OLYMON

Produced by the **Canadian Mathematical Society** and the **Department of Mathematics of the University of Toronto**.

Issue 9:6

September, 2008

Please send your solutions to

Edward Barbeau
Department of Mathematics
University of Toronto
40 St. George Street
Toronto, ON M5S 2E4

no later than October 15, 2008. Electronic files can be sent to *barbeau@math.utoronto.ca*.

It is important that your complete mailing address and your email address appear legibly on the front page. If you do not write your family name last, please underline it.

563. (a) Determine infinitely many triples (a, b, c) of integers for which a, b, c are not in arithmetic progression and $ab + 1$, $bc + 1$, $ca + 1$ are all squares.

(b) Determine infinitely many triples (a, b, c) of integers for which a, b, c are in arithemetic progression and $ab + 1$, $bc + 1$, $ca + 1$ are all squares.

(c) Determine infinitely many triples (u, v, w) of integers for which $uv - 1$, $vw - 1$, $wu - 1$ are all squares. (Can it be arranged that u, v, w are in arithmetic progression?)

564. Let $x_1 = 2$ and

$$x_{n+1} = \frac{2x_n}{3} + \frac{1}{3x_n}$$

for $n \geq 1$. Prove that, for all $n > 1$, $1 < x_n < 2$.

565. Let ABC be an acute-angled triangle. Points A_1 and A_2 are located on side BC so that the four points are ordered B, A_1, A_2, C; similarly B_1 and B_2 are on CA in the order C, B_1, B_2, A and C_1 and C_2 on side AB in order A, C_1, C_2, B. All the angles AA_1A_2, AA_2A_1, BB_1B_2, BB_2B_1, CC_1C_2, CC_2C_1 are equal to θ. Let $\mathfrak{T}_1$ be the triangle bounded by the lines AA_1, BB_1, CC_1 and $\mathfrak{T}_2$ the triangle bounded by the lines AA_2, BB_2, CC_2. Prove that all six vertices of the triangles are concyclic.

566. A deck of cards numbered 1 to n (one card for each number) is arranged in random order and placed on the table. If the card numbered k is on top, remove the kth card counted from the top and place it on top of the pile, not otherwise disturbing the order of the cards. Repeat the process. Prove that the card numbered 1 will eventually come to the top, and determine the maximum number of moves that is required to achieve this.

567. (a) Let A, B, C, D be four distinct points in a straight line. For any points X, Y on the line, let XY denote the *directed* distance between them. In other words, a positive direction is selected on the line and $XY = \pm|XY|$ according as the direction X to Y is positive or negative. Define

$$(AC, BD) = \frac{AB/BC}{AD/DC} = \frac{AB \times CD}{BC \times DA} \ .$$

Prove that $(AB, CD) + (AC, BD) = 1$.

(b) In the situation of (a), suppose in addition that $(AC, BD) = -1$. Prove that

$$\frac{1}{AC} = \frac{1}{2}\left(\frac{1}{AB} + \frac{1}{AD}\right) ,$$

and that

$$OC^2 = OB \times OD ,$$

where O is the midpoint of AC. Deduce from the latter that, if Q is the midpoint of BD and if the circles on diameters AC and BD intersect at P, $\angle OPQ = 90°$.

(c) Suppose that A, B, C, D are four distinct on one line and that P, Q, R, S are four distinct points on a second line. Suppose that AP, BQ, CR and DS all intersect in a common point V. Prove that $(AC, BD) = (PR, QS)$.

(d) Suppose that $ABQP$ is a quadrilateral in the plane with no two sides parallel. Let AQ and BP intersect in U, and let AP and BQ intersect in V. Suppose that VU and PQ produced meet AB at C and D respectively, and that VU meets PQ at W. Prove that

$$(AB, CD) = (PQ, WD) = -1 .$$

568. Let ABC be a triangle and the point D on BC be the foot of the altitude AD from A. Suppose that H lies on the segment AD and that BH and CH intersect AC and AB at E and F respectively.

Prove that $\angle FDH = \angle HDE$.

