point H (the circumcircle), (iii) the altitudes concur in a point K (the orthocentre).

   (b) Find the ratio $HG : GK$.

   (c) Draw a figure to show the poosition of these three points when $a = b$.

6. If the circle $x^2 + y^2 = r^2$ meets the $x-$axis in A_1 and A_2 and any chord parallel to the $y-$axis in P_1 and P_2, prove that the locus of the point of intersection of A_1P_1 and A_2P_2 is the rectangular hyperbola $x^2 - y^2 = r^2$.

7. A chord PQ of a parabola cuts the axis in R; tangents at P and Q intersect in T. Prove that RT is bisected by the tangent at the vertex.

8. Tangents are drawn to the circle $x^2 + y^2 = 1$ from any point (h, k) on the fixed line $x = h$, where $h > 1$, and also from the point $(-h, -k)$. Prove that the other two vertices of the parallelogram formed by these tangents lie on the ellipse

$$x^2 + y^2\left(1 - \frac{1}{h^2}\right) = 1$$

   for all values of k.

9. The inscribed circle of the triangle ABC touches the sides BC, CA, AB in the points P, Q, R, respectively. Show that

$$\cos\frac{A}{2} : \cos\frac{B}{2} : \cos\frac{C}{2} = QR : RP : PQ\ .$$

10. Given that one side of a triangle is twice as long as another side, show that the angle opposite the first-named side is more than twice the angle opposite the other side.

11. Two equal rectangles, both inscribed in the circle $x^2 + y^2 = 1$, with their axes of geometry along the $x-$axis and $y-$axis, cross each other, forming a square $ABCD$ which is common to both rectangles.

    (a) If θ is the acute angle between the diagonal of one rectangle and its major axis of symmetry , find, in terms of θ, the area of the four rectilinear figures exterior to $ABCD$.

    (b) Find the value of $\tan\theta$ when this total area is a maximum.

12. A cylinder of weight w pounds rests with its axis horizontal in a trough made of two planes which are perpendicular to each other and equally inclined to the vertical. A horizontal force of F pounds, applied at the top of this cylinder and perpendicular to its axis, is just on the point of turning it. Assuming that the cylinder in turning remains in contact with both faces of the trough, and that the coefficient of friction between the cylinder and each of the two planes is μ, show that

$$F = \frac{w\mu\sqrt{2}}{\mu^2(1+\sqrt{2})+1}\ .$$