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Problems 654-691

654. Let ABC be an arbitrary triangle with the points D, E, F on the sides BC, CA, AB respectively, so that
$$\frac{BD}{DC} \leq \frac{BF}{FA} \leq 1$$
and
$$\frac{AE}{EC} \leq \frac{AF}{FB} \quad .$$
Prove that $[DEF] \leq \frac{1}{4}[ABC]$, with equality if and only if two at least of the three points D, E, F are midpoints of the corresponding sides.
(*Note:* $[XYZ]$ denotes the area of triangle XYZ.)

655. (a) Three ants crawl along the sides of a fixed triangle in such a way that the centroid (intersection of the medians) of the triangle they form at any moment remains constant. Show that this centroid coincides with the centroid of the fixed triangle if one of the ants travels along the entire perimeter of the triangle.

(b) Is it indeed always possible for a given fixed triangle with one ant at any point on the perimeter of the triangle to place the remaining two ants somewhere on the perimeter so that the centroid of their triangle coincides with the centroid of the fixed triangle?

656. Let ABC be a triangle and k be a real constant. Determine the locus of a point M in the plane of the triangle for which
$$|MA|^2 \sin 2A + |MB|^2 \sin 2B + |MC|^2 \sin 2C = k \ .$$

657. Let a, b, c be positive real numbers for which $a + b + c = abc$. Find the minimum value of
$$\sqrt{1 + \frac{1}{a^2}} + \sqrt{1 + \frac{1}{b^2}} + \sqrt{1 + \frac{1}{c^2}} \ .$$

658. Prove that $\tan 20° + 4 \sin 20° = \sqrt{3}$.

659. (a) Give an example of a pair a, b of positive integers, not both prime, for which $2a - 1$, $2b - 1$ and $a + b$ are all primes. Determine all possibilities for which a and b are themselves prime.

(b) Suppose a and b are positive integers such that $2a - 1$, $2b - 1$ and $a + b$ are all primes. Prove that neither $a^b + b^a$ nor $a^a + b^b$ are multiples of $a + b$.

660. ABC is a triangle and D is a point on AB produced beyond B such that $BD = AC$, and E is a point on AC produced beyond C such that $CE = AB$. The right bisector of BC meets DE at P. Prove that $\angle BPC = \angle BAC$.

661. Let P be an arbitrary interior point of an equilateral triangle ABC. Prove that
$$|\angle PAB - \angle PAC| \geq |\angle PBC - \angle PCB| \quad .$$

662. Let n be a positive integer and $x > 0$. Prove that
$$(1 + x)^{n+1} \geq \frac{(n + 1)^{n+1}}{n^n} x \quad .$$

663. Find all functions $f : \mathbf{R} \longrightarrow \mathbf{R}$ such that

$$x^2y^2(f(x+y) - f(x) - f(y)) = 3(x+y)f(x)f(y)$$

for all real numbers x and y.

664. The real numbers x, y, and z satisfy the system of equations

$$x^2 - x = yz + 1;$$

$$y^2 - y = xz + 1;$$

$$z^2 - z = xy + 1.$$

Find all solutions (x, y, z) of the system and determine all possible values of $xy + yz + zx + x + y + z$ where (x, y, z) is a solution of the system.

665. Let $f(x) = x^3 + ax^2 + bx + b$. Determine all integer pairs (a, b) for which $f(x)$ is the product of three linear factors with integer coefficients.

666. Assume that a face S of a convex polyhedron $\mathfrak{P}$ has a common edge with every other face of $\mathfrak{P}$. Show that there exists a simple (nonintersecting) closed (not necessarily planar) polygon that consists of edges of $\mathfrak{P}$ and passes through all the vertices.

667. Let A_n be the set of mappings $f : \{1, 2, 3, \cdots, n\} \longrightarrow \{1, 2, 3, \cdots, n\}$ such that, if $f(k) = i$ for some i, then f also assumes all the values $1, 2, \cdots, i-1$. Prove that the number of elements of A_n is $\sum_{k=0}^{\infty} k^n 2^{-(k+1)}$.

668. The nonisosceles right triangle ABC has $\angle CAB = 90°$. The inscribed circle with centre T touches the sides AB and AC at U and V respectively. The tangent through A of the circumscribed circle meets UV produced in S. Prove that

(a) $ST \parallel BC$;

(b) $|d_1 - d_2| = r$, where r is the radius of the inscribed circle and d_1 and d_2 are the respective distances from S to AC and AB.

669. Let $n \geq 3$ be a natural number. Prove that

$$1989 | n^{n^{n^n}} - n^{n^n} \quad ,$$

i.e., the number on the right is a multiple of 1989.

670. Consider the sequence of positive integers $\{1, 12, 123, 1234, 12345, \cdots\}$ where the next term is constructed by lengthening the previous term at the right-hand end by appending the next positive integer. Note that this next integer occupies only one place, with "carrying"occurring as in addition. Thus, the ninth and tenth terms of the sequence are 123456789 and 1234567900 respectively. Determine which terms of the sequence are divisible by 7.

671. Each point in the plane is coloured with one of three distinct colours. Prove that there are two points that are unit distant apart with the same colour.

672. The Fibonacci sequence $\{F_n\}$ is defined by $F_1 = F_2 = 1$ and $F_{n+2} = F_{n+1} + F_n$ for $n = 0, \pm1, \pm2, \pm3, \cdots$. The real number τ is the positive solution of the quadratic equation $x^2 = x + 1$.

(a) Prove that, for each positive integer n, $F_{-n} = (-1)^{n+1}F_n$.

(b) Prove that, for each integer n, $\tau^n = F_n\tau + F_{n-1}$.

(c) Let G_n be any one of the functions $F_{n+1}F_n$, $F_{n+1}F_{n-1}$ and F_n^2. In each case, prove that $G_{n+3} + G_n = 2(G_{n+2} + G_{n+1})$.

673. ABC is an isosceles triangle with $AB = AC$. Let D be the point on the side AC for which $CD = 2AD$. Let P be the point on the segment BD such that $\angle APC = 90°$. Prove that $\angle ABP = \angle PCB$.

674. The sides BC, CA, AB of triangle ABC are produced to the poins R, P, Q respectively, so that $CR = AP = BQ$. Prove that triangle PQR is equilateral if and only if triangle ABC is equilateral.

675. ABC is a triangle with circumcentre O such that $\angle A$ exceeds $90°$ and $AB < AC$. Let M and N be the midpoints of BC and AO, and let D be the intersection of MN and AC. Suppose that $AD = \frac{1}{2}(AB + AC)$. Determine $\angle A$.

676. Determine all functions f from the set of reals to the set of reals which satisfy the functional equation

$$(x - y)f(x + y) - (x + y)f(x - y) = 4xy(x^2 - y^2)$$

for all real x and y.

677. For vectors in three-dimensional real space, establish the identity

$$[\mathbf{a} \times (\mathbf{b} - \mathbf{c})]^2 + [\mathbf{b} \times (\mathbf{c} - \mathbf{a})]^2 + [\mathbf{c} \times (\mathbf{a} - \mathbf{b})]^2 = (\mathbf{b} \times \mathbf{c})^2 + (\mathbf{c} \times \mathbf{a})^2 + (\mathbf{a} \times \mathbf{b})^2 + (\mathbf{b} \times \mathbf{c} + \mathbf{c} \times \mathbf{a} + \mathbf{a} \times \mathbf{b})^2 \ .$$

678. For $a, b, c > 0$, prove that

$$\frac{1}{a(b+1)} + \frac{1}{b(c+1)} + \frac{1}{c(a+1)} \geq \frac{3}{1 + abc} \ .$$

679. Let F_1 and F_2 be the foci of an ellipse and P be a point in the plane of the ellipse. Suppose that G_1 and G_2 are points on the ellipse for which PG_1 and PG_2 are tangents to the ellipse. Prove that $\angle F_1PG_1 = \angle F_2PG_2$.

680. Let $u_0 = 1$, $u_1 = 2$ and $u_{n+1} = 2u_n + u_{n-1}$ for $n \geq 1$. Prove that, for every nonnegative integer n,

$$u_n = \sum \left\{ \frac{(i + j + k)!}{i!j!k!} : i, j, k \geq 0, i + j + 2k = n \right\} .$$

681. Let $\mathbf{a}$ and $\mathbf{b}$, the latter nonzero, be vectors in $\mathbb{R}^3$. Determine the value of λ for which the vector equation

$$\mathbf{a} - (\mathbf{x} \times \mathbf{b}) = \lambda \mathbf{b}$$

is solvable, and then solve it.

682. The plane is partitioned into n regions by three families of parallel lines. What is the least number of lines to ensure that $n \geq 2010$?

683. Let $f(x)$ be a quadratic polynomial. Prove that there exist quadratic polynomials $g(x)$ and $h(x)$ for which

$$f(x)f(x + 1) = g(h(x)) \ ,$$

684. Let x, y, z be positive reals for which $xyz = 1$. Prove that

$$\frac{x + y}{x^2 + y^2} + \frac{y + z}{y^2 + z^2} + \frac{z + x}{z^2 + x^2} \leq \sqrt{x} + \sqrt{y} + \sqrt{z} \ .$$

685. Let $f : \mathbf{R} \rightarrow \mathbf{R}$ be defined by

$$f(x) = x - 4\lfloor x \rfloor + \lfloor 2x \rfloor \ ,$$

where $\lfloor \cdot \rfloor$ represents the greatest integer that does not exceed the argument. Determine $f(f(x))$ and show that f is a surjective (onto) function.

686. Solve the equation

$$\sqrt{6+3\sqrt{2+\sqrt{2+x}}}+\sqrt{2-\sqrt{2+\sqrt{2+x}}}=2x\ .$$

687. Prove that

$$\frac{(1+2+3+\cdots+n)!}{1!2!\ldots n!}$$

is a natural number for any positive integer n.

688. Solve the equation

$$2010^x+2010^{-x}=1+2x-x^2\ .$$

689. Let BC e a diameter of the circle $\mathfrak{C}$ and let A be an interior point. Suppose that BA and CA intersect the circle $\mathfrak{C}$ at D and E respectively. If the tangents to the circle $\mathfrak{C}$ at E and D intersect at the point M, prove that $AM \perp BC$.

690. Let m_a, m_b, m_c; h_a, h_b, h_c be the lengths of the medians and the heights of triangle ABC, where the notation is used conventionally.

(a) If $a \le b \le c$, prove that $h_a \ge h_b \ge h_c$ and that $m_a \ge m_b \ge m_c$.

(b) If

$$\left(\frac{h_a^2}{h_b\cdot h_c}\right)^{m_a}\cdot\left(\frac{h_b)^2}{h_c\cdot h_a}\right)^{m_b}\cdot\left(\frac{h_c)^2}{h_a\cdot h_b}\right)^{m_c}=1\ ,$$

prove that triangle ABC is equilateral.

691. Prove that

$$\sqrt{\sqrt[3]{x}+\sqrt[3]{y}+\sqrt[3]{z}}>\sqrt[3]{\sqrt{x}+\sqrt{y}+\sqrt{z}}$$

for positive integers x, y, z.

Solutions.

654. Let ABC be an arbitrary triangle with the points D, E, F on the sides BC, CA, AB respectively, so that

$$\frac{BD}{DC}\le\frac{BF}{FA}\le 1$$

and

$$\frac{AE}{EC}\le\frac{AF}{FB}\quad .$$

Prove that $[DEF] \le \frac{1}{4}[ABC]$, with equality if and only if two at least of the three points D, E, F are midpoints of the corresponding sides.
(*Note:* $[XYZ]$ denotes the area of triangle XYZ.)

Solution 1. Let $BF = \mu BA$, $BD = \lambda BC$ and $CE = \nu CA$.

The conditions are that

$$\lambda \le \mu \le \frac{1}{2} \qquad \text{and} \qquad 1-\nu \le 1-\mu \quad \text{or} \quad \mu \le \nu\ .$$

We observe that $[BDF] = \lambda\mu[ABC]$.

To see this, let $BG = \lambda BA$. Then

$$[BDF] = \frac{\mu}{\lambda}[BGD] = \frac{\mu}{\lambda}\lambda^2[ABC] = \mu\lambda[ABC] \ .$$

Similarly $[AFE] = (1-\mu)(1-\lambda)[ABC]$ and $[DEC] = \nu(1-\lambda)[ABC]$.

Hence

$$\begin{aligned}[DEF] &= (1 - \lambda\mu - (1-\mu)(1-\nu) - \nu(1-\lambda))[ABC] \\ &= (\mu - \mu\nu - \mu\lambda + \nu\lambda)[ABC] \\ &= \left(\frac{1}{4} - (\frac{1}{2} - \mu)^2 - (\mu - \lambda)(\nu - \mu)\right)[ABC] \le \frac{1}{4}[ABC]\end{aligned}$$

with equality if and only if $\mu = 1/2$ and either $\lambda = \mu = 1/2$ or $\nu = \mu = 1/2$. The result follows.

Solution 2. Let G be on AC so that $FG \| BC$. Then, since $\frac{AE}{EC} \le \frac{AF}{FB}$, E lies in the segment AG.

Since $\frac{BD}{DC} \le \frac{BF}{FA}$, DF produced is either parallel to AC or meets CA produced at a point X beyond A. Hence the distance from G to FD is not less than the distance from E to FD, so that $[DEF] \le [FGD]$. The area of $[FGD]$ does not change as D varies along BC. To maximize $[DEF]$ is suffices to consider the special case of triangle $[FGD]$. Let $AF = xAB$. Then $FG = xBC$ and the heights of ΔDFG and ΔABC are in the ratio $1 - x$. Hence

$$\frac{[DFG]}{[ABC]} = x(1-x)$$

which is maximized when $x = \frac{1}{2}$. The result follows from this, with $[DEF]$ being exactly one quarter of $[ABC]$ when F and G are the midpoints of AB and AC respectively.

Solution 3. Set up the situation as in the second solution. Let $BF = tFA$. Then $AB = (1+t)FA$, and the height of the triangle FGD is $t/(1+t)$ times the height of the triangle ABC. Hence

$$[DEF] \le [FGD] = \frac{t}{(1+t)^2}[ABC] \ .$$

Now

$$\frac{1}{4} - \frac{t}{(1+t)^2} = \frac{(1-t)^2}{4(1+t)^2} \ge 0$$

so that $t(1+t)^{-2} \le 1/4$ and the result follows. Equality occurs if and only if $t = 1$ and $E = G$, *i.e.*, F and E are both midpoints of their sides.