569. Let A, W, B, U, C, V be six points in this order on a circle such that AU, BV and CW all intersect in the common point P at angles of $60°$. Prove that

$$|PA| + |PB| + |PC| = |PU| + |PV| + |PW| .$$

Solutions

549. The set E consists of 37 two-digit natural numbers, none of them a multiple of 10. Prove that, among the elements of E, we can find at least five numbers, such that any two of them have different tens digits and different units digits.

Solution. Call a set of nine numbers with the same tens digit a *decade*. By the Pigeon-Hole Principle, there is at least one decade with at least five numbers of E in it; wolog, let it be the tens decade. There are at least 28 numbers in E that are not in the tens decade; one of the remaining decades, say the twenties, must have at least four members of E. There are at least 19 members of E that are not in the tens or twenties decade; at least one of the remaining decades, say the thirties, has at least three members of E. Similarly, a fourth decade, say the forties, has at least two members and a fifth decade, say the fifties, has at least one member.

We can select the five element subset of E working back from the fifth decade. Select any number from the fifties, a number from the forties with a different tens digit, a number from the thirties with a tens digit differing from the two already determined, a number from the twenties with a tens digit differing from the three already determined and finally a number from the tens with a fifth tens digit. This will serve the purpose.

550. The functions $f(x)$ and $g(x)$ are defined by the equations: $f(x) = 2x^2 + 2x - 4$ and $g(x) = x^2 - x + 2$.

(a) Find all real numbers x for which $f(x)/g(x)$ is a natural number.

(b) Find the solutions of the inequality

$$\sqrt{f(x)} + \sqrt{g(x)} \geq \sqrt{2} \ .$$

Solution. (a) We have that

$$\frac{f(x)}{g(x)} = \frac{2x^2+2x-4}{x^2-x+2} = \frac{2(x+2)(x-1)}{x^2-x+2} = 2 + \frac{4(x-2)}{x^2-x+2} \ .$$

Observe that $x^2 - x + 2 = (x - \frac{1}{2})^2 + \frac{7}{4} > 0$.

Since $x^2 - 5x + 10 = (x - \frac{5}{2})^2 + \frac{15}{4} > 0$, $4x - 8 < x^2 - x + 2$, so that $4(x-2)/(x^2-x+2) < 1$. Hence $f(x)/g(x)$ cannot take integer values exceeding 2.

$$f(x)/g(x) = 2 \Longleftrightarrow x = 2 \ ;$$

$$f(x)/g(x) = 1 \Longleftrightarrow 4x - 8 = -(x^2 - x + 2) \Longleftrightarrow x^2 + 3x - 6 = 0 \ .$$

Therefore, $f(x)/g(x)$ is a natural number if and only if $x = 2$ or $x = \frac{1}{2}(-3 \pm \sqrt{3})$.

Comment. It is not too hard to find all values of x for which $f(x)/g(x)$ assumes integer values. Note that, if $x \geq 2$ or $x \leq -5$, then $x^2+3x-6 > 0$, so that $4x-8 > -(x^2-x+2)$ and $4(x-2)/(x^2-x+2) > -1$. Thus, if $f(x)/g(x)$ assumes integer values, then $|x| \leq 5$ and

$$\left| \frac{4(x-2)}{x^2-x+2} \right| \leq \frac{12}{3/2} = 8 \ .$$

Thus, $f(x)/g(x)$ can takes only integer values between -6 and 2, and we can check each case.

(b) We require that $f(x) \geq 0$, so that $x \leq -2$ or $x \geq 1$. The functions $f(x) = 2(x+2)(x-1)$ and $g(x) = x(x-1)+2$ are both decreasing on $(-\infty, -2]$ and increasing on $[1, +\infty)$. Since $\sqrt{f(-2)} + \sqrt{g(-2)} = 0 + 2\sqrt{2} > 2$, $\sqrt{f(1)} + \sqrt{g(1)} = \sqrt{2}$, the inequality is satisfied on the set $(-\infty, -2] \cup [1, +\infty)$.

Note. There was an error in the statement of (b), where $\sqrt{2}$ was given as 2. In this case, the inequality is satisfied on the set $(\infty, -2] \cup [\alpha, \infty)$, where α lies between 1 and 2, and satisfies the equation $\sqrt{f(\alpha)} + \sqrt{g(\alpha)} = 2$.

