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Please send your solution to

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no later than February 29, 2008.

It is important that your complete mailing address and your email address appear on the front page. If you do not write your family name last, please underline it.

Notes. $\lfloor x \rfloor$, known as the "floor of x, is the largest integer n that does not exceed x, *i.e.*, that integer n for which $n \leq x < n+1$. The notation $[PQR]$ denotes the area of the triangle PQR. A *geometric progression* is a sequence for which the ratio of two successive terms is always the same; its nth term has the general form ar^{n-1}.

528. Let the sequence $\{x_n : n = 0, 1, 2, \cdots\}$ be defined by $x_0 = a$ and $x_1 = b$, where a and b are real numbers, and by

$$7x_n = 5x_{n-1} + 2x_{n-2}$$

for $n \geq 2$. Derive a formula for x_n as a function of a, b and n.

529. Let k, n be positive integers. Define $p_{n,1} = 1$ for all n and $p_{n,k} = 0$ for $k \geq n+1$. For $2 \leq k \leq n$, we define inductively

$$p_{n,k} = k(p_{n-1,k-1} + p_{n-1,k}) \ .$$

Prove, by mathematical induction, that

$$p_{n,k} = \sum_{r=0}^{k-1} \binom{k}{r} (-1)^r (k-r)^n \ .$$

530. Let $\{x_1, x_2, x_3, \cdots, x_n, \cdots\}$ be a sequence is distinct positive real numbers. Prove that this sequence is a geometric progression if and only if

$$\frac{x_1}{x_2} \sum_{k=1}^{n-1} \frac{x_n^2}{x_k x_{k+1}} = \frac{x_n^2 - x_1^2}{x_2^2 - x_1^2}$$

for all $n \geq 2$.

531. Show that the remainder of the polynomial

$$p(x) = x^{2007} + 2x^{2006} + 3x^{2005} + 4x^{2004} + \cdots + 2005x^3 + 2006x^2 + 2007x + 2008$$

is the same upon division by $x(x+1)$ as upon division by $x(x+1)^2$.

532. The angle bisectors BD and CE of triangle ABC meet AC and AB at D and E respectively and meet at I. If $[ABD] = [ACE]$, prove that $AI \perp ED$. is the converse true?

533. Prove that the number

$$1 + \lfloor (5+\sqrt{17}))^{2008} \rfloor$$

is divisible by 2^{2008}.

534. Let $\{x_n : n = 1, 2, \cdots\}$ be a sequence of distinct positive integers, with $x_1 = a$. Suppose that

$$2\sum_{k=1}^{n} \sqrt{x_i} = (n+1)\sqrt{x_n}$$

for $n \geq 2$. Determine $\sum_{k=1}^{n} x_k$.

Solutions

521. On a 8×8 chessboard, either $+1$ or -1 is written in each square cell. Let A_k be the product of all the numbers in the kth row, and B_k the product of all the numbers in the kth column of the board $(k = 1, 2, \cdots, 8)$. Prove that the number

$$A_1 + A_2 + \cdots + A_8 + B_1 + B_2 + \cdots + B_8$$

is a multiple of 4.

Solution 1. it is clear that the value of each A_k and B_k is $+1$ or -1. Assume that p of the eight A_k have the value 1 and $8-p$ have the value -1. Similarly, suppose that q of the eight B_k have the value 1 and $8-q$ have the value -1. The each product is the product of all the entries, the products $A_1A_2\cdots A_8$ and $B_1B_2\cdots B_8$ are equal, so that $(-1)^{8-p} = (-1)^{8-q}$ and p and q have the same parity. We have that

$$A_1 + A_2 + \cdots + A_8 + B_1 + B_2 + \cdots + B_8 = p + (8-p)(-1) + q + (8-q)(-1) = 2(p+q) - 16 \ .$$

Since $p+q$ is even, both terms on the right are divisible by 4 and the result follows.