655. (a) Three ants crawl along the sides of a fixed triangle in such a way that the centroid (intersection of the medians) of the triangle they form at any moment remains constant. Show that this centroid coincides with the centroid of the fixed triangle if one of the ants travels along the entire perimeter of the triangle.

(b) Is it indeed always possible for a given fixed triangle with one ant at any point on the perimeter of the triangle to place the remaining two ants somewhere on the perimeter so that the centroid of their triangle coincides with the centroid of the fixed triangle?

(a) *Solution.* Recall that the centroid lies two-thirds of the way along the median from a vertex of the triangle to its opposite side. Let ABC be the fixed triangle and let $PQ \| BC$, $RS \| AC$ and $TU \| BA$ with PQ, RS and TU intersecting in the centroid G.

Observe, for example, that if A, X, Y are collinear and X and Y lie on PQ and BC respectively, then $AX : XY = 2 : 3$. It follows from this that, if one ant is at A, then the centroid of the triangle formed by the three ants lies inside ΔAPQ (otherwise the midpoint of the side opposite the ant at A would not be in ΔABC). Similarly, if one ant is at B (respectively C) then the centroid of the ants' triangle lies within ΔBRS (respectively ΔCTU). Thus, if one ant traverses the entire perimeter, the centroid of the ants' triangle must lie inside the intersection of these three triangles, the singleton $\{G\}$. The result follows.

(b) *Solution 1.* Suppose the vertices of the triangle are given by the planar vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$; the centroid of the triangle is at $\frac{1}{3}(\mathbf{a}+\mathbf{b}+\mathbf{c})$. Suppose that one ant is placed at $t\mathbf{a}+(1-t)\mathbf{b}$ for $0 \le t \le 1$. Place the other two ants at $t\mathbf{b}+(1-t)\mathbf{c}$ and $t\mathbf{c}+(1-t)\mathbf{a}$. The centroid of the ants' triangle is at

$$\frac{1}{3}[(t\mathbf{a}+(1-t)\mathbf{b})+(t\mathbf{b}+(1-t)\mathbf{c})+(t\mathbf{c}+(1-t)\mathbf{a}) = \frac{1}{3}(\mathbf{a}+\mathbf{b}+\mathbf{c}) \ .$$

(b) *Solution 2.* If one ant is at a vertex, then we can replace the remaining ants at the other vertices of the fixed triangle. Suppose, wolog, the ant is at X in the side BC.

Let MN be the line joining the midpoints M and N of AB and AC respectively; $MN \| BC$. Let XG meet MN at W. Since $BG : BN (= CG : CM) = 2 : 3$, it follows, by considering the similar triangles BGX and NGW, that $XG : XW = 2 : 3$. Hence the midpoint of the segment joining the other two ants' positions must be at W. Thus, the problem now is to find points Y and Z on the perimeter of ΔABC such that W is the midpoint of YZ. We use a continuity argument.

Let UV be any segment containing W whose endpoints lie on the perimeter of ΔABC. Let Y travel counterclockwise around the perimeter from U to V, and let Z be a point on the perimeter such that W lies on YZ. When Y is at U, $YW : WZ = VW : WV$, while when Y is at V, $YW : WZ = VW : WU$. Hence $YW : WZ$ varies continuously from a certain ratio to its reciprocal, so there must be a position for which $YW = WZ$.

(b) *Solution 3.* [A. Panayotov] Suppose that the triangle has vertices at $(0,0)$, $(1,0)$ and (u,v), so that its centroid is at $(\frac{1}{3}(1+u), \frac{v}{3})$. Wolog, let one ant be at $(a,0)$ where $0 \le a \le 1$. Put the second ant at (u,v). Then we will place the third ant at a point $(b,0)$ on the $x-$axis. We require that $\frac{1}{3}(a+b+u) = \frac{1}{3}(1+u)$, so that $b = 1-a$. Clearly, $0 \le b \le 1$ and the result follows.

656. Let ABC be a triangle and k be a real constant. Determine the locus of a point M in the plane of the triangle for which

$$|MA|^2 \sin 2A + |MB|^2 \sin 2B + |MC|^2 \sin 2C = k \ .$$

Solution. Let O and R be the circumcentre and circumradius, respectively, of triangle ABC. We have that

$$\begin{aligned} |MA|^2 &= |\overrightarrow{MA}|^2 = |\overrightarrow{MO} + \overrightarrow{OA}|^2 \\ &= |\overrightarrow{MO}|^2 + |\overrightarrow{OA}|^2 + 2\overrightarrow{MO} \cdot \overrightarrow{OA} \\ &= |\overrightarrow{MO}|^2 + R^2 + 2\overrightarrow{MO} \cdot \overrightarrow{OA} \end{aligned}$$

with similar expressions for MB and MC. Therefore, we have that

$$\begin{aligned} |MA|^2 \sin 2A + |MB|^2 \sin 2B + |MC|^2 \sin 2C &= (|MO|^2 + R^2)(\sin 2A + \sin 2B + \sin 2C) \\ & 2\overrightarrow{MO} \cdot (\overrightarrow{OA} \sin 2A + \overrightarrow{OB} \sin 2B + \overrightarrow{OC} \sin 2C) \ . \end{aligned}$$

Now

$$\begin{aligned} \sin 2A + \sin 2B + \sin 2C &= \sin 2A + \sin 2B - \sin(2A + 2B) \\ &= \sin 2A(1 - \cos 2B) + \sin 2B(1 - \cos 2A) \\ &= 2 \sin A \cos A(2 \sin^2 B) + 2 \sin B \cos B(2 \sin^2 A) \\ &= 4 \sin A \sin B \sin(A + B) = 4 \sin A \sin B \sin C \\ &= \frac{2[ABC]}{R^2} \ , \end{aligned}$$

since $[ABC] = \frac{1}{2}ab\sin C = 2R^2 \sin A \sin B \sin C$.

Also, we have that

$$\overrightarrow{OA}\sin 2A + \overrightarrow{OB}\sin 2B + \overrightarrow{OC}\sin 2C = \overrightarrow{O} \ .$$

To see this, let P be the intersection of the line AO with the side BC of the triangle. Observe that $\angle BOP = 180° - 2\angle ACB$, $\angle COP = 180° - 2\angle ABC$, $\angle OBC = \angle OCB = 90° - \angle BAC$. Applying the Law of Sines to triangle OPC yields that

$$\frac{|OP|}{\sin(90° - A)} = \frac{|OC|}{\sin(2C + A - 90°)} \ .$$

Since $|OC| = R$, we find that

$$\begin{aligned} |OA| &= \frac{-\cos(2C + A)}{\cos A}|OP| = \frac{-2\sin A\cos(2C + A)}{2\sin A\cos A}|OP| \\ &= \frac{\sin 2B + \sin 2C}{\sin 2A}|OP| \ , \end{aligned}$$

so that

$$\overrightarrow{OA} = -\frac{\sin 2B + \sin 2C}{\sin 2A}\overrightarrow{OP} \ .$$

Applying the Law of Sines in triangle BOP and COP, we obtain that

$$\frac{|OP|}{\sin(90° - A)} = \frac{|BP|}{\sin 2C}$$

and

$$\frac{|OP|}{\sin(90° - A)} = \frac{|CP|}{\sin 2B} \ .$$

Therefore $|BP|\sin 2B = |CP|\sin 2C$, so that

$$\sin 2B\overrightarrow{PB} = -\sin 2C\overrightarrow{PC}$$

and

$$\begin{aligned} \overrightarrow{OA}\sin 2A + \overrightarrow{OB}\sin 2B + \overrightarrow{OC}\sin 2C &= -(\sin 2B + \sin 2C)\overrightarrow{OP} + \sin 2B\overrightarrow{OB} + \sin 2C\overrightarrow{OC} \\ &= \sin 2B(\overrightarrow{OB} - \overrightarrow{OP}) + \sin 2C(\overrightarrow{OC} - \overrightarrow{OP}) \\ &= \sin 2B\overrightarrow{PB} + \sin 2C\overrightarrow{PC} = \overrightarrow{O} \ . \end{aligned}$$

Therefore $(|MO|^2 + R^2)(2[ABC]/R^2) = k$ so that

$$|MO|^2 = \frac{k - 2[ABC]}{2[ABC]}R^2 \ .$$

Therefore, when $k < 2[ABC]$, the locus is the empty set. When $k = 2[ABC]$, the locus consists solely of the circumcentre. When $k > 2[ABC]$, the locus is a circle concentric with the circumcircle.

657. Let a, b, c be positive real numbers for which $a + b + c = abc$. Find the minimum value of

$$\sqrt{1 + \frac{1}{a^2}} + \sqrt{1 + \frac{1}{b^2}} + \sqrt{1 + \frac{1}{c^2}} \ .$$

Solution 1. By repeated squaring it can be shown that

$$\sqrt{x^2 + u^2} + \sqrt{y^2 + b^2} \geq \sqrt{(x + u)^2 + (y + v)^2} \ ,$$

for $x, y, u, v \geq 0$. Applying this inequality yields that

$$\sqrt{1+\frac{1}{a^2}}+\sqrt{1+\frac{1}{b^2}}+\sqrt{1+\frac{1}{c^2}} \geq \sqrt{(1+1)^2+(\frac{1}{a}+\frac{1}{b})^2}+\sqrt{1+\frac{1}{c^2}}$$
$$\geq \sqrt{(2+1)^2+(\frac{1}{a}+\frac{1}{b}+\frac{1}{c})^2} \; .$$

The given condition implies that $\frac{1}{ab}+\frac{1}{bc}+\frac{1}{ca}$, whereupon

$$\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)^2 \geq 2+\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2} \geq 2+\frac{1}{ab}+\frac{1}{bc}+\frac{1}{ca}=3 \; .$$

It follows that the given expression is not less than $2\sqrt{3}$, with equality occurring if and only if $a=b=c=\sqrt{3}$.

Solution 2. [S. Sun] Using the inequality $x^2+y^2+z^2 \geq xy+yz+zx$ for real x, y, z, we find that the square of the quantity in question is not less than

$$3\left(\sqrt{1+\frac{1}{a^2}}\sqrt{1+\frac{1}{b^2}}+\sqrt{1+\frac{1}{b^2}}\sqrt{1+\frac{1}{c^2}}+\sqrt{1+\frac{1}{c^2}}\sqrt{1+\frac{1}{a^2}}\right) \; .$$

From the Arithmetic-Geometric Means Inequality, we find that

$$\sqrt{1+\frac{1}{a^2}}\sqrt{1+\frac{1}{b^2}}=\sqrt{1+\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{a^2b^2}} \geq \sqrt{1+\frac{2}{ab}+\frac{1}{a^2b^2}}=1+\frac{1}{ab} \; ,$$

with similar inequalities for the other products. Since

$$\frac{1}{ab}+\frac{1}{bc}+\frac{1}{ca}=\frac{a+b+c}{abc}=1 \; ,$$

we find that the square of the quantity in question is not less than $3 \times 4 = 12$, so that the quantity has the minimum value $2\sqrt{3}$, attainable if and only is $a=b=c=\sqrt{3}$.

Solution 3. Let A, B, C be acute angles for which $a=\tan A$, $b=\tan B$ and $c=\tan C$. Then

$$c=-\frac{a+b}{1-ab}=-\frac{\tan A+\tan B}{1-\tan A\tan B}$$
$$=-\tan(A+B)=\tan(\pi-A-B) \; ,$$

so that $C=\pi-A-B$. Substituting these values fo a, b, c into the given expression yields

$$\csc A+\csc B+\csc C$$

. Since the cosecant function is convex in the interval $(0, \pi/2)$, by Jensen's inequality, we deduce that

$$\csc A+\csc B+\csc C \geq 3\csc\left(\frac{A+B+C}{3}\right)=3\csc\frac{\pi}{3}=2\sqrt{3} \; ,$$

with equality if and only if $A=B=C=\frac{\pi}{3}$. Thus, the minimum of the given expression is equal to $2\sqrt{3}$ with equality if and only is $a=b=c=\sqrt{3}$.

658. Prove that $\tan 20° + 4\sin 20° = \sqrt{3}$.

Solution 1. [CJ. Bao] Since

$$(\sqrt{3}/2)\cos 20° - (1/2)\sin 20° = \sin 60°\cos 20° - \cos 60°\sin 20° = \sin 40° = 2\sin 20°\cos 20° \; ,$$

it follows that

$$\sqrt{3}\cos 20^\circ = \sin 20^\circ + 4\sin 20^\circ \cos 20^\circ a \ .$$

Division by $\cos 20^\circ$ yields the desired result.

