Department of Education, Ontario

Annual Examinations, 1951

GRADE XIII

PROBLEMS

(To be taken only by candidates writing for certain University Scholarships involving Mathematics)

Ten questions constitute a full paper.

1. The side of a square and the radius of a circle vary in such a way that the sum of the perimeters of the square and the circle is a constant k. Find the radius of the circle when the sum of the areas of the square and the circle is a minimum.

2. Let S_n be the sum of the first n terms of a geometric progression. Express S_{4n} as a fractional rational function of S_{2n} and S_n.

3. For an ellipse $b^2x^2 + a^2y^2 = a^2b^2$, the length of the subtangent corresponding to the point $(3, 12/5)$ is $16/3$. Find the eccentricity of the ellipse.

4. Find the equation of the locus of the middle points of the chords of the ellipse $b^2x^2 + a^2y^2 = a^2b^2$ which are drawn from the positive end of the minor axis.

5. From the top of a hill the angle of depression of a point P on the level plain below is 30 degrees, and from a point three-quarters of the way down the hill the angle of the depression of P is 15 degrees. Assuming the data to be exact, find, to the nearest minute, the inclination of the hill.

6. A body of weight W rests on a rough plane which is inclined at a constant angle to the horizontal. Two separate experiments show the following results:

   (i) the least horizontal force that will cause the body to move up the plane is P pounds;

   (ii) the least force acting up the plane that will cause the body to move up the plane is Q pounds.

   If θ is the angle of friction, prove that

$$\cos\theta = \frac{PW}{Q\sqrt{P^2 + W^2}} \ .$$

7. Show that the equation

$$\frac{1}{x+2}+\frac{1}{y+2}=\frac{1}{2}+\frac{1}{z+2}$$

is not satisfied by any set of positive integers x, y, z in which x is equal or greater than four. Hence find all the sets of positive integers x, y, z which satisfy the given equation.

8. Show that

$$\left(1+\frac{1}{3}\right)\left(1+\frac{1}{3^2}\right)\left(1+\frac{1}{3^4}\right)\left(1+\frac{1}{3^8}\right)\left(1+\frac{1}{3^{16}}\right)\left(1+\frac{1}{3^{32}}\right)$$

differs from 1.6 by less than 10^{-30}.

9. Given that the equation $ax^2+2hxy+by^2+2gx+2fy+c=0$ represents two parallel straight lines, show that $h^2=ab$, $bg^2=af^2$.

10. A hyperbola is met by a diameter in a point A and the conjugate hyperbola is met by a conjugate diameter in a point B. Show that the line AB is parallel to one asymptote and is bisected by the other.

11. If A, B, and C are three acute angles such that $\cos A = \tan B$, $\cos B = \tan C$, $\cos C = \tan A$, prove that

$$A = B = C = \sin^{-1}\left(2\sin\frac{\pi}{10}\right) .$$

12. Find all the values of x in the range $0^\circ \le x \le 180^\circ$ which satisfy the equation

$$2\cos x - \sqrt{3}\sin x = \sin 2x \ .$$