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Please send your solutions to

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no later than November 21, 2008. Electronic files can be sent to *barbeau@math.utoronto.ca*. However, please do not send scanned files; they use a lot of computer space, are often indistinct and can be difficult to download.

It is important that your complete mailing address and your email address appear legibly on the front page. If you do not write your family name last, please underline it.

570. Let a be an integer. Consider the diophantine equation

$$\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=\frac{a}{xyz}$$

where x, y, z are integers for which the greatest common divisor of xyz and a is 1.

(a) Determine all integers a for which there are infinitely many solutions to the equation that satisfy the condition.

(b) Determine an infinite set of integers a for which there are solutions to the equation for which the condition is satisfied and x, y, z are all positive. [Optional: Given N ¿ 0, are there infinitely many a for which there are at least N positive solutions satisfying the condition?]

571. Let ABC be a triangle and U, V, W points, not vertices, on the respective sides BC, CA, AB, for which the segments AU, BV, CW intersect in a common point O. Prove that

$$\frac{|OU|}{|AU|}+\frac{|OV|}{|BV|}+\frac{|OW|}{|CW|}=1 ,$$

and

$$\frac{|AO|}{|OU|}\cdot\frac{|BO|}{|OV|}\cdot\frac{|CO|}{|OW|}=\frac{|AO|}{|OU|}+\frac{|BO|}{|OV|}+\frac{|CO|}{|OW|}+2 .$$

572. Let $ABCD$ be a convex quadrilateral that is not a parallelogram. On the sides AB, BC, CD, DA, construct isosceles triangles KAB, MBC, LCD, NDA exterior to the quadrilateral $ABCD$ such that the angles K, M, L, N are right. Suppose that O is the midpoint of BD. Prove that one of the triangles MON and LOK is a $90°$ rotation of the other around O.

What happens when $ABCD$ is a parallelogram?

573. A point O inside the hexagon $ABCDEF$ satisfies the conditions $\angle AOB = \angle BOC = \angle COD = \angle DOE = \angle EOF = 60°$, $OA > OC > OE$ and $OB > OD > OF$. Prove that $|AB| + |CD| + |EF| < |BC| + |DE| + |FA|$.

574. A fair coin is tossed at most n times. The tossing stops before n tosses if there is a run of an odd number of heads followed by a tail. Determine the expected number of tosses.

575. A partition of the positive integer n is a set of positive integers (repetitions allowed) whose sum is n. For example, the partitions of 4 are (4), (3,1), (2,2), (2,1,1), (1,1,1,1); of 5 are (5), (4,1), (3,2), (3,1,1), (2,2,1), (2,1,1,1), (1,1,1,1,1); and of 6 are (6), (5,1), (4,2), (3,3), (4,1,1), (3,2,1), (2,2,2), (3,1,1,1), (2,2,1,1), (2,1,1,1), (1,1,1,1,1,1).

Let $f(n)$ be the number of 2's that occur in all partitions of n and $g(n)$ the number of times a number occurs exactly once in a partition. For example, $f(4) = 3$, $f(5) = 4$, $f(6) = 8$, $g(4) = 4$, $g(5) = 8$ and $g(6) = 11$. Prove that, for $n \geq 2$, $f(n) = g(n-1)$.

576. (a) Let $a \geq b > c$ be the radii of three circles each of which is tangent to a common line and is tangent externally to the other two circles. Determine c in terms of a and b.

(b) Let a, b, c, d be the radii of four circles each of which is tangent to the other three. Determine a relationship among a, b, c, d

Solutions

556. Let x, y, z be positive real numbers for which $x + y + z = 4$. Prove the inequality

$$\frac{1}{2xy + xz + yz} + \frac{1}{xy + 2xz + yz} + \frac{1}{xy + xz + 2yz} \leq \frac{1}{xyz} \ .$$

