Department of Education, Ontario

Annual Examinations, 1942

GRADE XIII

PROBLEMS

*(To be taken only by candidates writing for certain University Scholarships involving Mathematics)*

Ten questions constitute a full paper.

1. Prove that

$$(n+1)^{r+1} - (n+1) = (r+1)S_r + \frac{r(r+1)}{2}S_{r-1} + \cdots + (r+1)S_1$$

where $S_r = 1^r + 2^r + \cdots + n^r$.

2. Find all the three-digit numbers which satisfy the following conditions. The sum of the digits is 12. The sum of the cube of the first and the cube of the sum of the other two is 756. The square of the second digit is greater or less than the square of the third digit by 9.

3. A loan of \$5000 with compound interest at 5% is to be repaid, principal and interest, by annual payments, beginning one year after the loan is made. Each annual payment is to be \$500 except the last, which is to be less than \$500. Find the amount of the last payment.

4. Of the 9! numbers formed by permuting the digits 1, 2, 3, 4, 5, 6, 7, 8, 9, how many are between 678,000,000 and 859,000,000?

5. The vertices of a triangle are $A(a,l)$, $B(b,m)$, and $C(c,n)$, and $O$ is the origin. The line $OA$ cuts $BC$ at $P$, $OB$ cuts $CA$ at $Q$, and $OC$ cuts $AB$ at $R$. Find with its proper sign the ratio of $BP$ to $PC$ in terms of the given co-ordinates, and deduce that

$$\frac{BP}{PC} \cdot \frac{CQ}{QA} \cdot \frac{AR}{RB} = 1$$

(Ceva's Theorem).

6. Two line segments $AQ$ and $BR$ each of fixed length turn in one plane about the fixed points $A$ and $B$ with the same angular velocity thus making a constant angle with each other. Show that a point dividing $QR$ in a constant ratio traces a circle.

7. On the parabola $y^2 = 4px$, find the position of a point $A$ the normal of which cuts the parabola again at a point whose ordinate exceeds the ordinate of $A$ by twice the latus recturm.

8. Through two points $A$ and $B$ two parallel secants are drawn to meet a curve $c$ of the second degree at $P$ and $Q$, and $R$ and $S$, respectively. Through the same points $A$ and $B$ are drawn two other parallel secants, meeting $c$ at $P'$ and $Q'$, and $R'$ and $S'$, respectively. Prove that
$$\frac{AP \cdot AQ}{BR \cdot BS} = \frac{AP' \cdot AQ'}{BR' \cdot BS'} \ .$$

9. If $m = \csc\theta - \sin\theta$, and $n = \sec\theta - \cos\theta$, show that
$$m^{2/3} + n^{2/3} = (mn)^{-2/3} \ .$$

10. If $A$, $B$, and $C$ are angles of any triangle, prove that
$$sin(B + 2C) + \sin(C + 2A) + \sin(A + 2B) = 4\sin\frac{B - C}{2}\sin\frac{C - A}{2}\sin\frac{A - B}{2} \ .$$

11. On sides $BC$, $CA$, $AB$, respectively, of triangle $ABC$, points $P$, $Q$, $R$ are taken such that
$$\angle BAP + \angle CBQ = \angle ACR = a \ .$$
Show that $AP$, $BQ$, $CR$ intersect to form another triangle, similar to the former, and that the ratio of a side of the first triangle to the corresponding side of the second is
$$1 : (\cos a - \sin a(\cot A + \cot B + \cot C)) \ .$$

12. A tower situated on a horizontal plane leans towards the north; at two points due south and distant $a$ and $b$ respectively from the base, the angular altitudes of the tower are $\alpha$ and $\beta$. Show that, if $\theta$ is the incination of the tower to the horizontal and $h$ is the vertical height,
$$\tan\theta = \frac{b - a}{b\cot\alpha - a\cot\beta} \ , \qquad h = \frac{b - a}{\cot\beta - \cot\alpha} \ .$$