Solution 2. The proof is by induction on the number of negative entries in the square array. If all of the entries are equal to $+1$, then the sum in the problem is equal to 16, which is divisible by 4. Let n be a positive integer, and suppose that the result holds when there are $n-1$ entries in the array equal to -1. Let an array U be given for which there are exactly n entries equal to -1. Let V be the array obtained from U by changing exactly one of the entries -1 to $+1$, say the entry in the rth row and sth column. Then the numbers A_i and B_j are the same for both arrays when $i \neq r$ and $j \neq s$.

If A_k and B_k denote the row and column products for the matrix V, then the sum of the problem for the array U is obtained from that for the matrix V by the addition of $-2A_r - 2B_s = -2(A_r + B_s)$. Since (A_r, B_s) has one of the values $(+1,+1), (+1,-1), (-1,+1), (-1,-1)$, it follows that the sum is altered by a multiple of 4. Since by the induction hypothesis, the sum for U is divisible by 4, then so also must be the sum for V.

522. (a) Prove that, in each scalene triangle, the angle bisector from one of its vertices is always "between" the median and the altitude from the same vertex.

(b) Find the measures of the angles of a triangle if the lengths of the median, the angle bisector and the altitude from one of its vertices are in the ratio $\sqrt{5} : \sqrt{2} : 1$.

Solution 1. (a) Let ABC be a triangle and let P, K and M be the respective intersections of the altitude, angle bisector and median from A in the side BC. Suppose, wolog, $AB < AC$. Then (by Pythagoras' Theorem, for example), $BP < CP$, so that $\angle BAP < \angle CAP$ and the bisector AK of angle A falls within the angle CAP. Hence, $BP < BK$. Since $KB : KC = AB : AC$, $KB < KC$ and the midpoint M of BC must lie in the segment KC. The result follows.

(b) Use the same notation as in (a). We may assume that $|AP| = 1$, $|AK| = \sqrt{2}$ and $|AM| = \sqrt{5}$. We first note that the altitude from A must lie outside of the triangle. Suppose, on the contrary, that P lies on the side BC. By Pythagoras' Theorem, we have that $|PK| = 1$, so that $\angle PAK = 45°$. Then

$$\angle BAP + 45° = \angle BAK = \angle CAK = \angle CAP - 45° ,$$

so that

$$\angle CAP = \angle BAP + 90° > 90° ,$$

which is impossible.

Hence P must lie on CB produced and B lies in the segment PK. Let $|PB| = x$, so that $|BK| = 1 - x$, $|PM| = 2$ (by Pythagoras' Theorem), $|KM| = 1$, $|MC| = 2 - x$ and $|PC| = 4 - x$. We have that

$$45° - \angle PAB = \angle BAK = \angle CAK = \angle PAC - 45° ,$$

so that $\angle PAC = 90° - \angle PAB$ and

$$4 - x = \tan \angle PAC = \cot \angle PAB = \frac{1}{x} .$$

Thus, $x^2 - 4x + 1 = 0$ and $x = 2 - \sqrt{3}$. We reject the larger root as it would be the reciprocal of the smaller and so it would be the tangent of $\angle PAC$ which is larger than $\angle PAB$.

Therefore, $\tan \angle PAB = 2 - \sqrt{3}$ and so, from the double angle formula, $\tan 2\angle PAB = 1/\sqrt{3}$ Thus, $\angle PAB = 15°$, $\angle PAC = 75°$ and $\angle BAC = 60°$. Since $\angle PBA = 75°$, it follows that $\angle ABC = 105°$ and $\angle BCA = 15°$.

(It can also be checked that $|AB| = 2\sqrt{2 - \sqrt{3}}$, $|AC| = 2\sqrt{2 + \sqrt{3}}$ and $|BC| = 2\sqrt{3}$.