Solution. It is straightforward to establish for $a, b > 0$ that $(a+b)^{-1} \leq \frac{1}{4}(a^{-1} + b^{-1})$. Therefore,

$$\begin{aligned} \frac{1}{2xy + xz + yz} &\leq \frac{1}{4}\left(\frac{1}{xy + xz} + \frac{1}{xy + yz}\right) \leq \frac{1}{4}\left[\frac{1}{4}\left(\frac{1}{xy} + \frac{1}{xz}\right) + \frac{1}{4}\left(\frac{1}{xy} + \frac{1}{yz}\right)\right] \\ &= \frac{1}{16}\left(\frac{2}{xy} + \frac{1}{xz} + \frac{1}{yz}\right) = \frac{1}{16}\left(\frac{2z + y + x}{xyz}\right) \ . \end{aligned}$$

Similarly,

$$\frac{1}{xy + 2xz + yz} \leq \frac{1}{16}\left(\frac{z + 2y + x}{xyz}\right)$$

and

$$\frac{1}{xy + xz + 2yz} \leq \frac{1}{16}\left(\frac{z + y + 2x}{xyz}\right) \ .$$

Adding the three inequalities yields that

$$\frac{1}{2xy + xz + yz} + \frac{1}{xy + 2xz + yz} + \frac{1}{xy + xz + 2yz} \leq \frac{1}{16}\left(\frac{4x + 4y + 4z}{xyz}\right) = \frac{1}{xyz} \ .$$

Equality holds if and only if $x = y = z = 4/3$.

557. Suppose that the polynomial $f(x) = (1 + x + x^2)^{1004}$ has the expansion $a_0 + a_1 x + a_2 x^2 + \cdots + a_{2008} x^{2008}$. Prove that $a_0 + a_2 + \cdots + a_{2008}$ is an odd integer.

Solution. Observe that

$$a_0 + a_2 + \cdots + a_{2008} = \frac{1}{2}(f(1) + f(-1)) = \frac{1}{2}(3^{1004} + 1) \ .$$

It remains to show that $3^{1004} + 1$ is congruent to 2 modulo 4.

558. Determine the sum

$$\sum_{m=0}^{n-1}\sum_{k=0}^{m}\binom{n}{k} .$$

Solution. Let $S_m = \binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{m}$. Then $S_0 + S_{n-1} = S_1 + S_{n-2} = \cdots = S_{n-1} + S_0 = 2^n$, so that $S = n2^{n-1}$.

Comment. In more detail,

$$\begin{aligned} S_k + S_{n-1-k} &= \left[\binom{n}{0} + \cdots + \binom{n}{k}\right] + \left[\binom{n}{0} + \cdots + \binom{n}{n-1-k}\right] \\ &= \left[\binom{n}{0} + \cdots + \binom{n}{k}\right] + \left[\binom{n}{n} + \cdots + \binom{n}{k+1}\right] = 2^n . \end{aligned}$$

559. Let ϵ be one of the roots of the equation $x^n = 1$, where n is a positive integer. Prove that, for any polynomial $f(x) = a_0 + a_x + \cdots + a_n x^n$ with real coefficients, the sum $\sum_{k=1}^{n} f(1/\epsilon^k)$ is real.

Solution. If $\epsilon = 1$, the result is clear. Let $\epsilon \neq 1$; we have that $\epsilon^n = 1$.

$$\begin{aligned} \sum_{k=1}^{n} f(1/\epsilon^k) &= \sum_{k=1}^{n}\sum_{j=0}^{n} a_j (1/\epsilon^k)^j = \sum_{k=1}^{n}\sum_{j=0}^{n} a_j (1/\epsilon^{jk}) \\ &= \sum_{j=0}^{n} a_j \sum_{k=1}^{n} (1/\epsilon^{jk}) = na_0 + \sum_{j=2}^{n-1} a_j (1/\epsilon^j)\left(\frac{1-\epsilon^{-jn}}{1-\epsilon^{-j}}\right) + na_n \\ &= na_0 + 0 + na_n = n(a_0 + a_n) . \end{aligned}$$