551. The numbers 1, 2, 3 and 4 are written on the circumference of a circle, in this order. Alice and Bob play the following game: On each turn, Alice adds 1 to two adjacent numbers, while Bob switches the places of two adjacent numbers. Alice wins the game, if after her turn, all numbers on the circle are equal. Does Bob have a strategy to prevent Alice from winning the game? Justify your answer.

Solution. Bob can prevent Alice from winning the game whenever Alice has the first move. The configuration of numbers begins with even and odd numbers alternating; Bob's strategy is to always present Alice with this situation. Then, whatever Alice does, she must leave two odd and two even numbers and therefore at least two distinct numbers.

To do this, Bob must switch whatever pair of numbers Alice selects to add 1 to. Alice's move changes the parity of the numbers in the two positions and Bob's move switches the parity back to what it was before.

552. Two real nonnegative numbers a and b satisfy the inequality $ab \geq a^3 + b^3$. Prove that $a + b \leq 1$.

Solution 1. Note that $a^3 + b^3 = (a+b)(a^2 - ab + b^2)$ and that $a^2 - ab + b^2$ is always positive. Then

$$\begin{aligned} 1 - (a+b) &= 1 - \frac{a^3+b^3}{a^2-ab+b^2} \geq 1 - \frac{ab}{a^2-ab+b^2} \\ &= \frac{a^2-2ab+b^2}{a^2-ab+b^2} = \frac{(a-b)^2}{a^2-ab+b^2} \geq 0 \ , \end{aligned}$$

from which the desired result follows.

Solution 2. We note two inequalities: (1) $(a+b)^2 \geq 4ab$ and (2) $4(a^3+b^3) \geq (a+b)^3$. The first is a consequence of the arithmetic-geometric means inequality, while the second can be obtained either from $(a+b)(a-b)^2 \geq 0$ or from the power-mean inequality $[\frac{1}{2}(a^3+b^3)]^{1/3} \geq \frac{1}{2}(a+b)$. It follows that

$$(a+b)^2 \geq 4ab \geq 4(a^3+b^3) \geq (a+b)^3 \ ,$$

from which the result is obtained.

553. The convex quadrilateral $ABCD$ is concyclic with side lengths $|AB| = 4$, $|BC| = 3$, $|CD| = 2$ and $|DA| = 1$. What is the length of the radius of the circumcircle of $ABCD$? Provide an exact value of the answer.

Solution. Let $\alpha = \angle DAB$ and $\beta = \angle ABC$, so that $\angle BCD = 180^\circ - \alpha$ and $\angle CDA = 180^\circ - \beta$. Then, by the Law of Cosines,

$$1 + 16 - 8\cos\alpha = |BD|^2 = 9 + 4 + 12\cos\alpha \ ,$$

whence $\cos\alpha = 1/5$ and $|BD| = \sqrt{77/5}$. The circumradius R of $ABCD$ satisfies $2R\sin\alpha = |BD|$, whence

$$R = \frac{\sqrt{77/5}}{2\sqrt{24/25}} = \frac{\sqrt{5\times 7\times 11}}{4\sqrt{6}} = \frac{\sqrt{385}}{4\sqrt{6}} \ .$$

As a check, we can find that

$$16 + 9 - 24\cos\beta = |AC|^2 = 4 + 1 + 4\cos\beta \ ,$$

whence $\cos\beta = 5/7$ and $|AC| = \sqrt{\frac{55}{7}}$. Thus, $2R\sin\beta = |AC|$, so that

$$R = \frac{\sqrt{55/7}}{2\sqrt{24/49}} = \frac{\sqrt{5\times 7\times 11}}{4\sqrt{6}} = \frac{\sqrt{385}}{4\sqrt{6}} \ .$$