Solution 2. (a) can be established as before. For (b), assume wolog that $AC > AB$. We first establish that $\angle ABC$ is obtuse. Let $\angle BAC = \alpha$, $\angle ABC = \beta$ and $\angle ACB = \gamma$. Since $\beta > \gamma$,

$$\angle AKC = \beta + \alpha/2 > \gamma + \alpha/2 = \angle AKB ,$$

so that $\angle AKB < 90°$ (which agrees with $\angle AKP = 45°$) and $\angle AKC > 90°$ (more precisely, $\angle AKC = 135°$). Hence $\beta + \alpha/2 = \angle AKC = 135°$, so that $180° - \beta - \gamma = \alpha = 270° - 2\beta$ and $\beta = \gamma + 90° > 90°$.

By Pythagoras's theorem, $|PK| = 1$, $|PM| = 2$ and $|KM| = 1$. Let $|PB| = x$ ($x < 1$), so that $|BK| = 1 - x$, $|BM| = 2 - x$, $|BC| = 2|BM| = 4 - 2x$, and $|PC| = 4 - x$.

The triangles ACP and BAP are similar since both are right and

$$\angle PAB = \angle ABC - 90° = \beta - 90° = \gamma = \angle ACP .$$

Therefore $AP : PC = BP : AP$, or, equivalently, $1 : (4 - x) = x : 1$. Therefore, x is the smaller of the roots of $x^2 - 4x + 1 = 0$, namely $2 - \sqrt{3}$.

Thus, $\tan \angle PAB = 2 - \sqrt{3}$, so that $\angle PAB = 15°$. (One way to check this is to use the double angle formula to find the tangent of $15°$.) Therefore, $\gamma = \angle ACB = \angle PAB = 15°$, $\beta = \angle ABC = \gamma + 90° = 105°$ and $\alpha = \angle BAC = 60°$.

Solution 3. [J. Schneider] Wolog, let $\angle B > \angle C$. we use the notation of the first solution. If B is obtuse, then B lies between P and K. Since $AB < AC$, $BK : KC = AB : AC$, so that $BK < KC$ and M lies between K and C.

Let the angle at B be acute. Then $BP : PC = \tan C : \tan B$, $BK : KC = c : b = \sin C : \sin B$ and $BM : MC = 1 : 1$. Since $\sin C < \sin B$ and $\cos C > \cos B$,

$$\frac{\tan C}{\tan B} = \frac{\cos B}{\cos C} \cdot \frac{\sin C}{\sin B} < \frac{\sin C}{\sin B} < 1 ,$$

and the result follows.

(b) Let $x = |MC|$ and coordinatize the situation by $A \sim (0, 1)$, $B \sim (0, 0)$, $K \sim (1, 0)$, $M \sim (2, 0)$, $C \sim (2 + x, 0)$ and $B \sim (2 - x, 0)$. The proportion $AB^2 : AC^2 = AK^2 : KC^2$ leads to the equation

$$\frac{(x+1)^2}{(x-1)^2} = \frac{x^2 + 4x + 5}{x^2 - 4x + 5} ,$$

which simplifies to $x(x^2 - 3) = 0$. Since $vertAK| < |AC|$, we reject $x = -\sqrt{3}$. Hence $x = \sqrt{3}$. Note that this places B to the right of the origin and so angle B is obtuse.

Thus $|AB| = 2\sqrt{2 - \sqrt{3}}$, $|AC| = 2\sqrt{2 + \sqrt{3}}$ and $|BC| = 2\sqrt{3}$. Angle A can be identified using the Law of Cosines and the remaining angles from their tangents.

523. Let ABC be an isosceles triangle with $AB = AC$. The segments BC and AC are used as hypotenuses to construct three right triangles BCM, BCN and ACP. Prove that, if $\angle ACP + \angle BCM + \angle BCN = 90°$, then the triangle MPN is isosceles.