560. Suppose that the numbers $x_1, x_2, \cdots, x_n$ all satisfy $-1 \leq x_i \leq 1$ $(1 \leq i \leq n)$ and $x_1^3 + x_2^3 + \cdots + x_n^3 = 0$. Prove that

$$x_1 + x_2 + \cdots + x_n \leq \frac{n}{3} .$$

Solution. Since $-1 \leq x_i \leq 1$, for $1 \leq i \leq n$, there exists θ_i with $0 \leq \theta_i \leq \pi$ such that $x_i = \cos\theta_i$. Therefore

$$\begin{aligned} \sum_{i=1}^{n} x_i &= \sum_{i=1}^{n} \cos\theta_i = \frac{1}{3}\left[4\sum_{i=1}^{n}\cos^3\theta_i - \sum_{i=1}^{n}\cos 3\theta_i\right] \\ &= -\frac{1}{3}\sum_{i=1}^{n}\cos 3\theta_i \leq \frac{n}{3} , \end{aligned}$$

as desired.

561. Solve the equation

$$\left(\frac{1}{10}\right)^{\log_{(1/4)}(\sqrt[4]{x}-1)} - 4^{\log_{10}(\sqrt[4]{x}+5)} = 6 ,$$

for $x \geq 1$.

Solution. Let $a = \log_{(1/4)}(\sqrt[4]{x} - 1)$ and $b = \log_{10}(\sqrt[4]{x} + 5)$. Then $(1/4)^a = \sqrt[4]{x} - 1$ and $10^b = \sqrt[4]{x} + 5$, whence $(1/4)^a + 1 = 10^b - 5$, or

$$\left(\frac{1}{4}\right)^a - 10^b = -6 .$$

On the other hand, the given equation is

$$\left(\frac{1}{10}\right)^a - 4^b = 6 \ .$$

Therefore

$$\left(\frac{1}{4}\right)^a - 4^b + \left(\frac{1}{10}\right)^a - 10^b = 0$$

which is equivalent to

$$(4^{-a} - 4^b) + (10^{-a} - 10^b) = 0 \ .$$

The left side is less than 0 when $-a < b$ and greater than 0 when $-a > b$. Therefore $-a = b$ and so $10^b - 4^b = 6$. One solution of this is $b = 1$.

We show that this solution is unique. Observe that the function $f(x) = 6(1/10)^x + (4/10)^x$ decreases as x increases from 0 and takes the value 1 when $x = 1$. Since $f(x) = 1$ is equivalent to $6 = 10^x - 4^x$, we see that $x = 1$ is the only solution of the latter equation.

562. The circles $\mathfrak{C}$ and $\mathfrak{D}$ intersect at the two points A and B. A secant through A intersects $\mathfrak{C}$ at C and $\mathfrak{D}$ at D. On the segments CD, BC, BD, consider the respective points M, N, K for which $MN \| BD$ and $MK \| BC$. On the arc BC of the circle $\mathfrak{C}$ that does not contain A, choose E so that $EN \perp BC$, and on the arc BD of the circle $\mathfrak{D}$ that does not contain A, choose F so that $FK \perp BD$. Prove that angle EMF is right.

Solution. We have that $BN : NC = DM : MC = DK : KB$. Let G be the point of intersection of FK and $\mathfrak{D}$. Then $\angle BGD = \angle BAD = \angle BEC$. In triangle BGD and CEB, we have that $\angle BGD = \angle CEB$. Compare triangles BGD and CEB: $\angle BGD = \angle CEB$; GK and EN are respective altitudes; $DK : KB = BN : NC$. There is a similarilty transformation with factor $|DK|/|BN|$ that takes $B \rightarrow D$, $C \rightarrow B$, $N \rightarrow K$ and E to a point E' on the line KG. Since $\angle BGD = \angle CEB = \angle BE'D$, we must have $E' = G$. Thus triangles BGD and CEB are similar, whence $\angle EBC = \angle GDB = \angle GFB$. As a result, triangles BNE and FKB are similar.

Since $MNBK$ is a parallelogram, $\angle MNB = \angle MKB$. Thus $\angle MNE = \angle MKF$. Since $MN : KF = BK : KF = EN : NB = EN : MK$, triangles ENM and MKF are similar. Therefore $\angle NME = \angle KFM$. But $MN \perp KF$. Therefore $EM \perp FM$.