Solution 2. Let ABC be a triangle with $\angle ABC = 60^\circ$ and $\angle CAB = 30^\circ$. Let ABD be a triangle on the same side of AB with $\angle ABD = 40^\circ$ and $\angle DAB = 50^\circ$. Suppose that AC and BD intersect at E, and that the length of BC is 1, so that the respective lengths of CA and AB are $\sqrt{3}$ and 2. Then

$$|AD| = |AB|\sin 40^\circ = 4\sin 20^\circ \cos 20^\circ$$

and

$$|AE| = |AD|\sec 20^\circ = |AB|\cos 50^\circ \sec 20^\circ = 2\sin 40^\circ \sec 20^\circ = 4\sin 20^\circ \ .$$

However, $|CE| = |BC|\tan 20^\circ = \tan 20^\circ$. Therefore

$$\tan 20^\circ + 4\sin 20^\circ = |CE| + |AE| = |AC| = \sqrt{3} \ .$$

Solution 3. [M. Essafty]

$$\begin{aligned}
\tan 20^\circ + 4\sin 20^\circ &= \frac{\sin 20^\circ + 4\sin 20^\circ \cos 20^\circ}{\cos 20^\circ} \\
&= \frac{\sin 20^\circ + 2\sin 40^\circ}{\cos 20^\circ} \\
&= \frac{\sin(30^\circ - 10^\circ) + 2\sin(30^\circ + 10^\circ}{\cos(30^\circ - 10^\circ} \\
&= \frac{3\sin 30^\circ \cos 10^\circ + \sin 10^\circ \cos 30^\circ}{\cos 30^\circ \cos 10^\circ + \sin 30^\circ \sin 10^\circ} \\
&= \frac{3\cos 10^\circ + \sqrt{3}\sin 10^\circ}{\sqrt{3}\cos 10^\circ + \sin 10^\circ} = \sqrt{3} \ .
\end{aligned}$$

Solution 4.

$$\begin{aligned}
\tan 20^\circ + 4\sin 20^\circ &= \frac{\sin 20^\circ + 4\sin 20^\circ \cos 20^\circ}{\cos 20^\circ} = \frac{\sin 20^\circ + 2\sin 40^\circ}{\cos 20^\circ} \\
&= \frac{\sin 40^\circ + 2\sin 30^\circ \cos 10^\circ}{\cos 20^\circ} = \frac{\sin 40^\circ + \sin 80^\circ}{\cos 20^\circ} \\
&= \frac{2\sin 60^\circ \cos 20^\circ}{\cos 20^\circ} = \sqrt{3} \ .
\end{aligned}$$

Solution 5.

$$\begin{aligned}
\tan 20^\circ + 4\sin 20^\circ &= \frac{\sin 20^\circ + 4\sin 20^\circ \cos 20^\circ}{\cos 20^\circ} = \frac{\sin 20^\circ + 2\sin 40^\circ}{\cos 20^\circ} \\
&= \frac{\sin 50^\circ \cos 30^\circ - (1/2)\cos 50^\circ + 2\sin 40^\circ}{\cos 20^\circ} \\
&= \frac{\sin 50^\circ \cos 30^\circ + (1/2)\cos 50^\circ + \cos 50^\circ}{\cos 20^\circ} \\
&= \frac{\sin 80^\circ + \cos 50^\circ}{\cos 20^\circ} = \frac{\cos 10^\circ + \cos 50^\circ}{\cos 20^\circ} \\
&\frac{2\cos 30^\circ \cos 20^\circ}{\cos 20^\circ} = \sqrt{3} \ .
\end{aligned}$$

Solution 6. Let $a = \cos 20°$. Then, using the de Moivre formula $\cos 3\theta + i \sin 3\theta = (\cos\theta + i\sin\theta)^3$ with $\theta = 20°$, we find that

$$\frac{1}{2} = \cos 60° = 4a^3 - 3a$$

and

$$\frac{\sqrt{3}}{2} = 3\sin 20° - 4\sin^3 20° = \sin 20°(3 - 4(1 - a^2)) = \sin 20°(4a^2 - 1) \ .$$

Therefore

$$\tan 20° + 4\sin 20° - \sqrt{3} = \sin 20°[(1/a) + 4 - 8a^2 + 2) = a^{-1}\sin 20°(1 + 6a - 8a^3) = 0 \ .$$

Solution 7. [B. Wu]

$$\begin{aligned} \tan 60° - \tan 20° &= \frac{\sin 60°}{\cos 60°} - \frac{\sin 20°}{\cos 20°} \\ &= \frac{\sin 40°}{\cos 60° \cos 20°} = 4\sin 20° \cos 40° \mathit{over} \cos 20° = 4\sin 20° \ , \end{aligned}$$

whence $\tan 20° + 4\sin 20° = \sqrt{3}$.

659. (a) Give an example of a pair a, b of positive integers, not both prime, for which $2a-1$, $2b-1$ and $a+b$ are all primes. Determine all possibilities for which a and b are themselves prime.

(b) Suppose a and b are positive integers such that $2a-1$, $2b-1$ and $a+b$ are all primes. Prove that neither $a^b + b^a$ nor $a^a + b^b$ are multiples of $a+b$.

(a) *First solution.* $(a,b) = (3,2)$ yields $2a-1 = 5$, $2b-1 = 3$ and $a+b = 5$; $(a,b) = (3,4)$ yields $2a-1 = 5$, $2b-1 = 7$ and $a+b = 7$. Suppose that a and b are primes. Then for $a+b$ to be prime, $a+b$ must be odd, so that one of a and b, say b, is equal to 2. Thus, we require the $a+2$ and $2a-1$, along with a, to be prime. This is true when $a = 3$.

Now suppose a is an odd prime exceeding 3. Then $a \equiv \pm 1 \pmod 6$, so the only way a and $a+2$ can both be prime is for $a \equiv -1 \pmod 6$, whence $2a - 1 \equiv -3 \pmod 6$. Thus, 3 divides $2a-1$, and since $2a - 1 \geq 9$, $2a-1$ must be composite.

(b) *Solution 1.* We first recall a bit of theory. Let p be a prime. By Fermat's Little Theorem, $a^{p-1} \equiv 1 \pmod p$ whenever $\gcd(a,p) = 1$. Let d be the smallest positive integer for which $a^d \equiv \pm 1 \pmod p$. Then d divides $p-1$, and indeed divides any positive integer k for which $a^k \equiv \pm 1 \pmod p$. Now to the problem.

Since $a+b$ is prime, $a \neq b$. Wolog, let $a > b$ and let $p = a+b$. Then $a \equiv -b \pmod p$, so that

$$a^b + b^a \equiv (-b)^b + b^a \equiv b^b((-1)^b + b^{a-b}) \ .$$

Suppose, if possible, that p divides $a^b + b^a$. Then, since $b < p$, $\gcd(b,p) = 1$ and so $b^{a-b} \equiv (-1)^{b+1} \pmod p$. It follows that

$$b^{2b-1} = b^{(p-1)-(a-b)} \equiv (-1)^{b+1} \mod p \ .$$

Now $2b-1$ is prime, so that $2b-1$ must be the smallest exponent d for which $b^d \equiv \pm 1 \pmod p$. Hence $2b-1$ divides $a-b$, so that for some positive integer c, $a - b = c(2b-1)$, whence $a = b + 2bc - c$ and so

$$2a - 1 = 2b - 1 + (2b-1)2c = (2b-1)(2c+1) \ .$$

But $2a-1$ is prime and $2b - 1 > 1$, so $2c+1 = 1$ and $c = 0$. This is a contradiction. Hence p does not divide $a^b + b^a$.

Similarly, using the fact that $a^b + b^a \equiv (-b)^a + b^b \equiv b^b((-1)^a b^{a-b} + 1)$, we can show that p does not divide $a^a + b^b$.

(b) *Solution 2.* [M. Boase] Suppose that a and b exist as specified. Exactly one of a and b is odd, since $a + b$ is prime. Let it be a. Modulo $a + b$, we have that

$$0 \equiv a^b + b^a = a^b + (-a)^a \equiv a^b - a^a \equiv a^a(a^{b-a} - 1) \quad \text{or} \quad a^b(1 - a^{a-b})$$

according as $a < b$ or $a > b$. Hence $a^{|b-a|} - 1 \equiv 0 \pmod{a+b}$. Now $a + b - 1 \pm |b - a| = 2a - 1$ or $2b - 1$, and $a^{a+b-1} \equiv 1 \pmod{a+b}$ (by Fermat's Little Theorem). Hence $a^{2a-1} \equiv a^{2b-1} \equiv 1 \pmod{a+b}$. Both $2a - 1$ and $2b - 1$ exceed 1 and are divisible by the smallest value of m for which $a^m \equiv 1 \pmod{a+b}$. Since both are prime, $2a - 1 = 2b - 1 = m$, whence $a = b$, a contradiction. A similar argument can be applied to $a^a + b^b$.

(c) *Solution 3.* Suppose, if possible, that one of $a^b + b^a$ and $a^a + b^b$ is divisible by $a + b$. Then $a + b$ divides their product $a^{a+b} + (ab)^a + (ab)^b + b^{a+b}$. By Fermat's Little Theorem, $a^{a+b} + b^{a+b} \equiv a + b \equiv 0 \pmod{a+b}$, so that $(ab)^a + (ab)^b \equiv 0 \pmod{a+b}$. Since $a + b$ is prime, it is odd and so $a \neq b$. Wolog, let $a > b$. Then

$$(ab)^a + (ab)^b = (ab)^b[(ab)^{a-b} + 1]$$

and $\gcd(a, a + b) = \gcd(b, a + b) = 1$, so that $(ab)^{a-b} + 1 \equiv 0 \pmod{a+b}$. Since $(ab)^{a+b-1} \equiv 1 \pmod{a+b}$, it follows that $(ab)^{2a-1} \equiv (ab)^{2b-1} \equiv -1 \pmod{a+b}$. As in the foregoing solution, it follows that $a = b$, and we get a contradiction.

660. ABC is a triangle and D is a point on AB produced beyond B such that $BD = AC$, and E is a point on AC produced beyond C such that $CE = AB$. The right bisector of BC meets DE at P. Prove that $\angle BPC = \angle BAC$.

Solution 1. Let the lengths a, b, c, u and the angles α, β, γ, λ, μ, ν be as indicated in the diagram.

In the solution, we make use of the fact that if $p/q = r/s$, then both fractions are equal to $(p+r)/(q+s)$. Since $\angle DBP = 90° + \lambda - 2\beta$, it follows that

$$2\mu = 180° - (90° - \alpha) - (90° + \lambda - 2\beta) = \alpha + 2\beta - \lambda \ .$$

Similarly, $2\nu = \alpha + 2\gamma - \lambda$. Using the Law of Sines, we find that

$$\frac{a}{\sin 2\alpha} = \frac{b}{\sin 2\beta} = \frac{c}{\sin 2\gamma} = \frac{b+c}{\sin 2\beta + \sin 2\gamma} = \frac{b+c}{2\sin(\beta+\gamma)\cos(\beta-\gamma)}$$
$$= \frac{b+c}{2\cos\alpha\cos(\beta-\gamma)} \ .$$

Hence

$$\frac{a}{\sin\alpha} = \frac{b+c}{\cos(\beta-\gamma)} \ .$$

Since $a = 2u\sin\lambda$ and, by the Law of Sines,

$$\frac{u}{\sin(90° - \alpha)} = \frac{b}{\sin 2\mu} \quad \text{and} \quad \frac{u}{\sin(90° - \alpha)} = \frac{c}{\sin 2\nu} \ ,$$

we have that

$$\frac{a}{2\sin\lambda\cos\alpha} = \frac{u}{\cos\alpha} = \frac{b}{\sin 2\mu} = \frac{c}{\sin 2\nu} = \frac{b+c}{\sin 2\mu + \sin 2\nu}$$
$$= \frac{b+c}{2\sin(\mu+\nu)\cos(\mu-\nu)} = \frac{b+c}{2\cos\lambda\cos(\beta-\gamma)} = \frac{a}{2\cos\lambda\sin\alpha} \ .$$

Hence $\tan\alpha = \tan\lambda$ and so $\alpha = \lambda$.

Solution 2. Let M be the midpoint of BC. A rotation of $180°$ about M interchanges B and C and takes E to G, D to F and P to Q. Then $AB = CE = BG$ and $AC = BD = CF$. Join GA and FA. Let $2\alpha = \angle BAC$. Since $AE \| BG$ and AB is a transversal, $\angle GBA = \angle BAC = 2\alpha$. Since $AB = BG$, $\angle BGA = 90° - \alpha$. But $\angle BGF = \angle CED = 90° - \alpha$. Thus, G, A, F are collinear.

Since GF and DE are equidistant from M, we can use Cartesian coordinates with the origin at M, the line $y = 1$ as GF and the line $y = -1$ as DE. Let $A \sim (a, 1)$, $B \sim (-u, -mu)$, $C \sim (u, mu)$. Then $P \sim (m, -1)$, $Q \sim (-m, 1)$,

$$D \sim (a - \frac{2(a+u)}{1+mu}, -1), \qquad E \sim (a + \frac{2(a+u)}{1+mu}, -1) \ .$$

Since $|AC| = |BD|$, we find that $u - a = -u - a + \frac{2(a+u)}{1+mu}$, or $a = mu^2$. (We can check this by equating the slopes of AC and AE.)

The slope of AE is $-1/u$ and of AD is $1/u$, so that

$$\tan\angle BAC = \frac{-(2/u)}{1-(1/u^2)} = -\frac{2u}{u^2-1} \ .$$

The slope of CQ is $(mu-1)/(m+u)$ and of BQ is $(1+mu)/(u-m)$, so that

$$\begin{aligned}\tan\angle BPC = \tan\angle BQC &= \frac{(mu-1)(u-m) - (mu+1)(u+m)}{(u-m)(u+m) + (mu-1)(mu+1)} \\ &= \frac{-2(m^2u+u)}{u^2-m^2+m^2u^2-1} = \frac{-2(m^2+1)u}{(1+m^2)(u^2-1)} = \frac{-2u}{u^2-1} \ .\end{aligned}$$

The result follows.

Solution 3. [M. Boase] Let XAY be drawn parallel to DE.

Since M is the midpoint of BC, the distance from M to DE is the average of the distances from B and C to DE. Similarly, the distance from M to XY is the average of the distances from B and C to XY. The distance of B (resp. C) to DE equals the distance of C (resp. B) to XY. Hence, M is equidistant from DE and XY. If PM produced meets XY in Q, then $PM = MQ$ and so $\angle BQC = \angle BPC$.

Select R on MQ (possibly produced) so that $\angle BAC = \angle BRC$. Since $\Delta ADE ||| \Delta RBC$, $\angle RBC = \angle RCB = \angle ADE$. Since $BARC$ is a concyclic quadrilateral, $\angle BAR = 180° - \angle RCB = 180° - \angle ADE = 180° - \angle XAD = \angle BAQ$ from which it follows that $R = Q$ and so $\angle BPC = \angle BQC = \angle BRC = \angle BAC$.

Solution 4. [Jimmy Chui] Set coordinates: $A \sim (0, (m+n)b)$, $B \sim (-ma, nb)$, $C \sim (na, mb)$ $D \sim (-(m+n)a, 0)$ and $E \sim ((m+n)a, 0)$ where $m = |AB|$, $n = |AC|$ and $a^2 + b^2 = 1$. Then the line BC has the equation

$$\frac{m-n}{a}x - \frac{m+n}{b}y + m^2 + n^2 = 0$$

and the right bisector of BC has equation

$$\frac{m+n}{b}x + \frac{m-n}{a}y + \frac{(a^2-b^2)(m^2-n^2)}{2ab} = 0 \ .$$

Thus

$$P \sim \left(\frac{(b^2-a^2)(m-n)}{2a}, 0\right) \ .$$

Now

$$|BC|^2 = m^2 + n^2 + 2mn(a^2-b^2)$$

and

$$|BP|^2 = \frac{m^2 + n^2 + 2mn(a^2 - b^2)}{4a^2}$$

so that $|BC|/|BP| = 2a$. Also $|DE|/|AD| = 2(m+n)a/(m+n) = 2a$ so that ΔBPC is similar to ΔADE and the result follows.