Solution 1. Clearly, M and N are points on a circle whose diameter is BC. Let O be the midpoint of BC and the centre of this circle, and Q the intersection point of the ray PO and the circle. We have that

$$\angle BCM + \angle BCN = \frac{1}{2}\text{arc } BM + \text{arc } BN) \ . \qquad (1)$$

Observe that, as triangle ABC is isosceles with O the midpoint of its base BC, $AO \perp BC$. Therefore, O and P are on the circle with diameter AC, so that

$$90° - \angle ACP = \angle PAC = \angle POC = \angle BOQ \ . \qquad (2)$$

We are given that $90° - \angle ACP = \angle BCM + \angle BCN$, so that (1) and (2) yield

$$\text{arc } BQ = \angle BOQ = \frac{1}{2}(\text{arc } BM + \text{ arc } BN)$$

or

$$\text{arc } BN \ + \text{arc } NQ \ + \ \text{arc } BQ = \frac{1}{2}(\text{arc } BN \ + \text{arc } MN \ + \ \text{arc } BN)$$

which in turn is equivalent to arc $NQ = \frac{1}{2}(\text{arc } MN)$. Thus, Q is the midpoint of the arc MN, so that PQ is the right bisector of the segment MN. The result follows.

Solution 2. [J. Schneider] Note that triangle MPN is isosceles with $PM = PN$ if and only if the right bisector of MN passes through P.

Let D be the midpoint of BC. Since ABC is isosceles, $AD \perp BC$ and D lies on the circle with diameter AC. Thus, $APCD$ is concyclic and $\angle ADP = \angle ACP$.

Since D is the centre of the circle with diameter BC that contains M and N, $\angle BDN = 2\angle BCN$ and $\angle BDM = 2\angle BCM$. Let X be the midpoint of MN. Then DX right bisects MN and bisects angle MDN. Hence

$$\angle BDX = \frac{1}{2}(\angle BDM + \angle BDN) = \angle BCN + \angle BCM \ .$$

Suppose that $\angle BCN + \angle BCM + \angle ACP = 90°$, as hypothesized. Then

$$\angle PDC = 90° - \angle ADP = 90° - \angle ACP = \angle BCM + \angle BCN = \angle BDX \ .$$

Hence X, D, P are collinear. But DX is the right bisector of MN, and so is DP. Hence triangle MPN is isosceles.

Comment. The above argument applies when all triangles are external to triangle ABC. It can be adapted to the other cases.

524. Solve the irrational equation

$$\frac{7}{\sqrt{x^2-10x+26}+\sqrt{x^2-10x+29}+\sqrt{x^2-10x+41}} = x^4-9x^3+16x^2+15x+26 \ .$$

Solution. Observe that

$$x^4-9x^3+16x^2+15x+26 = (x^2+x+1)(x-5)^2+1 \ .$$

Since $x^2+x+1 = (x+\frac{1}{2})^2+\frac{3}{4} > 0$ for all x, the quartic on the right side of the equation is never less than 1 and is equal to 1 if and only if $x = 5$.

Since $x^2-10x+25+k = (x-5)^2+k$ for $k = 1, 4, 16$, the left side of the equation is never greater than 1 and is equal to 1 if and only if $x = 5$. It follows that $x = 5$ is the only solution of the equation.

525. The circle inscribed in the triangle ABC divides the median from A into three segments of the same length. If the area of ABC is $6\sqrt{14}$, calculate the lengths of its sides.

Solution. Let the median from A meet the side BC at M. Let a, b, c denote the side lengths of ABC as usual, and let the length of the median AM be $3u$. Suppose that the incircle of triangle ABC touches sides BC, CA, AB at U, V, W, respectively. Suppose, wolog, that $AB < AC$, so that U lies between B and M.

