Department of Education, Ontario

Annual Examinations, 1944

GRADE XIII

PROBLEMS

(To be taken only by candidates writing for certain University Scholarships involving Mathematics)

Ten questions constitute a full paper.

1. Find all functions $f(x)$ of the form

$$f(x) = \frac{a+bx}{b+x}$$

with constant coefficients a and b, for which

$$\frac{f(2)}{f(5)} = 2 \qquad \text{and} \qquad f(0) + 3f(-2) = 0 \ .$$

2. If the polynomial $a_3x^3 + a_2x^2 + a_1x + a_0$ is the third power of a linear function, prove that

$$9a_0a_3 = a_1a_2 \ , \qquad a_2^2 = 3a_1a_3 \ .$$

Prove the converse: if these two conditions are satisfied, then the polynomial is the third power of a linear function. (All numbers are assumed to be real.)

3. In how many different ways is it possible for a family of three to own a total of less than n dollars, if each member of the family owns an integral number of dollars, positive or zero?

4. In the series

$$-1 + x + 4x^2 + \cdots + a_kx^k \cdots$$

every coefficient a_k is obtained from the three preceding coefficients a_{k-1}, a_{k-2}, a_{k-3} by means of the formula

$$a_k = 3a_{k-1} - 3a_{k-2} + a_{k-3}$$

for $k \geq 3$. ($a_0 = -1$, $a_1 = 1$, $a_2 = 4$.) Prove that the series represents a rational function with denominator $(1-x)^3$. By expanding this function directly in a series, obtain an explicit formula for a_k. [Hint: Multiply the given series by $(1-x)^3$.]

5. A line moves in the plane so that the directed distances d_1, d_2, $\cdots$, d_n of n fixed points from the line satisfy an equation $c_1 d_1 + c_2 d_2 + \cdots + c_n d_n = 0$, where $c_1, c_2, \cdots, c_n$ are given constants whose sum is not zero. Prove that the line passes through a fixed point.

6. Find the equations of two circles each of which passes through the points $(3, 1)$ and $(3, -1)$ and touches the line $x = y$.

7. A triangle has a given base and base angles differing by a right angle. Prove that the locus of the variable vertex is a rectangular hyperbola.

8. Find a point A on the parabola $y^2 = 4px$ such that the part of the normal at A which is terminated by the curve has $1/10$ of its length on the same side of the axis as A.

9. Show that all the real values of x which satisfy the equation $\tan(\pi \cot x) = \cot(\pi \tan x)$ are given by

$$\tan x = \frac{2n + 1 \pm \sqrt{4n^2 + 4n - 15}}{4}$$

where n is a positive or negative integer different from -2, ± 1.

10. Given the product p of the sines of the angles of a triangle, and the product q of their cosines, show that the tangents of the angles are the roots of the equation

$$qx^3 - px^2 + (1 + q)x - p = 0 \ .$$

11. In the ambiguous case in which two triangles are determined by the given parts a, b, A, show that the distance between the centres of the circumcircles of the two triangles is given by $\sqrt{a^2 \csc^2 A - b^2}$.

12. Observations of the position of a ship are made from a fixed stations. At one instant the bearing of the ship is α west of north and t minutes later the ship is due north. After an additional interval of t minutes, the bearing of the ship is β east of north. Assuming that the speed and the course of the ship has not changed, show that the course of the ship is θ east of north where

$$\theta = \tan^{-1}\left(\frac{2 \sin \alpha \sin \beta}{\sin(\alpha - \beta)}\right) \ .$$