Department of Education, Ontario

Annual Examinations, 1959

GRADE XIII

PROBLEMS

(To be taken only by candidates writing for certain University Scholarships involving Mathematics)

Ten questions constitute a full paper.

1. Prove the following identity:

$$\binom{a}{0}\cdot\binom{b}{n}+\binom{a}{1}\cdot\binom{b}{n-1}+\cdots+\binom{a}{n-1}\cdot\binom{b}{1}+\binom{a}{n}\cdot\binom{b}{0}=\binom{a+b}{n}.$$

2. Prove that $\sqrt[3]{5}$ is not a rational number.

3. The following advertisement appeared in a newspaper:

Our car	\$2295.00
Your car	\$1295.00
Balance	\$1000.00
4% finance charge for 18 months	\$67.50
	\$1067.50
18 equal instalments of	\$59.31

Explain why the interest rate of 4% is inaccurate, and find the rate of interest a purchaser would really be paying. (Assume interest is compounded monthly.)

4. Let $A_0(0,0)$, $A_1(1/n,0)$, $A_2(2/n,0)$, $\cdots$, $A_{n-1}((n-1)/n,0)$, $A_n(1,0)$ be $n+1$ points on the $x-$axis. Ordinates are drawn to these $n+1$ points to meet the parabola $y=1+x^2$ at the points B_0, B_1, B_2, $\cdots$, B_{n-1}, B_n, respectively. Horizontal lines B_0C_1, B_1C_2, B_2C_3, $\cdots$, $B_{n-1}C_n$ are then drawn so that C_1 lies on A_1B_1, C_2 lies on A_2B_2, *etc.*.

(a) Write an expression for the sum of the areas of all the rectangles $A_0B_0C_1A_1$, $A_1B_1C_2A_2$, $\cdots$, $A_{n-1}B_{n-1}C_nA_n$.

(b) Show that this expression can be written as

$$f(n)=\frac{8n^2-3n+1}{6n^2}.$$

(c) Find what number is approximated by $f(n)$ as n becomes larger and larger, and explain what this number represents.

5. (a) If the line $y=mx+c$ is tangent to the curve $b^2x^2-a^2y^2=a^2b^2$, show that $c=\pm\sqrt{a^2m^2-b^2}$.

(b) Chords of the circle $x^2 + y^2 = r^2$ touch the hyperbola $b^2x^2 - a^2y^2 = a^2b^2$. Find the equation of the locus of their midpoints.

6. Show that the portion of any tangent included between the asymptotes of a hyperbola is bisected at the point of contact, and forms with the asymptotes a triangle of contant area.

7. Prove that all lines which make intercepts on the x and y axes, the sum of whose reciprocals is a constant k, pass through a fixed point. Determine the coordinates of this point.

8. The transformation known as *inversion* with respect to the circle $C: x^2 + y^2 = r^2$ associates with a given point $P(x, y)$ a point $P_1(x_1, y_1)$ where

$$x_1 = \frac{r^2 x}{x^2 + y^2}\,, \qquad y_1 = \frac{r^2 y}{x^2 + y^2}\,.$$

(i) Prove that O, P, P_1 are collinear and expressed x, y in terms of x_1, y_1.

(ii) Show tht every point of C remains fixed under the transformation.

(iii) What is the inverse of an arbitrary circle throught the origin? What is the inverse of an arbitrary circle in the plane?

(iv) Describe carefully the inverse of a family of lines through $(0, r)$.

9. Consider the function $\sin(\pi \cos x)$.

(a) Is the function periodic? If so, what is its period?

(b) Give all the values of x for which the function attains its maximum and all values for which it attains its minimum.

(c) Draw the graph of the function for $-\pi \leq x \leq \pi$.

(d) Describe the symmetry of the completel graph.

10. Show that $2\cot\theta - 2\cot 2\theta - \csc\theta$ is positive for all angles θ for which $0 < 2\theta < \pi$.

11. Explain how a boat is able to sail in a direction nearly opposite to that of the wind. If the sail be considered as a rigid plane, show that it should be set so as to bisect the angle between the keel and the apparent direction of the wind in order that the force to urge the boat forward may be a maximum.

12. A uniform heavy bar 20 feet long, with its lower end on a floor, rests on a railing with 5 feet of its length projecting over the railing. The inclination of the bar to the horizontal is $45°$. Given that the contact between the bar and the railing is without friction, find the least value of the coefficient of friction between the bar and the floor, in order that the bar should not slide.