By the power of a point, we have that $|AV|^2 = 2u^2 = |MU|^2$, so that

$$(1/2)(b+c-a) = |AV| = |MU| = (1/2)a-(1/2)(a+c-b) = (1/2)(b-c) \ ,$$

and $8u^2 = b^2-2bc+c^2$. Hence $b+c-a = b-c$, whence $a = 2c$ and $|BM| = |MC| = |AB| = c$. By the Law of Cosines applied to triangles ABM and AMC, with $\alpha = \angle AMB$,

$$c^2 = c^2+(3u)^2-6uc\cos\alpha$$

and

$$b^2 = c^2+(3u)^2+6uc\cos\alpha \ ,$$

whence

$$b^2 = c^2+18u^2 = (9/4)(b^2-2bc+c^2) \ .$$

This simplifies to

$$0 = 5b^2-18bc+13c^2 = (b-c)(5b-13c) \ .$$

Since $b \neq c$ (otherwise, the median from A would be the angle bisector of A and the incircle would touch BC at M), we must have $b = 13c/5$. Hence $(a,b,c) = (2c, 13c/5, c)$, the semiperimeter of the triangle is $14c/5$ and the square of its area is $(1/5^4)(14c)(4c)c(9c) = (c^4/5^4)(14)(36)$. Since we are given that the square of the area is $(14)(36)$, $c = 5$ and the dimensions of the triangle are $(10, 5, 13)$.

Comment. All triangles described in the first sentence of the problem have a common property, in that their sides are in the ratio $10 : 5 : 13$. This is, in fact, the essence of the problem. There are many modifications with the same core idea; for example, instead of giving the area of the triangle, we could give the length of the altitude from B, of the angle bisector from C or of the median from A. Recall that these last three quantities are given respectively by

$$h_b = \frac{2}{b}\sqrt{s(s-a)(s-b)(s-c)}$$

$$l_c = \frac{2}{a+b}\sqrt{abs(s-c)}$$

$$m_a = \frac{1}{2}\sqrt{2b^2 + 2c^2 - a^2}$$

where $s = \frac{1}{2}(a+b+c)$ is the semiperimeter of the triangle.

526. For the non-negative numbers a, b, c, prove the inequality

$$4(a+b+c) \geq 3(a + \sqrt{ab} + \sqrt[3]{abc}) \ .$$

When does equality hold?

Solution 1. Equality holds when $a = b = c = 0$. The inequality clearly holds when $a = 0$ and when $b = 0$, so henceforth we will assume that $ab \neq 0$. Define the nonnegative numbers u and v by

$$u^2 = \frac{b}{a} \qquad \text{and} \qquad v^3 = \frac{bc}{a^2} \ .$$

Dividing the inequality through by a, we see that it is equivalent to

$$4\left(1 + u^2 + \frac{v^3}{u^2}\right) \geq 3(1 + u + v)$$

or

$$4(u^2 + u^4 + v^3) \geq 3(u^2 + u^3 + u^2 v) \ .$$

The difference between the two members of the last inequality is

$$4u^4 - 3u^3 + u^2 + 4v^3 - 3u^2 v = u^2(2u-1)^2 + (2v-u)^2(v+u) \ .$$

Because of the square terms, it is always nonnegative, and it is equal to zero if and only if $(u, v) = (1/2, 1/4)$. This is achieved when $a : b : c = 16 : 4 : 1$. Therefore, the inequality always holds and equality occurs when $(a, b, c) = (16t, 4t, t)$ for some nonnegative value of t.

Comment. Since the genesis of the solution is far from obvious, it might be worth commenting on how it was arrived at. It is straightforward to dispose of the cases in which any of the variables vanish, so we may as well suppose that all are positive. We observe that the left and right sides of the inequality are homogeneous of degree 1, so that any scalar mutiple of a solution vector is also a solution. Thus, we might as well assume that $a = 1$. The next step is to get rid of the radicals, which we can do by assuming the quantity under the square root sign is u^2 and under the cube root sign is v^3; it is now a matter of backtracking to define these in terms of a, b and c. Some manipulation gives an equivalent polynomial inequality in terms of u and v. We now look at the difference between the two sides and investigate the possibility of getting some representation of this difference in terms of squares and things known to be positive. However, all these machinations can be avoided by a little insight, as we shall see in the next solution.

