Department of Education, Ontario

Annual Examinations, 1954

GRADE XIII

PROBLEMS

(To be taken only by candidates writing for certain University Scholarships involving Mathematics)

Ten questions constitute a full paper.

1. If P and Q are two points on an ellipse whose centre is C, show that the area of the triangle CPQ is greatest when P and Q are the extremeties of a pair of conjugate diameters.

2. The following lines form the sides of a quadrilateral: $L_1 \equiv x - 2y = 0$; $L_2 \equiv 2x - 3y + 4 = 0$; $L_3 \equiv 2x - y - 4 = 0$; $L_4 \equiv x + y = 0$.

   (a) Prove that $L_1L_2 + kL_3L_4 = 0$ is a family of curves passing through the vertices of the quadrilateral.

   (b) Find the member of this family in whose equation the coefficient of xy is 0.

3. A straight line cuts a hyperbola at the points P and P' and its asymptotes at the points Q and Q'. Prove that the midpoint of PP' is also the midpoint of QQ'.

4. (a) The base BC of a triangle ABC is fixed and the angle B is double the angle C. Find the locus of A.

   (b) On the base of the triangle of (a) a segment of a circle containing the angle $180° - A$ is drawn. Show that the angle A can be trisected by the use of this segment and the locus referred to in (a).

5. It is given that $\sin(y + z - x)$, $\sin(z + x - y)$ and $\sin(x + y - z)$ are in arithmetic progression. Show that, when $\tan x$, $\tan y$, and $\tan z$ exist, they are in arithmetic progression.

6. Given an acute angle A, find the value of θ in the range $[0, \pi]$ for which $\sin\theta\cos(A - \theta)$ is greatest.

7. A semi-circle is described on the diameter AB of length $2a$, and from the centre O a

radius OC is drawn, making an angle 2θ with OA. Circles are inscribed in the triangles OBC and OAC. Show that the distance between the centres of the inscribed circles is

$$\sqrt{\frac{2 - \sin 2\theta}{(1 + \sin\theta)(1 + \cos\theta)}} .$$

8. A uniform plank of length $2a$ and weight W is balanced on a fixed circular cylinder whose axis is horizontal and perpendicular to the length of the plank. A weight W' is attached to one end of the plank, which now seeks a new position of equilibrium. Show that the plank will not slide off the cylinder, provided W' is less than $Wb\theta/(a - b\theta)$, where b is the radius of the cylinder and $\tan\theta = \mu$ is the coefficient of friction between the plank and the cylinder.

9. Show that the sum of n terms of the geometric progression $a + ar + ar^2 + \cdots$ and the sum of n terms of the geometric progression $a - ar + ar^2 - \cdots$ have as product the sum of n terms of the geometric progression $a^2 + a^2r^2 + a^2r^4 + \cdots$, provided n is odd.

10. Given that x is positive but different from 1, and also that n is a positive integer, show that

$$\frac{x^n - 1}{n} < \frac{x^{n+1} - 1}{n + 1} .$$

11. (a) Given that $x + a + \sqrt{a^2 - b} = 0$, where x is not 0, verify that

$$x + \frac{b}{x} + 2a = 0 .$$

(b) Given that $y = px + q$, where $x + a + \sqrt{a^2 - b} = 0$, verify that

$$y + (ap - q) + \sqrt{(ap - q)^2 - (bp^2 - 2apq + q^2)} = 0 .$$

12. (a) Given that

$$T_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1 + \sqrt{5}}{2}\right)^n - \left(\frac{1 - \sqrt{5}}{2}\right)^n\right]$$

for $n = 1, 2, 3, \cdots$, show that $T_n + T_{n+1} = T_{n+2}$.

(b) Verify that $T_1 = 1$, $T_2 = 1$, $T_3 = 2$, and deduce that $T_4 = 3$ and $T_5 = 5$.