Department of Education, Ontario

Annual Examinations, 1957

GRADE XIII

PROBLEMS

(To be taken only by candidates writing for certain University Scholarships involving Mathematics)

Ten questions constitute a full paper.

1. The sum to infinity of a geometric progression is 128; the sum of the fourth, fifth and sixth terms is -18. Find the first four terms of the series.

2. Develop a formula for the sum S_n of n terms of the series
$$\frac{1}{1\cdot 4}+\frac{1}{4\cdot 7}+\frac{1}{7\cdot 10}+\frac{1}{10\cdot 13}+\cdots .$$
Show that, for n very large, the values of S_n approach a limit, and find this.

3. A sequence of positive numbers, x_1, x_2, x_3, $\cdots$, is defined by the relations $x_1 = 1$ and $x_{n+1} = \sqrt{1 + x_n}$ for $n \geq 1$. Show that (a) $x_{n+1} > x_n$, (b) $x_n < 2$, and (c) for n very large, the values of x_n approach the limit $\frac{1}{2}(1+\sqrt{5})$.

4. Let a, b, and c be the roots of the equation $x^3 - 2x^2 + x + 5 = 0$. Compute the value of $a^4 + b^4 + c^4$.

5. (a) Perpendiculars are drawn from the point $P(5, 0)$ to the sides of the triangle whose vertices are at the points $(4, 3)$, $(-4, 3)$ and $(0, -5)$. Show that the feet of these three perpendiculars lie in a straight line.

   (b) Give three other positions of P for which this property holds.

6. (a) A normal to the parabola $y^2 = 4ax$ at $P(x_1, y_1)$ on the parabola has a slope m. Show that the equation of the normal is $y = mx - 2am - am^3$.

   (b) The normal in (a) intersects the parabola again at the point Q. If PQ subtends a right angle at the vertex, find the value of m.

7. A circle is described with a focus of the hyperbola $9x^2-16y^2 = 144$ as centre and radius equal to 1/4 of the latus rectum. Show that the lines joining the points of intersection of the circle and the hyperbola to the focus are parallel to the asymptotes.

8. A normal to the hyperbola $x^2/a^2 - y^2/b^2 = 1$ meets the axes at M and N and the lines MP and NP are drawn are drawn perpendicular to the axes. Find the equation of the locus of P.

9. If θ and ϕ are two angles such that $\theta + \phi = \alpha$ (α a constant), prove that the numerical value of $a\cos\theta + b\cos\phi$ (a and b constants), never exceeds $\sqrt{a^2 + 2ab\cos\alpha + b^2}$.

10. The bisectors of thye interior angles of a triangle ABC make angles α, β, γ, with the sides a, b, c, respectively. Prove that

$$a\sin 2\alpha + b\sin 2\beta + c\sin 2\gamma = 0\ .$$

11. The cross-section of a right prism is a triangle ABC with an obtuse angle at C. The prism is placed on a smooth horizontal table, with the face containing AC in contact with the table top. Prove that this position of the prism will be unstable if

$$0 = \cot^{-1}\left(-\frac{1}{2}\cot A\right)\ .$$

12. A uniform solid hemisphere rests with its curved surface touching a rough horizontal floor and a rough horizontal wall, in such a mannner that its plane makes an angle θ with the horizontal. If the solid is on on the point of slipping, show that

$$\sin\theta = \frac{8\mu(1+\mu)}{3(1+\mu^2)}$$

where μ is the coefficient of friction for both points of contact. (Assume that the centre of gravity of the hemisphere is located on the rdius perpendicular to the bases at a distance from the base equal to three-eighths of the radius.)