Solution 5. Determine points L and N on DE such that $BL \| AE$ and $LN = NE$. Now

$$\frac{LE}{LD} = \frac{AB}{BD} = \frac{CE}{CA}$$

so that $CL \| AD$ and $CL : AD = CE : AE$. Since $AD = DE$, $CL = CE$ and so $CN \perp LE$. Consider the trapezoid $CBLE$. The line MN joins the midpoints of the nonparallel opposite sides and so $MN \| BL$. $MPNC$ is a quadrilateral with right angles at M and N, and so is concyclic. Hence

$$\angle BPC = 2\angle MPC = 2\angle MNC = 2\angle NCE = \angle LCE = \angle BAC \quad .$$

Solution 6. [C. So] Let F, N, G be the feet of the perpendiculars dropped from B, M, C respectively to DE. Note that $FN = NG$, so that $MF = MG$. Let $\angle ADE = \angle AED = \theta$, $|AB| = c$, $|AC| = b$ and h be the altitude of ΔADE. Then

$$|MN| = \frac{1}{2}[|BF| + |CG|] = \frac{1}{2}(b+c)\sin\theta = \frac{h}{2}$$

and

$$|DF| = b\cos\theta \ , \quad |GE| = c\cos\theta \ , \quad |DE| = 2(b+c)\cos\theta \quad .$$

Hence $|FG| = |DE| - |DF| - |GE| = \frac{1}{2}|DE|$. Since ΔADE and ΔMFG are isosceles triangles with heights and beses in proportion, they are similar so that $\angle MFG = \angle ADE = \theta$. Since $\angle BFP = \angle BMG = 90°$, the quadrilateral $BFPM$ is concyclic and so $\angle CBP = \angle MFP = \theta$ (we are supposing that the configuration is labelled so P lies between F and E). Hence ΔADE is similar to ΔPCB and so $\angle BPC = \angle BAC$.

Solution 7. [A. Chan] Let $\angle ADE = \angle AED = \theta$, so $\angle BAC = 180° - 2\theta$. Suppose that $\angle ACB = \phi$, $\angle CPE = \sigma$ and $\angle BCP = \rho$. By the Law of Sines for triangles ABC and PCE, we find that

$$\frac{2|PC|\cos\rho}{\sin 2\theta} = \frac{|AB|}{\sin\phi}$$

whence

$$\frac{\sin\sigma}{\sin\theta} = \frac{|CE|}{|PC|} = \frac{|AB|}{|PC|} = \frac{2\cos\rho\sin\phi}{\sin 2\theta}$$

and

$$\sin\sigma\cos\theta = \sin\phi\cos\rho \quad .$$

Therefore

$$\sin(\theta + \sigma) + \sin(\sigma - \theta) = \sin(\phi + \rho) + \sin(\phi - \rho).$$

Since $\theta + \sigma = \phi + \rho$, $\sin(\sigma - \theta) = \sin(\phi - \rho)$. Either $(\sigma - \theta) + (\phi - \rho) = \pm 180°$ or $\sigma - \theta = \phi - \rho$. In the first case, since $\theta + \sigma = \phi + \rho$, $|\sigma - \rho| = 90°$, which is false.

Hence $\sigma - \theta = \phi - \rho$, so, with $\theta + \sigma = \phi + \rho$, we have that

$$2\theta = \theta + (\rho + \sigma - \phi) = \theta + (\rho + \rho - \sigma) = 2\rho$$

and the result follows.

Solution 8. [A. Murali] Let F be the midpoint of BC. Observe that triangles ADE and PBC are isosceles with $AD = AE$ and $PB = PC$. Suppose that the line parallel to AC through D and the line parallel to AD through C meet at N, and let CN intersect DE at M. Since $ACND$ is a parallelogram, $DN = AC$. Since triangle CME is similar to triangle ADE, it is isosceles with $CM = CE = AB$. Since $AD = CN$, $BMND$ is a parallelogram. In fact, $MN = BD = AC = DN = BM$, so that $BMND$ is a rhombus.

Since P is a point on a diagonal of the rhombus $BMND$, $PB = PN$ and so triangles PBM and PNM are congrunent, from which we see that $\angle PBM = \angle PNM$. Since $PC = PB = PN$, it follows that $\angle PBM = \angle PNC = \angle PCM$ and quadrilateral $BCMP$ is concyclic. Therefore, $\angle BPC = \angle BMC = \angle BAC$ ($ABMC$ being a quadrilateral).

Solution 9. [C. Deng] If BC were parallel to DE, then BC would be a midline of triangle ADE and P would be the reflection of A in the axis BC yielding the desired result. Suppose that BC and DE are not parallel. Let R be the circumradius of triangle ADE, R_1 the circumradius of triangle BDP and R_2 the circumradius of triangle CEP. Observe that $AD = AE$ and $PB = PC$.

Let the circumcircles of triangles BDP and CEP intersect at O. The point O lies inside triangle ADE. By the Extended Sine Law,

$$\frac{OP}{\sin \angle PBO} = 2R_1 = \frac{PB}{\sin \angle ADE} = \frac{PC}{\sin \angle AED} = 2R_2 = \frac{OP}{\sin \angle PCO} \ .$$

Since $\angle PCO = \angle PEO < \angle PEA < 90°$, the angle PCO is acute. Similarly, angle PBO is acute. Therefore $\angle PBO = \angle PCO$, so that $\angle OBC = \angle OCB$ and O is on the right bisector of BC. Since

$$DO = 2R_1 \sin \angle DPO = 2R_2 \sin \angle OPE = EO$$

, the point O is on the right bisector of DE, which is also the angle bisector of $\angle BAC$.

Since the quadrilaterals $OBDP$ and $OCEP$ are concyclic,

$$\begin{aligned} \angle BOC &= 360° - \angle BOP - \angle COP \\ &= 36° - (180° - \angle BDP) - (180° - \angle CEP) \\ &= \angle ADE + \angle AED = 180° - \angle BAC \ . \end{aligned}$$

Hence quadrilaterla $ABOC$ is concyclic. Also $\angle BCO = \angle CBO = \frac{1}{2}\angle BAC$.

From Ptolemy's Theorem, we have that

$$BC \cdot AO = AB \cdot CO + AC \cdot BO = (AB + AB \cdot BO = AD \cdot BO \ .$$

Therefore

$$AO = AD \cdot \frac{BO}{BC} = AD \cdot \frac{\sin \angle BCO}{\sin BOC} = AD \cdot \frac{\sin \frac{1}{2}\angle BAC}{2 \sin \angle BAC} = \frac{AD}{2 \cos \frac{1}{2}\angle BAC} = R \ .$$

Since O is on the right bisector of DE and $AO = R$, O is the circumcentre of triangle ADE. Therefore

$$\angle BPC = \angle BPO + \angle CPO = \angle BDO + \angle CEO = \angle OAB + \angle OAC = \angle A \ .$$

661. Let P be an arbitrary interior point of an equilateral triangle ABC. Prove that

$$|\angle PAB - \angle PAC| \geq |\angle PBC - \angle PCB| \ .$$

Solution. The result is clear if P is on the bisector of the angle at A, since both sides of the inequality are 0.

Wolog, let P be closer to AB than AC, and let Q be the image of P under reflection in the bisector of the angle A. Then

$$\angle PAQ = \angle PAC - \angle QAC = \angle PAC - \angle PAB$$

and

$$\angle PCQ = \angle QCB - \angle PCB = \angle PBC - \angle PCB \ .$$

Thus, it is required to show that $\angle PAQ \geq \angle PCQ$.

Produce PQ to meet AB in R and AC in S. Consider the reflection $\mathfrak{R}$ with axis RS. The circumcircle $\mathfrak{C}$ of ΔARS is carried to a circle $\mathfrak{C}'$ with chord RS. Since $\angle RCS < 60^\circ = \angle RAS$ and the angle subtended at the major arc of $\mathfrak{C}'$ by RS is 60°, the point C must lie outside of $\mathfrak{C}'$. The circumcircle $\mathfrak{D}$ of ΔAPQ is carried by $\mathfrak{R}$ to a circle $\mathfrak{D}'$ with chord PQ. Since $\mathfrak{D}$ is contained in $\mathfrak{C}$, $\mathfrak{D}'$ must be contained in $\mathfrak{C}'$, so C must lie outside of $\mathfrak{D}'$. Hence $\angle PCQ$ must be less than the angle subtended at the major arc of $\mathfrak{D}'$ by PQ, and this angle is equal to $\angle PAQ$. The result follows.

662. Let n be a positive integer and $x > 0$. Prove that

$$(1+x)^{n+1} \geq \frac{(n+1)^{n+1}}{n^n} x \ .$$

Solution 1. By the Arithmetic-Geometric Means Inequality, we have that

$$\frac{1+x}{n+1} = \frac{n(1/n)+x}{n+1} \geq \left[\left(\frac{1}{n}\right)^n x\right]^{\frac{1}{n+1}}$$

so that

$$\frac{(1+x)^{n+1}}{(n+1)^{n+1}} \geq \frac{x}{n^n}$$

and the result follows.

Solution 2. (by calculus) Let

$$f(x) = n^n(1+x)^{n+1} - (n+1)^{n+1}x \quad \text{ for } \ x > 0 \ .$$

Then

$$f'(x) = (n+1)[n^n(1+x)^n - (n+1)^n] = (n+1)n^n[(1+x)^n - (1+\frac{1}{n})^n]$$

so that $f'(x) < 0$ for $0 < x < 1/n$ and $f'(x) > 0$ for $1/n < x$. Thus $f(x)$ attains its minimum value 0 when $x = 1/n$ and so $f(x) \geq 0$ when $x > 0$. The result follows.

Solution 3. (by calculus) Let $g(x) = (1+x)^{n+1}x^{-1}$. Then $g'(x) = (1+x)^n x^{-2}[nx-1]$, so that $g(x) < 0$ for $0 < x < 1/n$ and $g'(x) > 0$ for $x > 1/n$. Therefore $g(x)$ assumes its minimum value of $(n+1)^{n+1}n^{-n}$ when $x = 1/n$, and the result follows.

Solution 4. [G. Ghosn] We make the substituion $t = (nx)^{1/(n+1)} \Leftrightarrow x = t^{n+1}/n$. Then it is required to prove that

$$1 + \frac{t^{n+1}}{n} \geq \frac{(n+1)t}{n} \ .$$

Observe that

$$\begin{aligned} t^{n+1} - (n+1)t - n &= t(t^n - 1) - n(t-1) = (t-1)(t^n + t^{n-1} + \cdots + t - n) \\ &= (t-1)[(t^n - 1) + (t^{n-1} - 1) + \cdots + (t-1)] \\ &= (t-1)^2[t^{n-1} + 2t^{n-2} + \cdots + (n-1)] \geq 0 \ , \end{aligned}$$

for $t > 0$. The desired result follows.

Solution 5. Let $u = nx - 1$ so that $x = (1+u)/n$. Then

$$\begin{aligned}(1+x)^{n+1} - \frac{(n+1)^{n+1}}{n^n}x &= (1+\frac{1}{n}+\frac{u}{n})^{n+1} - (1+\frac{1}{n})^{n+1}(1+u)\\ &= (1+\frac{1}{n})^{n+1} + (n+1)(1+\frac{1}{n})^n\frac{u}{n} + \binom{n+1}{2}(1+\frac{1}{n})^{n-1}(\frac{u}{n})^2\\ &\quad + \binom{n+1}{3}(1+\frac{1}{n})^{n-2}(\frac{u}{n})^3 + \cdots - (1+\frac{1}{n})^{n+1}(1+u)\\ &= \binom{n+1}{2}(1+\frac{1}{n})^{n-1}(\frac{u}{n})^2 + \binom{n+1}{3}(1+\frac{1}{n})^{n-2}(\frac{u}{n})^3 + \cdots \ .\end{aligned}$$

This is clearly nonnegative when $u \geq 0$. Suppose that $-1 < u < 0$. For $1 \leq k \leq n/2$, we have that

$$\binom{n+1}{2k}(1+\frac{1}{n})^{n-2k+1}(\frac{u}{n})^{2k} + \binom{n+1}{2k+1}(1+\frac{1}{n})^{n-2k}(\frac{u}{n})^{2k+1}$$

$$= \frac{(n+1)!(1+1/n)^{n-2k}}{(2k+1)!(n+1-2k)!}\left(\frac{u}{n}\right)^{2k}[(2k+1)(1+\frac{1}{n}) + (n+1-2k)(\frac{u}{n})] \ .$$

This will be nonnegative if and only if the quantity in square brackets is nonnegative. Since $u > -1$, this quantity exceeds

$$(2k+1)(1+\frac{1}{n}) - (n+1-2k)(\frac{1}{n}) = \left(\frac{n+1}{n}\right)(2k+1-1) - \frac{2k}{n} = 2k > 0 \ .$$

Thus, each consecutive pair of terms in the sequence

$$\binom{n+1}{2}(1+\frac{1}{n})^{n-1}(\frac{u}{n})^2 + \binom{n+1}{3}(1+\frac{1}{n})^{n-2}(\frac{u}{n})^3 + \cdots$$

has a positive sum and so the desired result follows.

663. Find all functions $f : \mathbf{R} \longrightarrow \mathbf{R}$ such that

$$x^2y^2(f(x+y) - f(x) - f(y)) = 3(x+y)f(x)f(y)$$

for all real numbers x and y.

Solution. An obvious solution if $f(x) \equiv 0$. A less obvious solution is that $f(0)$ is arbitrary and $f(x) = 0$ when $x \neq 0$. Henceforth, assume that $f(x) \neq 0$ for at least one nonzero value of x.

Setting $y = 0$ yields that $0 = 3xf(x)f(0)$ for all x, whence $f(0) = 0$. Setting $y = -x$ yields that $x^4[-f(x) - f(-x)] = 0$, so that $f(x) = -f(-x)$ for all nonzero x.