554. Determine all real pairs (x,y) that satisfy the system of equations:

$$3\sqrt[3]{x^2y^5} = 4(y^2 - x^2)$$
$$5\sqrt[3]{x^4y} = y^2 + x^2 \ .$$

Solution. Multiply the two equations to obtain

$$15x^2y^2 = 4(y^4 - x^4) \Leftrightarrow 0 = 4x^4 + 15x^2y^2 - 4y^4 = (4x^2 - y^2)(x^2 + 4y^2)$$
$$\Leftrightarrow y^2 = 4x^2 \Leftrightarrow y = \pm 2x \ .$$

Substituting $y = 2x$ into the first equation yields that

$$3\sqrt[3]{2^5x^2x^5} = 12x^2 \Longrightarrow 2^5\times 3^3\times x^7 = 2^6\times 3^3\times x^6 \Longrightarrow x = 0 \text{ or } x = 2 \ .$$

Similarly, substituting $y = -2x$ into the first equation yields the additional solution $x = -2$. There are three solutions to the system, namely, $(x,y) = (0,0), (2,4), (-2,4)$. All check out.

555. Let ABC be a triangle, all of whose angles do not exceed 90°. The points K on side AB, M on side AC and N on side BC are such that $KM \perp AC$ and $KN \perp BC$. Prove that the area $[ABC]$ of triangle

ABC is at least 4 times as great as the area $[KMN]$ of triangle KMN, *i.e.*, $[ABC] \geq 4[KMN]$. When does equality hold?

Solution 1. Let $b = |AC|$, $a = |BC|$, $m = |KM|$, $n = |KN|$ and $\theta = \angle ACB$, so that $\angle MKN = 180° - \theta$. Then, by the arithmetic-geometric means inequality,

$$[ABC] = [AKC] + [AKB] = \frac{1}{2}(bm + an) \geq \sqrt{abmn}\ .$$

Therefore

$$\begin{aligned} [ABC]^2 &\geq abmn \geq abmn \sin^2 \theta \\ &\geq (ab \sin \theta)(mn \sin(180° - \theta)) \\ &= 2[ABC] \cdot 2[KMN]\ , \end{aligned}$$

whence $[ABC] \geq 4[KMN]$, as desired.

Equality holds if and only if $bm = an$ and $\sin \theta = 1$, if and only if $a : b = m : n$ and $\theta = 90°$, if and only if triangle ABC is right and similar to triangles AKM and KBN. In this case, $KN \| AC$ and $KM \| BC$ and the linear dimensions of KMN are half those of CAB; thus, $AC = 2NK$, $BC = 2MK$ and K is the midpoint of AB.

Solution 2. Let the angles at A, B and C in triangle ABC be respectively α, β, γ and the sides of this triangle be, conventionally, a, b, c. Suppose that $m = |KM|$, $n = |KN|$, and $x = |AK|$, so that $m = x \sin \alpha$ and $n = (c - x) \sin \beta$.

Then $[ABC] = \frac{1}{2} ab \sin \gamma$ and

$$[KMN] = \frac{1}{2} mn \sin(180° - \gamma) = \frac{1}{2} x(c - x) \sin \alpha \sin \beta \sin \gamma\ .$$

Thus

$$\begin{aligned} \frac{[KMN]}{[ABC]} &= \frac{x(c - x) \sin \alpha \sin \beta}{ab} = x(c - x) \left(\frac{\sin \alpha}{a} \right) \left(\frac{\sin \beta}{b} \right) \\ &= x(c - x) \left(\frac{1}{2R} \right) \left(\frac{1}{2R} \right) = \frac{x(c - x)}{4R^2}\ , \end{aligned}$$

where R is the circumradius of triangle ABC. By the Arithmetic-Geometric Means Inequality, $x(c - x) \leq (\frac{1}{2}[x + (c - x)])^2 = c^2/4$, so that

$$\begin{aligned} \frac{[KMN]}{[ABC]} &= \frac{x(c - x)}{4R^2} \leq \left(\frac{c^2}{4} \right) \left(\frac{1}{4R^2} \right) = \frac{1}{4} \left(\frac{c}{2R} \right)^2 \\ &= \frac{1}{4} \sin^2 \gamma \leq \frac{1}{4}\ , \end{aligned}$$

as desired. Equality holds if and only if $x = c - x$ and $\sin \gamma = 1$, *i.e.*, when ABC has a right angle at C and K is the midpoint of AB.