Department of Education, Ontario

Annual Examinations, 1947

GRADE XIII

PROBLEMS

(To be taken only by candidates writing for certain University Scholarships involving Mathematics)

Ten questions constitute a full paper.

1. Solve the system of equations

$$x^2 - (y - z)^2 = a^2 \ ,$$

$$y^2 - (z - x)^2 = b^2 \ ,$$

$$z^2 - (x - y)^2 = c^2 \ ,$$

in which a, b, c are constants different from 0.

2. A candidate at an examination writes four papers. If the maximum number of marks obtainable on each paper is m, show that the number of ways of obtaining a total of $2m$ marks is

$$\frac{1}{3}(m+1)(2m^2+4m+3) \ .$$

3. If a polynomial is divided by $(x-a)(x-b)(x-c)$, where a, b, c are all different, prove that the remainder is given by

$$\frac{(x-b)(x-c)}{(a-b)(a-c)}f(a) + \frac{(x-c)(x-a)}{(b-c)(b-a)}f(b) + \frac{(x-a)(x-b}{(c-a)(c-b)}f(c) \ .$$

4. Prove that the integer next above $\sqrt{3}+1)^{2n}$ is divisible by 2^{n+1}. n being a positive integer.

5. Find the coordinates of the points on the parabola $y^2 = 4x$, the normals at which pass through the point

$$\left(\frac{15}{4}, -\frac{3}{4}\right) \ .$$

6. F, F_1 are the foci of an ellipse and P is any point on the ellipse. Prove that the circles described on FP, F_1P as diameters touch the circle which has the major axis of the ellipse as diameter.

7. Two variable points Q and R move on the lines $y = x$, $y = -x$, respectively, in such a way that QR always passes through the point $(2, 0)$. Find the equation of the locus of the middle point of the segment QR.

8. Two circles with different centres are drawn in a plane.

   (a) If the two circles do not intersect, show that a system of rectangular axes and a unit of length may be chosen so that the circles shall be members of the family

   $$x^2 + y^2 + kx + 1 = 0 \ .$$

   (b) If the two circles intersect in two points, show that it can be arranged that they are members of the family

   $$x^2 + y^2 + ky - 1 = 0 \ .$$

9. Express $\cos(6 \tan^{-1} x)$ as a rational function of x.

10. The bisectors of the exterior angles of a triange ABC are drawn, forming the sides of a triangle DEF. If S is the area of the triangle ABC and S' is the area of the triangle DEF, show that

    $$S' = \frac{1}{2} S \csc \frac{A}{2} \csc \frac{B}{2} \csc \frac{C}{2} \ .$$

11. Given that $\tan A$ and $\tan B$ are the roots of the equation $x^2 + px + q = 0$, find the value of

    $$\sin^2(A + B) + p \sin(A + B) \cos(A + B) + q \cos^2(A + B) \ .$$

12. The upper edge of a smooth hemispherical bowl of radius 14 inches is horizontal. A thin uniform rod of length 34 inches rests in equilibrium partly within and partly without the bowl. Find the cosine of the angle which the rod makes with the horizontal plane.