Setting $y = x$ yields that

$$f(2x) = \frac{6}{x^3}f(x)^2 + 2f(x)$$

for all nonzero x, while the sum $x = 2x + (-x)$ leads to

$$4x^4[2f(x) - f(2x)] = 3xf(2x)f(-x) = -3xf(2x)f(x) \ .$$

Therefore

$$4x^3\left[\frac{6}{x^3}f(x)^2\right] = 3\left[\frac{6}{x^3}f(x)^2 + 2f(x)\right]f(x)$$

so that

$$8x^3f(x)^2 = 6f(x)^3 + 2x^3f(x)^2$$

or

$$f(x)^3 = x^3f(x)^2 \ .$$

Therefore, for each real x, either $f(x) = 0$ or $f(x) = x^3$.

Suppose that $f(z) = 0$ for some real z. Select x so that $f(x) \neq 0$ and let $y = z - x$. Then, since $x^2y^2[-f(x) - f(y)] = 3zf(x)f(y)$, $f(y) \neq 0$. Thus $f(x) = x^3$, $f(y) = y^3$ so that

$$-x^2y^2(x^3 + y^3) = 3(x + y)x^3y^3 \ .$$

This simplifies to

$$0 = x^2y^2(x + y)(x^2 + 2xy + y^2) = x^2y^2(x + y)^3$$

with the result that $z = x + y = 0$. Therefore $f(x) = x^3$ for all real x (including 0).

664. The real numbers x, y, and z satisfy the system of equations

$$x^2 - x = yz + 1;$$

$$y^2 - y = xz + 1;$$

$$z^2 - z = xy + 1.$$

Find all solutions (x, y, z) of the system and determine all possible values of $xy + yz + zx + x + y + z$ where (x, y, z) is a solution of the system.

Solution. First we dispose of the situation that not all the variables takes distinct values. If $x = y = z$, then the equations reduce to $x = -1$, so that $(x, y, z) = (-1, -1, -1)$ is a solution and $x+y+z+xy+yz+zx = 0$.

By subtracting equations in pairs, we find that

$$0 = (x - y)(x + y + z - 1) = (y - z)(x + y + z - 1) = (z - x)(x + y + z - 1) \ .$$

Suppose that $x \neq y = z$. Then we must have $x + 2y = 1$ and $x^2 - x = y^2 + 1$, so that $0 = 3y^2 - 2y - 1 = (3y + 1)(y - 1)$. This leads to the two soutions $(x, y, z) = (-1, 1, 1), (\frac{5}{3}, -\frac{1}{3}, -\frac{1}{3}$. Symmetric permutations of these also are solutions and we find that $x + y + z + xy + yz + zx = 0$.

Henceforth, assume that the values of x, y, z are distinct. Any solution x, y, z of the system must satisfy the cubic equation

$$t^3 - t^2 - t = xyz \ .$$

In particular, from the coefficients, we find that $x + y + z = 1$ and $xy + yz + zx = -1$ whence $xy + yz + zx + x + y + z = 1$.

Conversely, suppose that we take any real number w. Let x, y, z be the roots of the cubic equation

$$t^3 - t^2 - t = w \ .$$

Then $xyz = w$. If $w = 0$, then the cubic equation has the roots $\{0, \frac{1}{2}(1 + \sqrt{5}), \frac{1}{2}(1 - \sqrt{5})\}$ and it can be checked that assigning these as the values of x, y and z any order will yields a solution to the given equation. If $w \neq 0$, then plugging the roots into the equation and dividing by it will yield the given system.

All that remains is to discover which values of w will yield three real roots for the cubic. Let $f(t) = t^3 - t^2 - t$. This function assumes a maximum value of $5/27$ at $t = -1/3$ and a minimum value of -1 when $t = 1$. Thus $f(t)$ assumes each value in the closed interval $[-1, 5/27]$ three times, counting multiplicity, and each other real value exactly once.

Thus, the solutions of the system are the roots of the cubic equation $t^3 - t^2 - t = w$, where w is any real number selected from the interval $[-1, 5/27]$.

(Note, that the "extreme" solutions are $(x, y, z) = (1, 1, -1), (-1/3, -1/3, 5/3)$. The only solution not related to the cubic is $(x, y, z) = (-1, -1, -1)$.)

Comment. G. Ajjanagadde, in the case of distinct values of x, y and z, obtained the equations $x+y+z = 1$ and $xy + yz + zx = -1$, whence, for given value of x, we get the system $y + z = 1 - x$ and $yz = x^2 - x - 1$, so that y and z are solutions of the quadratic equation

$$t^2 - (1 - x)t + (x^2 - x - 1) = 0 \ .$$

The discriminant of this quadratic is

$$(1 - x)^2 - 4(x^2 - x - 1) = -3x^2 + 2x + 5 = -(3x - 5)(x + 1) \ .$$

Thus, we will obtain real values of x, y, z if and only if x, y and z lies between -1 and $5/3$ inclusive.

665. Let $f(x) = x^3 + ax^2 + bx + b$. Determine all integer pairs (a, b) for which $f(x)$ is the product of three linear factors with integer coefficients.

Solution. If $b = 0$, then the polynomial becomes $x^2(x + a)$, which satisfies the condition for all values of a. This covers the situation for which x is a factor of the polynomial. Since the leading coefficient of $f(x)$ is 1, the same must be true (up to sign) of its factors. Assume that $f(x) = (x + u)(x + v)(x + w)$ for integers u, v and w with $uvw \neq 0$. Since $uvw = uv + vw + wu = b$,

$$\frac{1}{u} + \frac{1}{v} + \frac{1}{w} = 1 \ .$$

It is clearly not possible for all of u, v and w to be negative. Nor can it occur that two of them, say v and w can be negative, for then the left side would be less than $1/u \leq 1$. Suppose that u and v are positive, while w is negative. One possibility is that $u = 1$ and $v = -w$ in which case $f(x) = (x + 1)(x^2 - v^2) = x^3 + x^2 - v^2x - v^2$. If neither u nor v is equal to 1, then $1/u + 1/v + 1/w < 1/u + 1/v \leq 1$, and this case is not possible. Finally, suppose that u, v and w are all positive, with $u \leq v \leq w$. Then $1 \leq 3/u$, so that $u \leq 3$. A little trial and error leads to the possibilities $(u, v, w) = (3, 3, 3), (2, 4, 4)$ and $(2, 3, 6)$. Thus the possibilities for (a, b) are $(u, 0)$, $(1, -v^2)$, $(9, 27)$, $(10, 32)$ and $(11, 36)$. Indeed, $x^3 + 9x^2 + 27x + 27 = (x + 3)^3$, $x^3 + 10x^2 + 32x + 32 = (x + 2)(x + 4)^2$ and $x^3 + 11x^2 + 36x + 36 = (x + 2)(x + 3)(x + 6)$.

666. Assume that a face S of a convex polyhedron $\mathfrak{P}$ has a common edge with every other face of $\mathfrak{P}$. Show that there exists a simple (nonintersecting) closed (not necessarily planar) polygon that consists of edges of $\mathfrak{P}$ and passes through all the vertices.

Solution. Suppose that the face S has m vertices A_1, A_2, $\cdots$, A_m listed in order, and that there are n vertices of $\mathfrak{P}$ not contained in S. We prove the result by induction on n. If $n = 1$, then every face abutting S is a triangle. Let X be the vertex off S; then $A_1 \cdots A_m X A_1$ is a polygonal path of the desired type. Suppose that the result holds for any number of vertices m of S and for n vertices off S where $1 \leq n \leq k$. Consider the case $n = k + 1$.

Consider the graph G of all vertices of $\mathfrak{P}$ and those edges of $\mathfrak{P}$ not bounding S. Since there are no faces bounded solely by these edges, the graph must be a *tree* (*i.e.*, it contains no loops and there is a unique path joining any pair of points). We show that there is at least one vertex X not in S for which every edge but one must connect X to a vertex of S. Suppose otherwise. Then, let us start with such a vertex X and form a sequence $X_1, X_2, \cdots$ of vertices not in S such that $X_i X_{i+1}$ are edges of $\mathfrak{P}$. Since the number of vertices off S is finite, there must be $i < j$ for which $X_i = X_j$ so that $X_i X_{i+1} \cdots X_{j-1} X_j$ is a loop in G. But this contradicts the fact that G is a tree.

Hence there is a vertex X with at most one adjacent edge not connecting it to S. If there were no such edge, then X would be the only vertex not in S, contradicting $k+1 \geq 2$. Hence there is a vertex Y not in S such that XY is an edge of $\mathfrak{P}$. We may assume that Y is further from the plane of S than S. (If not, suppose that S is in the plane $z = 0$ and that $\mathfrak{P}$ lies in the quadrant $z > 0$, $y > 0$ with Y further than X from the plane $y = 0$. We can transform $\mathfrak{P}$ by a mapping of the type $(x, y, z) \rightarrow (x, y, z + \lambda y)$ for suitable positive λ. This will not alter the configuration of vertices and edges.) Extend YX to a point Z in the plane of S. Let $\mathfrak{Q}$ be the convex hull of (smallest closed convex set containing) Z and $\mathfrak{P}$. This will have a side T containing S of the form $A_1A_2 \cdots A_rZA_s \cdots A_m$ where $r < s$. The triangles XZA_r and XZA_s will be coplanar with faces of $\mathfrak{P}$, and the convex hull will have at most k vertices not on T. Every face of $\mathfrak{Q}$ will abut T. By the induction hypothesis, we can construct a polygon containing each vertex of $\mathfrak{Q}$. If an edge of this polygon is YZ and so includes X, and if one edge is say ZA_r, then we can replace these two edges by $YXA_sA_{s-1} \cdots A_{r+1}A_r$. If YZ is not an edge of this polygon, but A_rZ and ZA_s are, then we can replace these edges by $A_rXA_{r+1} \cdots A_s$. In both cases, we obtain a polygon of the required type for $\mathfrak{P}$.

667. Let A_n be the set of mappings $f : \{1, 2, 3, \cdots, n\} \longrightarrow \{1, 2, 3, \cdots, n\}$ such that, if $f(k) = i$ for some i, then f also assumes all the values $1, 2, \cdots, i-1$. Prove that the number of elements of A_n is $\sum_{k=0}^{\infty} k^n 2^{-(k+1)}$.

Solution 1. Let $u_0 = 1$ and, for $n \geq 1$, let u_n be the number of elements in A_n. Let $1 \leq r \leq n$. Consider the set of mappings in A_n for which the value 1 is assumed exactly r times. Then $1 \leq r \leq n$. Then each such mapping takes a set of $n - r$ points *onto* a set of the form $\{2, 3, \cdots, s\}$ where $s - 1 \leq n - r \leq n - 1$. Hence, there are u_{n-r} such mappings. Since there are $\binom{n}{r}$ possible sets on which a mapping may assume the value 1 r times,

$$u_n = \sum_{r=1}^{n} \binom{n}{r} u_{n-r} = \sum_{r=0}^{n-1} \binom{n}{r} u_r \ .$$

Now $u_0 = 1 = \sum_{k=0}^{\infty} 1/2^{k+1}$. Assume, as an induction hypothesis, that $u_r = \sum_{k=0}^{\infty} k^r/2^{k+1}$. Then

$$\begin{aligned} u_n &= \sum_{r=0}^{n-1} \binom{n}{r} u_r = \sum_{r=0}^{n-1} \binom{n}{r} \sum_{k=0}^{\infty} \frac{k^r}{2^{k+1}} \\ &= \sum_{k=0}^{\infty} \frac{1}{2^{k+1}} \sum_{r=0}^{n-1} \binom{n}{r} k^r = \sum_{k=0}^{\infty} \frac{1}{2^{k+1}} [(1+k)^n - k^n] \\ &= \sum_{k=0}^{\infty} \frac{(1+k)^n}{2^{k+1}} - \sum_{k=0}^{\infty} \frac{k^n}{2^{k+1}} = \sum_{k=1}^{\infty} \frac{k^n}{2^k} - \sum_{k=1}^{\infty} \frac{k^n}{2^{k+1}} \\ &= \sum_{k=1}^{\infty} \frac{k^n}{2^{k+1}} \end{aligned}$$

and the result follows. (The interchange of the order of summation and rearrangement of terms in the infinite sum can be justified by the absolute convergence of the series.)

Solution 2. For $1 \leq i$, let v_i be the number of mappings of $\{1, 2, \cdots, n\}$ *onto* a set of exactly i elements. Observe that $v_i = 0$ when $i \geq n + 1$. There are k^n mappings of $\{1, 2, \cdots, n\}$ into $\{1, 2, \cdots, k\}$, of which v_k belong to A_n. The other $k^n - v_k$ mappings will leave out i numbers in the range for some $1 \leq i \leq k - 1$, and the i numbers not found can be selected in $\binom{k}{i}$ ways. Thus

$$k^n = \sum_{i=1}^{k} \binom{k}{i} v_i \quad .$$

Hence

$$\begin{aligned} \sum_{k=0}^{\infty} \frac{k^n}{2^{k+1}} &= \sum_{k=0}^{\infty} \sum_{i=1}^{k} \frac{\binom{k}{i} v_i}{2^{k+1}} = \sum_{k=0}^{\infty} \sum_{i=1}^{n} \frac{\binom{k}{i} v_i}{2^{k+1}} \\ &= \sum_{i=1}^{n} \left(\sum_{k=0}^{\infty} \frac{\binom{k}{i}}{2^{k+1}} \right) v_i = \sum_{i=1}^{n} \left(\sum_{k=i}^{\infty} \frac{\binom{k}{i}}{2^{k+1}} \right) v_i \ . \end{aligned}$$

We evaluate the inner sum. Fix the positive integer i. Suppose that we flip a fair coin an indefinite number of times, and consider the event that the $(i+1)$th head occurs on the $(k+1)$th toss. Then the previous i heads could have occurred in $\binom{k}{i}$ posible positions, so that the probability of the event is $\binom{k}{i}2^{-(k+1)}$. Since the $(i+1)$th head must occur on *some* toss with probability 1, $\sum_{k=i}^{\infty}\binom{k}{i}2^{-(k+1)} = 1$. Hence

$$\sum_{k=0}^{\infty} \frac{k^n}{2^{k+1}} = \sum_{i=1}^{n} v_i = \#A_n \ .$$

Solution 3. [C. Deng] Let $s_n = \sum_{k=0}^{\infty} k^n 2^{-(k+1)}$; note that $s_0 = s_1 = 1$. Let $w_0 = 1$ and $w_n = \#A_n$ for $n \geq 1$, so that, in particular, $w_1 = 1$.

