Department of Education, Ontario

Annual Examinations, 1945

GRADE XIII

PROBLEMS

(To be taken only by candidates writing for certain University Scholarships involving Mathematics)

Ten questions constitute a full paper.

1. Find a polynomial $f(x)$ of degree five such that $f(x) - 1$ is divisible by $(x-1)^3$ and $f(x)$ itself is divisible by x^3.

2. Find all four-digit numbers with the following properties: if the second and the fourth digit of the number are interchanged (the first and third remaining unchanged) the number is diminished by 792. The sum of the four digits is 17. The sum of the squares of the four digits is 111.

3. Find the coefficient of x^6yzt^5 in the expression
$$(x+y+z+t)^{10}(x+z)(x^2+y^2) \ .$$

4. Show that every polynomial $f(x)$ of degree n can be written in the form
$$\begin{aligned} f(x) = c_0 + c_1\frac{x}{1} + c_2\frac{x(x-1)}{1\cdot 2} + c_3\frac{x(x-1)(x-2)}{1\cdot 2\cdot 3} \\ + \cdots + c_n\frac{x(x-1)(x-2)\cdots(x-n+1)}{1\cdot 2\cdot 3\cdots n} \ , \end{aligned}$$
where c_0, c_1, c_2, $\cdots$, c_n are constants. Show further that if $f(x)$ has integral values for integral values of x, then c_0, c_1, c_2, $\cdots$, c_n are integers.

5. If, in the triangle ABC, $\tan A \tan(B/2) = 2$ and AB is fixed, show that the locus of C is a parabola with its vertex at A and its focus at B.

6. Two perpendicular chords of a parabola intersect at the vertex. Tangents are drawn to the parabola at the other two points where it is met by these chords. Prove that, when the directions of the chords change, the locus of the intersection of the pair of tangents is a straight line.

7. Find the conditions to be satisfied by the coefficients a, b, c, d in order that the three lines represented by $ax^3 + bx^2y + cxy^2 + dy^3 = 0$ should intersect a circle with centre at the origin in points which form the vertices of a regular hexagon.

8. The base of a triangle is fixed and the opposite vertex C moves so that one base angle is twice the other. Find the locus of C and name it.

9. In the ambiguous case in which two triangles are determined by the given parts a, b, A, show that

$$rr' = b(b-a)\sin^2\frac{A}{2}$$

where r and r' are the radii of the inscribed circles of the two triangles.

10. At a point P, a north-south road changes from the horizontal to an incline making an angle θ with the horizontal. A man observes that when he had walked a feet from P up the inclined road the angle of depression of an object on the horizontal road is α, and that when he has walked an additional distance b feet up the inclined road the angle of depression of the same object is β. Show that the angle θ is given by the formula

$$\cot\theta = \frac{(a+b)\cot\beta - a\cot\alpha}{b} \ .$$

11. The equation $\tan 3x = k$, where k is any constant, has three different roots which lie between $-\pi/2$ and $\pi/2$. If α, β, γ are these roots, show that

$$\sin\alpha\sin\beta\sin\gamma\cos(\alpha+\beta+\gamma) + \cos\alpha\cos\beta\cos\gamma\sin(\alpha+\beta+\gamma) = 0 \ .$$

12. A uniform rod of negligible weight rests wholly within a smooth hemispherical bowl of radius r. A weight is clamped to the rod at a point whose distance from the ends are a and b, where $b > a$. Show that when the rod is in a position of equilibrium, its inclination θ to the horizontal is given by

$$\sin\theta = \frac{b-a}{2\sqrt{r^2+ab}} \ .$$