$4u^4 - 3u^3 + u^2$ is almost a square, so we might as well complete it by subtracting u^3 and adding it to the rest of the expression to get $(2u^2 - u)^2 + (4v^3 - 3u^2 v + u^3)$. We notice that the expression in the second parentheses vanished when $v = -u$, which makes $v + u$ a factor of it. The remaining factor turns out to be $(2v - u)^2$ and we are finished.

Solution 2. [J. Schneider] Let $a = u$, $b = v/4$ and $c = w/16$. The inequality is equivalent to

$$4\left(u + \frac{v}{4} + \frac{w}{16}\right) \geq 3\left(u + \frac{1}{2}\sqrt{uv} + \frac{1}{4}\sqrt[3]{uvw}\right) \ .$$

Since $3u \geq 3u$, $(3/4)(u+v) \geq (3/2)\sqrt{uv}$ and $(1/4)(u+v+w) \geq 3\sqrt[3]{uvw}$ (the last two by the arithmetic-geometric means inequality), the desired inequality follows. Equality occurs if and only if $u = v = w$, or $a = 4b = 16c$.

Solution 3. The left side of the inequality can be rewritten

$$4(a+b+c) = 3a + \frac{3}{4}(a+4b) + \frac{1}{4}(a+4b+16c) \ .$$

Using the arithmetic-geometric means inequality, we have that

$$a + 4b \geq 2\sqrt{a(4b)} = 4\sqrt{ab}$$

and

$$a + 4b + 16c \geq 3\sqrt[3]{a(4b)(16c)} = 12\sqrt[3]{abc} \ ,$$

from which the desired result follows. Equality occurs if and only if $a = 4b = 16c$.

527. Consider the set A of the $2n-$digit natural numbers, with 1 and 2 each occurring n times as a digit, and the set B of the $n-$digit numbers all of whose digits are 1, 2, 3, 4 with the digits 1 and 2 occurring with equal frequency. Show that A and B contain the same number of elements (*i.e.*, have the same cardinality).

Solution 1. We show that A and B have the same number of elements by pairing off the elements of one set with elements of the other. Suppose that a number in A is given; separate it into n consecutive pairs of digits; these pairs will be one of 11, 12, 21, 22. Observe that, since the digits 1 and 2 occur equally frequently, the pairs 11 and 22 must occur equally frequently. Moving from left to right, we construct an $n-$ digit number by replacing each pair 11 by the digit 1, 22 by the digit 2, 12 by the digit 3 and 21 by the digit 4. Thus, for example, the number 1222112112112212 corresponds to 32143123. Because the number in A has equally many pairs 11 and 22, the corresponding number will have 1 and 2 occurring equally often and will lie in B.

Conversely, given an $n-$digits number in B, construct a $2n-$digit number by replacing each 1 by 11, 2 by 22, 3 by 12 and 4 by 21. Because 1 and 2 occur equally often, the pairs 11 and 22 will occur equally often in the resulting number, which will then belong to A. Thus the correspondence is one-one and the result follows.

Comment. The number of elements in A is $\binom{2n}{n}$, the number of ways of selecting the places for the n ones. To select numbers in B with $r \leq n/2$ digits equal to 1, we can choose the places for the ones in $\binom{n}{r}$ ways, the places for the twos in $\binom{n-r}{r}$ ways. This leaves $n - 2r$ places left over, which can be filled with either threes or fours in 2^{2n-r} ways. Thus, the number of elements in B is

$$\sum_{r=0}^{\lfloor n/2 \rfloor} \binom{n}{r}\binom{n-r}{r} 2^{2n-2} \ .$$

The current problem provides a combinatorial way of verifying the equality of these two expressions. Can you find an alegebraic demonstration?