For $n \geq 0$,

$$\begin{aligned} s_{n+1} = 2s_{n+1} - s_{n+1} &= 2\sum_{k=0}^{\infty} k^{n+1}2^{-(k+1)} - \sum_{k=0}^{\infty} k^{n+1}2^{-(k+1)} \\ &= \sum_{k=0}^{\infty}[(k+1)^{n+1} - k^{n+1}]2^{-(k+1)} \\ &= \sum_{k=0}^{\infty}\left(\sum_{i=0}^{n}\binom{n+1}{i}k^i\right)2^{-(k+1)} \\ &= \sum_{i=0}^{n}\left(\sum_{k=0}^{\infty}\binom{n+1}{i}k^i 2^{-(k+1)}\right) \\ &= \sum_{i=0}^{n}\binom{n+1}{i}s_i \ . \end{aligned}$$

We now show that w_n satisfies the same recursion. Suppose that g is an arbitrary element of A_{n+1} and that its maximum appears $n+1-i$ times, where $0 \leq i \leq n$. Then there are $\binom{n+1}{i}$ ways to fill in i remaining slots with numbers without leaving gaps in the range, and then we can fill in the remaining $n+1-i$ slots with one more than the largest number in the range of the i slots. Thus, we find that $w_{n+1} = \sum_{i=0}^{n}\binom{n+1}{i}w_i$. The desired result now follows, since $s_0 - w_0$.

668. The nonisosceles right triangle ABC has $\angle CAB = 90°$. The inscribed circle with centre T touches the sides AB and AC at U and V respectively. The tangent through A of the circumscribed circle meets UV produced in S. Prove that

(a) $ST \parallel BC$;

(b) $|d_1 - d_2| = r$, where r is the radius of the inscribed circle and d_1 and d_2 are the respective distances from S to AC and AB.

(a) *Solution 1.* Wolog, suppose that the situation is as diagrammed. $\angle BAC = \angle AUT = \angle AVT = 90°$, so that $AUVT$ is a rectangle with $AU = AV$ and $UT = VT$. Hence $AUTV$ is a square with diagonals AT and UV which right-bisect each other at W. Since SW right-bisects AT, by reflection in the line SW, we see that $\Delta ASU \equiv \Delta UST$, and so $\angle UTS = \angle UAS$.

Let M be the midpoint of BC. Then M is the circumcentre of ΔABC, so that $MA = MC$ and $\angle MCA = \angle MAC$. Since AS is tangent to the circumcircle of ΔABC, $AS \perp AM$. Hence

$$\angle UTS = \angle UAS = \angle SAM - \angle BAM = 90° - \angle BAM = \angle MAC = \angle MCA \ .$$

Now $UT \perp AB$ implies that $UT \| AC$. Since $\angle UTS = \angle ACB$, it follows that $ST \| BC$.

Solution 2. Wolog, suppose that S is on the opposite side of AB to C.

BT, being a part of the diameter produced of the inscribed circle, is a line of reflection that takes the circle to itself and takes the tangent BA to BC. Hence $\angle UBT = \frac{1}{2}\angle ABC$. Let $\alpha = \angle ABT$. By the tangent-chord theorem applied to the circumscribed circle, $\angle XAC = \angle ABC = 2\alpha$, so that $\angle SAU = 90° - 2\alpha$.

Consider triangles SAU and STU. Since $AUTV$ is a square (see the first solution), $AU = UT$ and $\angle AUV = \angle TUV = 45°$ so $\angle SUA = \angle SUT = 135°$. Also SU is common. Hence $\Delta SAU \equiv \Delta STU$, so $\angle STU = \angle SAU = 90° - 2\alpha$. Therefore,

$$\angle STB = \angle UTB - \angle STU = (90° - \alpha) - (90° - 2\alpha) = \alpha = \angle TBC$$

from which it results that $ST \| BC$.

Solution 3. As before $\Delta AUS \equiv \Delta TUS$, so $\angle SAU = \angle STU$. Since $UT \| AC$, $\angle STU = \angle SYA$. Also, by the tangent-chord theorem, $\angle SAB = \angle ACB$. Hence $\angle SYA = \angle STU = \angle SAB = \angle ACB$, so $ST \| BC$.

Solution 4. In the Cartesian plane, let $A \sim (0,0)$, $B \sim (0,-b)$, $C \sim (c,0)$. The centre of the circumscribed circle is at $M \sim (c/2, -b/2)$. Since the slope of AM is $-b/c$, the equation of the tangent to the circumscribed circle through A is $y = (c/b)x$. Let r be the radius of the inscribed circle. Since $AU = AV$, the equation of the line UV is $y = x - r$. The abscissa of S is the solution of $x - r = (cx)/b$, so $S \sim (\frac{br}{b-c}, \frac{cr}{b-c})$. Since $T \sim (r, -r)$, the slope of ST is b/c and the result follows.

(b) *Solution 1.* $[\cdots]$ denotes area. Wolog, suppose that $d_1 > d_2$, as diagrammed.

Let r be the inradius of ΔABC. Then $[AVU] = \frac{1}{2}r^2$, $[AVS] = \frac{1}{2}rd_1$ and $[AUS] = \frac{1}{2}rd_2$. From $[AVU] = [AVS] - [AUS]$, it follows that $r^2 = rd_1 - rd_2$, whence $r = d_1 - d_2$.

Solution 2. [F. Crnogorac] Suppose that the situation is as diagrammed. Let P and Q be the respective feet of the perpendiculars from S to AC and AB. Since $\angle PVS = 45°$ and $\angle SPV = 90°$, ΔPSV is isosceles and so $PS = PV = PA + AV = SQ + AV$, *i.e.*, $d_1 = d_2 + r$.

Solution 3. Using the coordinates of the fourth solution of (a), we find that

$$d_1 = \left|\frac{cr}{b-c}\right| \qquad \text{and} \qquad d_2 = \left|\frac{br}{b-c}\right|$$

whence $|d_2 - d_1| = r$ as desired.

(b) *Solution.* [M. Boase] Wolog, assume that the configuration is as diagrammed.

Since $\angle SUB = \angle AUV = 45°$, SU is parallel to the external bisector of $\angle A$. This bisector is the locus of points equidistant from AB and CA produced. Wolog, let PS meet this bisector in W, as in the diagram. Then $PW = PA$ so that $PS - PA = PS - PW = SW = AU$ and thus $d_1 - d_2 = r$.

669. Let $n \geq 3$ be a natural number. Prove that

$$1989 | n^{n^{n^n}} - n^{n^n} \quad ,$$

i.e., the number on the right is a multiple of 1989.

Solution 1. Let $N = n^{n^{n^n}} - n^{n^n}$. Since $1989 = 3^2 \cdot 13 \cdot 17$,

$$N \equiv 0 \pmod{1989} \Leftrightarrow N \equiv 0 \pmod{9, 13 \;\&\; 17} .$$

We require the following facts:

(i) $x^u \equiv 0 \pmod 9$ whenever $u \geq 2$ and $x \equiv 0 \pmod 3$.
(ii) $x^6 \equiv 1 \pmod 9$ whenever $x \not\equiv 0 \pmod 3$.
(iii) $x^u \equiv 0 \pmod{13}$ whenever $x \equiv 0 \pmod{13}$.

(iv) $x^{12} \equiv 1 \pmod{13}$ whenever $x \not\equiv 0 \pmod{13}$, by Fermat's Little Theorem.
(v) $x^u \equiv 0 \pmod{17}$ whenever $x \equiv 0 \pmod{17}$.
(vi) $x^{16} \equiv 1 \pmod{17}$ whenever $x \not\equiv 0 \pmod{17}$, by FLT.
(vii) $x^4 \equiv 1 \pmod{16}$ whenever $x = 2y+1$ is odd. (For, $(2y+1)^4 = 16y^3(y+2) + 8y(3y+1) + 1 \equiv 1 \pmod{16}$.)

Note that
$$N = n^{n^n}\left[n^{(n^{n^n}-n^n)} - 1\right] = n^{n^n}\left[n^{n^n(n^{n^n-n}-1)} - 1\right] .$$

Modulo 17. If $n \equiv 0 \pmod{17}$, then $n^{n^n} \equiv 0$, and so $N \equiv 0 \pmod{17}$.

If n is even, $n \geq 4$, then $n^n \equiv 0 \pmod{16}$, so that
$$n^{n^n(n^{n^n-n}-1)} \equiv 1^{(n^{n^n-n}-1)} \equiv 1$$
so $N \equiv 0 \pmod{17}$.

Suppose that n is odd. Then $n^n \equiv n \pmod 4$
$$\begin{aligned}
&\Rightarrow n^n - n = 4r \quad \text{for some} \quad r \in \mathbf{N} \\
&\Rightarrow n^{n^n-n} = n^{4r} \equiv 1 \pmod{16} \\
&\Rightarrow n^{n^n-n} - 1 \equiv 0 \pmod{16} \\
&\Rightarrow n^{n^n(n^{n^n-n}-1)} \equiv 1 \pmod{17} \\
&\Rightarrow N \equiv 0 \pmod{17} \quad .
\end{aligned}$$
Hence $N \equiv 0 \pmod{17}$ for all $n \geq 3$.

Modulo 13. If $n \equiv 0 \pmod{13}$, then $n^{n^n} \equiv 0$ and $N \equiv 0 \pmod{13}$.

Suppose that n is even. Then $n^n \equiv 0 \pmod 4$, so that $n^{n^n} - n^n \equiv 0 \pmod 4$. Suppose that n is odd. Then $n^{n^n-n} - 1 \equiv 0 \pmod{16}$ and so $n^{n^n} - n^n \equiv 0 \pmod 4$.

If $n \equiv 0 \pmod 3$, then $n^n \equiv 0$ so $n^n(n^{n^n-n} - 1) \equiv 0 \pmod 3$. If $n \equiv 1 \pmod 3$, then $n^{n^n-n} \equiv 1$ so $n^n(n^{n^n-n}-1) \equiv 0 \pmod 3$. If $n \equiv 2 \pmod 3$, then, as $n^n - n$ is always even, $n^{n^n-n} \equiv 1$ so $n^n(n^{n^n-n}-1) \equiv 0 \pmod 3$. Hence, for all n, $n^{n^n} - n^n \equiv 0 \pmod 3$.

It follows that $n^{n^n} - n^n \equiv 0 \pmod{12}$ for all values of n. Hence, when n is not a multiple of 13, $n^{(n^{n^n}-n)} \equiv 1$ so $N \equiv 0 \pmod{13}$.

Modulo 9. If $n \equiv 0 \pmod 3$, then $n^{n^n} \equiv 0 \pmod 9$, so $N \equiv 0 \pmod 9$. Let $n \not\equiv 0 \pmod 9$. Since $n^{n^n} - n^n$ is divisible by 12, it is divisible by 6, and so $n^{(n^{n^n}-n^n)} \equiv 1$ and $N \equiv 0 \pmod 9$. Hence $N \equiv 0 \pmod 9$ for all n.

The required result follows.

670. Consider the sequence of positive integers $\{1, 12, 123, 1234, 12345, \cdots\}$ where the next term is constructed by lengthening the previous term at the right-hand end by appending the next positive integer. Note that this next integer occupies only one place, with "carrying"occurring as in addition. Thus, the ninth and tenth terms of the sequence are 123456789 and 1234567900 respectively. Determine which terms of the sequence are divisible by 7.

Solution 1. For positive integer n, let x_n be the nth term of the sequence, and let $x_0 = 0$. Then, for $n \geq 0$, $x_{n+1} = 10x_n + (n+1)$ so that $x_{n+1} \equiv 3x_n + (n+1) \pmod 7$. Suppose that m is a nonnegative integer and that $x_{7m} = a$. Then
$$\begin{array}{llll}
x_{7m+1} \equiv 3a+1 & x_{7m+2} \equiv 2a+5 & x_{7m+3} \equiv 6a+4 & x_{7m+4} \equiv 4a+2 \\
x_{7m+5} \equiv 5a+4 & x_{7m+6} \equiv a+4 & x_{7m+7} \equiv 3a+5 &
\end{array}$$

In particular, we find that, modulo 7, $\{x_{7m}\}$ is periodic with the values $\{0, 5, 6, 2, 4, 3\}$ repeated, so that $0 \equiv x_0 \equiv x_{42} \equiv x_{84} \equiv \cdots$. Hence, modulo 7, $x_{7m+1} \equiv 0$ iff $a \equiv 2$, $x_{7m+2} \equiv 0$ iff $a \equiv 1$, $x_{7m+3} \equiv 0$ iff $a \equiv 4$, $x_{7m+4} \equiv 0$ iff $a \equiv 3$, $x_{7m+5} \equiv 0$ iff $a \equiv 2$ and $x_{7m+6} \equiv 0$ iff $a \equiv 3$. Putting this all together, we find that $x_n \equiv 0 \pmod 7$ if and only if $n \equiv 0, 22, 26, 31, 39, 41 \pmod{42}$.

Solution 2. [C. Deng] Recall the formula

$$r^{n-1} + 2r^{n-2} + \cdots + (n-1)r + n = \frac{r^{n+1} - r - (r-1)n}{(r-1)^2} \ .$$

[Derive this.] Noting that

$$a_n = 1 \cdot 10^{n-1} + 2 \cdots 10^{n-2} + \cdots + (n-1) \cdot 10 + n \ ,$$

we find that

$$81a_n = 10^{n+1} - 10 - 9n$$

for each positive integer n. Therefore

$$81(a_{n+42} - a_n) = 10^{n+1}((10^6)^7 - 1) - 9(42)$$

for each positive integer n. Since $10^6 \equiv 1$ (modulo 7), it follows that $a_{n+42} \equiv a_n$ (modulo 7), so that the sequence has period 42 (modulo 7). Thus, the value of n for which a_n is divisible by 7 are the solutions of the congruence $3^{n+1} \equiv 2n + 3$ (modulo 7). These are $n \equiv 22, 26, 31, 39, 41, 42$ (modulo 7).

671. Each point in the plane is coloured with one of three distinct colours. Prove that there are two points that are unit distant apart with the same colour.

Solution 1. Suppose that the points in the plane are coloured with three colours. Select any point P.

We form two rhombi $PQSR$ and $PUWV$, one the rotated image of the other for which all of the following segments have unit length: PQ, PR, SQ, SR, QR, PU, PV, WU, WV, UV, SW. If P, Q, R are all coloured differently, then either the result holds or S must have the same colour as P. If P, U, V are all coloured differently, then either the result holds or W must have the same colour as P. Hence, either one of the triangles PQR and PUV has two vertices the same colour, or else S and W must be coloured the same.

Solution 2. Suppose, if possible, the planar points can be coloured without two points unit distance apart being coloured the same. Then if A and B are distant $\sqrt{3}$ apart, then there are distinct points C and D such that ACD and BCD are equilateral triangles ($ABCD$ is a rhombus). Since A and B must be coloured differently from the two colours of C and D, A and B must have the same colour. Hence, if O is any point in the plane, every point on the circle of radius $\sqrt{3}$ consists of points coloured the same as O. But there are two points on this circle unit distant apart, and we get a contradiction of our initial assumption.

Solution 3. Suppose we can colour the points of the plane with three colours, red, blue and yellow so that the result fails. We show that three collinear points at unit distance are coloured with three different colours. Let P, Q, R be three such points, and let P, R be opposite sides of a unit hexagon $ABPCDR$ whose centre is Q.

If, say, Q is red, B and A must be coloured differently, as are A and R, R and D, D and C, C and P, P and B. Thus, B, R, C, are one colour, say, blue, and A, D, P the other, say yellow. The preliminary result follows.

Now consider any isosceles triangle UVW with $|UV| = |UW| = 3$ and $|VW| = 2$. It follows from the preliminary result that U and V must have the same colour, as do U and W. But V and W cannot have the same colour and we reach a contradiction.

Solution 4. [D. Arthur] Suppose that the result is false. Let A, B be two points with $|AB| = 3$. Within the segment AB select PQ with $|AP| = |PQ| = |QB| = 1$, and suppose that R and S are points on the same side of AB with ΔRAP and ΔSPQ equilateral. Then $|RS| = 1$. Suppose if possible that A and Q have the same colour. Then P must have a second colour and R and S the third, leading to a contradiction. Hence A must be coloured differently from both P and Q. Similarly B must be coloured differently from both P and Q. Since P and Q are coloured differently, A and B must have the same colour.

Now consider a trapezoid $ABCD$ with $|CB| = |AB| = |AD| = 3$ and $|CD| = 1$. By the foregoing observation, C, A, B, D must have the same colour. But this yields a contradiction. The result follows.

672. The Fibonacci sequence $\{F_n\}$ is defined by $F_1 = F_2 = 1$ and $F_{n+2} = F_{n+1} + F_n$ for $n = 0, \pm 1, \pm 2, \pm 3, \cdots$. The real number τ is the positive solution of the quadratic equation $x^2 = x + 1$.

(a) Prove that, for each positive integer n, $F_{-n} = (-1)^{n+1} F_n$.

(b) Prove that, for each integer n, $\tau^n = F_n \tau + F_{n-1}$.

(c) Let G_n be any one of the functions $F_{n+1}F_n$, $F_{n+1}F_{n-1}$ and F_n^2. In each case, prove that $G_{n+3} + G_n = 2(G_{n+2} + G_{n+1})$.

(a) *Solution.* Since $F_0 = F_2 - F_1 = 0$, the result holds for $n = 0$. Since $F_{-1} = F_1 - F_0 = 1$, the result holds for $n = 1$. Suppose that we have established the result for $n = 0, 1, 2, \cdots r$. Then

$$F_{-(r+1)} = F_{-r-1} = F_{-r+1} - F_{-r} = (-1)^r F_{r-1} - (-1)^{r+1} F_r = (-1)^{r+2}(F_{r-1} + F_r) = (-1)^{r+2} F_{r+1} \ .$$

The result follows by induction.

(b) *Solution 1.* The result holds for $n = 0$, $n = 1$ and $n = 2$. Suppose that it holds for $n = 0, 1, 2, \cdots, r$. Then

$$\tau^{r+1} = \tau^r + \tau^{r-1} = (F_r + F_{r-1})\tau + (F_{r-1} + F_{r-2}) = F_{r+1}\tau + F_r\tau \ .$$

This establishes the result for positive values of n. Now $\tau^{-1} = \tau - 1 = F_{-1}\tau + F_{-2}$, so the result holds for $n = -1$. Suppose that we have established the result for $n = 0, -1, -2, \cdots, -r$. Then

$$\tau^{-(r+1)} = \tau^{-(r-1)} - \tau^{-r} = (F_{-(r-1)} - F_{-r})\tau + (F_{-r} - F_{-(r+1)}) = F_{-(r+1)}\tau + F_{-(r+2)} \ .$$

Solution 2. The result holds for $n = 1$. Suppose that it holds for $n = r \geq 0$. Then

$$\begin{aligned} \tau^{r+1} &= \tau^r \cdot \tau = (F_r\tau + F_{r-1})\tau = F_r\tau^2 + F_{r-1}\tau \\ &= (F_r + F_{r-1})\tau + F_r = F_{r+1}\tau + F_r \ . \end{aligned}$$

Now consider nonpositive values of n. We have that $\tau^0 = 1$, $\tau^{-1} = \tau - 1$, $\tau^{-2} = 1 - \tau^{-1} = 2 - \tau$. Suppose that we have shown for $r \geq 0$ that $\tau^{-r} = F_{-r}\tau + F_{-r-1}$. Then

$$\begin{aligned} \tau^{-(r+1)} &= \tau^{-1}\tau^{-r} = F_{-r} + F_{-r-1}(\tau - 1) = F_{-r-1}\tau + (F_{-r} - F_{-r-1}) \\ &= F_{-r-1}\tau + F_{-r-2} = F_{-(r+1)}\tau + F_{-(r+1)-1} \ . \end{aligned}$$

By induction, it follows that the result holds for both positive and negative values of n.

(c) *Solution.* Let $G_n = F_n F_{n+1}$. Then

$$\begin{aligned} G_{n+3} + G_n &= F_{n+4}F_{n+3} + F_{n+1}F_n \\ &= (F_{n+3} + F_{n+2})(F_{n+2} + F_{n+1}) + (F_{n+3} - F_{n+2})(F_{n+2} - F_{n+1}) \\ &= 2(F_{n+3}F_{n+2} + F_{n+2}F_{n+1}) = 2(G_{n+2} + G_{n+1}) \ . \end{aligned}$$

Let $G_n = F_{n+1}F_{n-1}$. Then

$$\begin{aligned} G_{n+3} + G_n &= F_{n+4}F_{n+2} + F_{n+1}F_{n-1} \\ &= (F_{n+3} + F_{n+2})(F_{n+1} + F_n) + (F_{n+3} - F_{n+2})(F_{n+1} - F_n) \\ &= 2(F_{n+3}F_{n+1} + F_{n+2}F_n) = 2(G_{n+2} + G_{n+1}) \ . \end{aligned}$$

Let $G_n = F_n^2$. Then

$$G_{n+3} + G_n = F_{n+3}^2 + F_n^2 = (F_{n+2} + F_{n+1})^2 + (F_{n+2} - F_{n+1})^2$$
$$= F_{n+2}^2 + 2F_{n+2}F_{n+1} + F_{n+1}^2 + F_{n+2}^2 - 2F_{n+2}F_{n+1} + F_{n+1}^2 = 2(G_{n+2} + G_{n+1}) \ .$$

Comments. Since $F_n^2 = F_nF_{n-1} + F_nF_{n-2}$, the third result of (c) can be obtained from the first two. J. Chui observed that, more generally, we can take $G_n = F_{n+u}F_{n+v}$ where u and v are integers. Then

$$\begin{aligned} &G_{n+3} + G_n - 2(G_{n+1} + G_{n+2}) \\ &\qquad = (F_{n+3+u}F_{n+3+v} + F_{n+u}F_{n+v}) - 2(F_{n+2+u}F_{n+2+v} + F_{n+1+u}F_{n+1+v}) \\ &\qquad = (2F_{n+1+u} + F_{n+u})(2F_{n+1+v} + F_{n+v}) + F_{n+u}F_{n+v} \\ &\qquad\qquad - 2(F_{n+1+u} + F_{n+u})(F_{n+1+v} + F_{n+v}) - 2F_{n+1+u}F_{n+1+v} \\ &\qquad = 0 \ \ , \end{aligned}$$

so that $G_{n+3} + G_n = 2(G_{n+2} + G_{n+1})$.

673. ABC is an isosceles triangle with $AB = AC$. Let D be the point on the side AC for which $CD = 2AD$. Let P be the point on the segment BD such that $\angle APC = 90°$. Prove that $\angle ABP = \angle PCB$.

Solution 1. Produce BA to E so that $BA = AE$ and join EC. Then D is the centroid of ΔBEC and BD produced meets EC at its midpoint F. Since $AE = AC$, ΔCAE is isosceles and so $AF \perp EC$. Also, since A and F are midpoints of their respective segments, $AF \| BC$ and so $\angle AFB = \angle DBC$. Because $\angle AFC$ and $\angle APC$ are both right, $APCF$ is concyclic so that $\angle AFP = \angle ACP$.

Hence $\angle ABP = \angle ABC - \angle DBC = \angle ABC - \angle AFB = \angle ACB - \angle ACP = \angle PCB$.

Solution 2. Let E be the midpoint of BC and let F be a point on BD produced so that $AF \| BC$. Since triangle ADF and CDB are similar and $CD = 2AD$, then $AF = EC$ and $AECF$ is a rectangle.

Since $\angle APC = \angle AFC = 90°$, the quadrilateral $APCF$ is concyclic, so that $\angle AFB = \angle ACB$. Since $AF \| BC$, $\angle AFB = \angle FBC$. Therefore

$$\angle ABP = \angle ABC - \angle PBC = \angle ABC - \angle FBC = \angle ACB - \angle ACP = \angle PCB \ .$$

Solution 3. [S. Sun] The circle with diameter AC has as its centre the midpoint O of AC. It intersects BC at the midpoint E (since $AB = AC$ and $AE \perp BC$). Let EO produced meet the circle again at F; then $AECF$ is concyclic.

Suppose FB meets AC at G. The triangles AGF and CGB are similar. Since $BC = 2AF$, then $CG = 2GA$, so that G and D coincide. Because $AF \| BC$ and $AFCP$ is concyclic, $\angle DBC = \angle DFA = \angle PFA = \angle PCA$. Therefore

$$\angle ABP = \angle ABC - \angle DBC = \angle ABC - \angle PCA = \angle PCB \ .$$

Solution 4. Assign coordinates: $A \sim (0, a)$, $B \sim (-1, 0)$, $C \sim (1, 0)$. Then $D \sim (\frac{1}{3}, \frac{2a}{3})$. Let $P \sim (p, q)$. Then, since P lies on the lines $y = \frac{a}{2}(x + 1)$, $q = \frac{a}{2}(p + 1)$. The relation $AP \perp PC$ implies that

$$-1 = \left(\frac{q - a}{p}\right)\left(\frac{q}{p - 1}\right) = \left[\frac{a(p - 1)}{2p}\right]\left[\frac{a(p + 1)}{2(p - 1)}\right] = \frac{a^2(p + 1)}{4p} = \frac{aq}{2p}$$

whence $p = -a^2/(a^2 + 4)$ and $q = 2a/(a^2 + 4)$. Now

$$\tan \angle ABP = \frac{a - (a/2)}{1 + (a^2/2)} = \frac{a}{2 + a^2}$$

while

$$\tan \angle PCB = \frac{-q}{p-1} = \frac{-2a}{-a^2-(a^2+4)} = \frac{a}{a^2+2} = \tan \angle ABP \quad .$$

The result follows.

Solution 5. [C. Deng] Let $A \sim (0, b)$, $B \sim (-a, 0)$, $C \sim (a, 0)$ so that $D \sim (a/3, 2b/3)$. The midpoint M of AC has coordinates $(a/2, b/2)$. It can be checked that the point with coordinates

$$\left(\frac{-ab^2}{4a^2+b^2}, \frac{2a^2b}{4a^2+b^2} \right)$$

is the same distance from M as the points AB so that it is on the circle with diameter AC and $AP \| CP$. Since this point also lies on the line with equation $2ay = bx + ba$ through B and D, it is none other than the point P. The circle with equation

$$x^2 + \left(y + \frac{a^2}{b} \right)^2 = a^2 + \frac{a^4}{b^2}$$

is tangent to AB and AC at B and C respectively and contains the point P. Hence $\angle PCB = \angle PBA = \angle DBA$, as desired.

674. The sides BC, CA, AB of triangle ABC are produced to the poins R, P, Q respectively, so that $CR = AP = BQ$. Prove that triangle PQR is equilateral if and only if triangle ABC is equilateral.

Solution . Suppose that triangle ABC is equilateral. A rotation of $60°$ about the centroid of ΔABC will rotate the points R, P and Q. Hence ΔPQR is equilateral. On the other hand, suppose, wolog, that $a \geq b \geq c$, with $a > c$. Then, for the internal angles of ΔABC, $A \geq B \geq C$. Suppose that $|PQ| = r$, $|QR| = p$ and $|PR| = q$, while s is the common length of the extensions. Then

$$p^2 = s^2 + (a+s)^2 + 2s(a+s)\cos B$$

and

$$r^2 = s^2 + (c+s)^2 + 2s(c+s)\cos A \ .$$

Since $a > c$ and $\cos B \geq \cos A$, we find that $p > r$, and so ΔPQR is not equilateral.

675. ABC is a triangle with circumcentre O such that $\angle A$ exceeds $90°$ and $AB < AC$. Let M and N be the midpoints of BC and AO, and let D be the intersection of MN and AC. Suppose that $AD = \frac{1}{2}(AB + AC)$. Determine $\angle A$.

Solution. Assign coordinates: $A \sim (0, 0)$, $B \sim (2\cos\theta, 2\sin\theta)$, $C \sim (2u, 0)$ where $90° < \theta < 180°$ and $u > 1$. First, we determine O as the intersection of the right bisectors of AB and AC. The centre of AB has coordinates $(\cos\theta, \sin\theta)$ and its right bisector has equation

$$(\cos\theta)x + (\sin\theta)y = 1 \ .$$

The centre of segment AC has coordinates $(u, 0)$ and its right bisector has equation $x = u$. Hence, we find that

$$O \sim \left(u, \frac{1 - u\cos\theta}{\sin\theta} \right)$$

$$N \sim \left(\frac{1}{2}u, \frac{1 - u\cos\theta}{2\sin\theta} \right)$$

$$M \sim (u + \cos\theta, \sin\theta)$$

and

$$D \sim (u+1, 0) \quad .$$

The slope of MD is $(\sin\theta)/(\cos\theta - 1)$. The slope of ND is $(u\cos\theta - 1)/((u+2)\sin\theta)$. Equating these two leads to the equation

$$u(\cos^2\theta - \sin^2\theta - \cos\theta) = 2\sin^2\theta + \cos\theta - 1$$

which reduces to

$$(u+1)(2\cos^2\theta - \cos\theta - 1) = 0 \; .$$

Since $u + 1 > 0$, we have that $0 = 2\cos^2\theta - \cos\theta - 1 = (2\cos\theta + 1)(\cos\theta - 1)$. Hence $\cos\theta = -1/2$ and so $\angle A = 120°$.

676. Determine all functions f from the set of reals to the set of reals which satisfy the functional equation

$$(x-y)f(x+y) - (x+y)f(x-y) = 4xy(x^2 - y^2)$$

for all real x and y.

Solution. Let u and v be any pair of real numbers. We can solve $x + y = u$ and $x - y = v$ to obtain

$$(x, y) = \left(\frac{1}{2}(u+v), \frac{1}{2}(u-v)\right) \quad .$$

From the functional equation, we find that $vf(u) - uf(v) = (u^2 - v^2)uv$, whence

$$\frac{f(u)}{u} - u^2 = \frac{f(v)}{v} - v^2 \quad .$$

Thus $(f(x)/x) - x^2$ must be some constant a, so that $f(x) = x^3 + ax$. This checks out for any constant a.

677. For vectors in three-dimensional real space, establish the identity

$$[\mathbf{a} \times (\mathbf{b} - \mathbf{c})]^2 + [\mathbf{b} \times (\mathbf{c} - \mathbf{a})]^2 + [\mathbf{c} \times (\mathbf{a} - \mathbf{b})]^2 = (\mathbf{b} \times \mathbf{c})^2 + (\mathbf{c} \times \mathbf{a})^2 + (\mathbf{a} \times \mathbf{b})^2 + (\mathbf{b} \times \mathbf{c} + \mathbf{c} \times \mathbf{a} + \mathbf{a} \times \mathbf{b})^2 \quad .$$

Solution 1. Let $\mathbf{u} = \mathbf{b} \times \mathbf{c}$, $\mathbf{v} = \mathbf{c} \times \mathbf{a}$ and $\mathbf{w} = \mathbf{a} \times \mathbf{b}$. Then, for example, $\mathbf{a} \times (\mathbf{b} - \mathbf{c}) = \mathbf{a} \times \mathbf{b} - \mathbf{a} \times \mathbf{c} = \mathbf{a} \times \mathbf{b} + \mathbf{c} \times \mathbf{a} = \mathbf{v} + \mathbf{w}$. The left side is equal to

$$(\mathbf{v}+\mathbf{w}) \cdot (\mathbf{v}+\mathbf{w}) + (\mathbf{u}+\mathbf{w}) \cdot (\mathbf{u}+\mathbf{w}) + (\mathbf{u}+\mathbf{v}) \cdot (\mathbf{u}+\mathbf{v}) = 2[(\mathbf{u} \cdot \mathbf{u}) + (\mathbf{v} \cdot \mathbf{v}) + (\mathbf{w} \cdot \mathbf{w}) + (\mathbf{u} \cdot \mathbf{v}) + (\mathbf{v} \cdot \mathbf{w}) + (\mathbf{w} \cdot \mathbf{u})]$$

while the right side is equal to

$$(\mathbf{u} \cdot \mathbf{u}) + (\mathbf{v} \cdot \mathbf{v}) + (\mathbf{w} \cdot \mathbf{w}) + (\mathbf{u} + \mathbf{v} + \mathbf{w})^2$$

which expands to the final expression for the left side.

Solution 2. For vectors $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$, we have the identities

$$(\mathbf{u} \times \mathbf{v}) \times \mathbf{w} = (\mathbf{u} \cdot \mathbf{w})\mathbf{v} - (\mathbf{v} \cdot \mathbf{w})\mathbf{u}$$

and

$$\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = (\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w} \; .$$

Using these, we find for example that

$$\begin{aligned}
[\mathbf{a} \times (\mathbf{b} - \mathbf{c})] \cdot [\mathbf{a} \times (\mathbf{b} - \mathbf{c})] &= [\mathbf{a} \times (\mathbf{b} - \mathbf{c}) \times \mathbf{a}] \cdot (\mathbf{b} - \mathbf{c}) \\
&= \{(\mathbf{a} \cdot \mathbf{a})(\mathbf{b} - \mathbf{c}) - [(\mathbf{b} - \mathbf{c}) \cdot \mathbf{a}]\mathbf{a}\} \cdot (\mathbf{b} - \mathbf{c}) \\
&= |\mathbf{a}|^2[|\mathbf{b}|^2 + |\mathbf{c}|^2 - 2(\mathbf{b} \cdot \mathbf{c})] - [(\mathbf{b} \cdot \mathbf{a} - \mathbf{c} \cdot \mathbf{a}]^2 \\
&= |\mathbf{a}|^2[|\mathbf{b}|^2 + |\mathbf{c}|^2 - 2(\mathbf{b} \cdot \mathbf{c})] - (\mathbf{b} \cdot \mathbf{a})^2 - (\mathbf{c} \cdot \mathbf{a})^2 + 2(\mathbf{b} \cdot \mathbf{a})(\mathbf{c} \cdot \mathbf{a}) \quad .
\end{aligned}$$

Also

$$\begin{aligned}(\mathbf{b}\times\mathbf{c})\cdot(\mathbf{b}\times\mathbf{c}) &= [(\mathbf{b}\cdot\mathbf{b})\mathbf{c}-(\mathbf{c}\cdot\mathbf{b})\mathbf{b}]\cdot\mathbf{c}\\ &= |\mathbf{b}|^2|\mathbf{c}|^2-(\mathbf{c}\cdot\mathbf{b})^2\end{aligned}$$

and

$$(\mathbf{b}\times\mathbf{c})\cdot(\mathbf{c}\times\mathbf{a}) = [(\mathbf{b}\times\mathbf{c})\times\mathbf{c}]\cdot\mathbf{a} = (\mathbf{b}\cdot\mathbf{c})(\mathbf{c}\cdot\mathbf{a})-(\mathbf{c}\cdot\mathbf{c})(\mathbf{b}\cdot\mathbf{a})\ .$$

From these the identity can be checked.

678. For $a, b, c > 0$, prove that

$$\frac{1}{a(b+1)}+\frac{1}{b(c+1)}+\frac{1}{c(a+1)} \geq \frac{3}{1+abc}\ .$$

Solution 1. It is easy to verify the following identity

$$\frac{1}{a(1+b)}+\frac{1}{1+abc} = \frac{1}{1+abc}\left(\frac{1+a}{a(1+b)}+\frac{b(1+c)}{1+b}\right)\ .$$

This and its analogues imply that

$$\frac{1}{a(b+1)}+\frac{1}{b(c+1)}+\frac{1}{c(a+1)}+\frac{3}{1+abc} =$$

$$\frac{1}{1+abc}\left(\frac{1+a}{a(1+b)}+\frac{b(1+c)}{1+b}+\frac{1+b}{b(1+c)}+\frac{c(1+a)}{1+c}+\frac{1+c}{c(1+a)}+\frac{a(1+b)}{1+a}\right)\ .$$

The arithmetic-geometric means inequality yields

$$\frac{1}{a(b+1)}+\frac{1}{b(c+1)}+\frac{1}{c(a+1)}+\frac{3}{1+abc} \geq 6\times\frac{1}{1+abc}\ .$$

Miraculously, subtracting $3/(1+abc)$ from both sides yields the required inequality. ♡

Solution 2. Multiplying the desired inequality by $(1+abc)a(b+1)b(c+1)c(a+1)$, after some manipulation, produces the equivalent inequality:

$$\begin{aligned}abc(bc^2+ca^2+ab^2)+(bc+ca+ab)+(abc)^2(a+b+c)+(bc^2+ca^2+ab^2)\\ \geq 2abc(a+b+c)+2abc(bc+ca+ab)\ .\end{aligned}$$

Pairing off the terms of the left side and applying the arithemetic-geometric means inequality, we get

$$\begin{aligned}&(a^2b^3c+bc)+(ab^2c^3+ac)+(a^3bc^2+ab)+(a^3b^2c^2+ab^2)\\ &\qquad+(a^2b^3c^2+bc^2)+(a^2b^2c^3+ca^2)\\ &\geq 2ab^2c+2abc^2+2a^2bc+2a^2b^2c+2ab^2c^2+2a^2bc^2\\ &= 2abc(a+b+c)+2abc(ab+bc+ca)\end{aligned}$$

as required.

Solution 3. [C. Deng] Taking the difference between the two sides yields, where the summation is a cyclic one,

$$\begin{aligned}
\sum\left(\frac{1}{a(b+1)}-\frac{1}{1+abc}\right) &= \sum\frac{1+abc-a(b+1)}{a(b+1)(1+abc)} \\
&= \frac{1}{1+abc}\sum\left(\frac{b}{b+1}(c-1)-\frac{1}{a(b+1)}(a-1)\right) \\
&= \frac{1}{1+abc}\sum\left(\frac{c}{c+1}(a-1)-\frac{1}{a(b+1)}(a-1)\right) \\
&= \frac{1}{1+abc}\sum(a-1)\left(\frac{c}{c+1}-\frac{1}{a(b+1)}\right) \\
&= \frac{1}{1+abc}\sum\left(\frac{a^2-1}{a}\right)\left(\frac{abc+ac-c-1}{(a+1)(b+1)(c+1)}\right) \\
&= \frac{1}{(1+abc)(1+a)(1+b)(1+c)}\sum\left(a^2bc+a^2c+\frac{c}{a}+\frac{1}{a}-ac-a-bc-c\right) \\
&= \frac{1}{(1+abc)(1+a)(1+b)(1+c)}\sum\left(a^2bc+a^2c-2ab-2a+\frac{b}{c}+\frac{1}{c}\right) \\
&= \frac{1}{(1+abc)(1+a)(1+b)(1+c)}\sum\frac{b+1}{c}(a^2c^2-2ac+1) \\
&= \frac{1}{(1+abc)(1+a)(1+b)(1+c)}\sum\frac{b+1}{c}(ac-1)^2 \geq 0\ ,
\end{aligned}$$

as desired.

Solution 4. [S. Seraj] Using the Arithmetic-Geometric Means Inequality, we obtain $a^2c+a^2b^2c^3 \geq 2a^2bc^2$ and $ab+a^3bc^2 \geq 2a^2bc$ and the two cyclic variants of each. Adding the six inequalities yields that

$$\begin{aligned}
&a^2c+a^2b^2c^3+ab^2+a^3b^2c^2+bc^2+a^2b^3c^2+ab+a^3bc^2+bc+a^2b^3c+ac+ab^2c^3 \\
&\qquad \geq 2a^2bc^2+2a^2b^2c+2ab^2c^2+2a^2bc+2ab^2c+2abc^2\ .
\end{aligned}$$

Adding the same terms to both sides of the equations, and then factoring the two sides leads to

$$\begin{aligned}
&(1+abc)(3abc+a^2bc+ab^2c+abc^2+a^2c+ab^2+bc^2+ab+bc+ca) \\
&\qquad \geq 3abc(abc+ac+bc+ab+a+b+c+1) = 3abc(a+1)(b+1)(c+1)\ .
\end{aligned}$$

Carrying out some divisions and strategically grouping terms in the numerator yields that

$$\frac{(abc^2+bc^2+abc+bc)+(a^2bc+a^2c+abc+ac)+(ab^2c+ab^2+abc+ab)}{abc(a+1)(b+1)(c+1)} \geq \frac{3}{1+abc}\ .$$

Factoring each bracket and simplifying leads to the desired inequality.

679. Let F_1 and F_2 be the foci of an ellipse and P be a point in the plane of the ellipse. Suppose that G_1 and G_2 are points on the ellipse for which PG_1 and PG_2 are tangents to the ellipse. Prove that $\angle F_1PG_1 = \angle F_2PG_2$.

Solution. Let H_1 be the reflection of F_1 in the tangent PG_1, and H_2 be the reflection of F_2 in the tangent PG_2. We have that $PH_1 = PF_1$ and $PF_2 = PH_2$. By the reflection property, $\angle PG_1F_2 = \angle F_1G_1Q = \angle H_1G_1Q$, where Q is a point on PG_1 produced. Therefore, H_1F_2 intersects the ellipse in G_1. Similarly, H_2F_1 intersects the ellipse in K_2. Therefore

$$\begin{aligned}
H_1F_2 &= H_1G_1+G_1F_2 = F_1G_1+G_1F_2 \\
&= F_1G_2+G_2F_2 = F_1G_2+G_2H_2 = H_2F_1\ .
\end{aligned}$$

Therefore, triangle PH_1F_2 and PF_1H_2 are congruent (SSS), so that $\angle H_1PF_2 = \angle H_2PF_1$. It follows that

$$2\angle F_1PG_1 = \angle H_1PF_1 = \angle H_2PF_2 = 2\angle F_2PG_2$$

and the desired result follows.

680. Let $u_0 = 1$, $u_1 = 2$ and $u_{n+1} = 2u_n + u_{n-1}$ for $n \geq 1$. Prove that, for every nonnegative integer n,

$$u_n = \sum \left\{ \frac{(i+j+k)!}{i!j!k!} : i,j,k \geq 0, i+j+2k = n \right\} .$$

Solution 1. Suppose that we have a supply of white and of blue coaches, each of length 1, and of red coaches, each of length 2; the coaches of each colour are indistinguishable. Let v_n be the number of trains of total length n that can be made up of red, white and blue coaches of total length n. Then $v_0 = 1$, $v_1 = 2$ and $v_2 = 5$ (R, WW, WB, BW, BB). In general, for $n \geq 1$, we can get a train of length $n+1$ by appending either a white or a blue coach to a train of length n or a red coach to a train of length $n-1$, so that $v_{n+1} = 2v_n + v_{n-1}$. Therefore $v_n = u_n$ for $n \geq 0$.

We can count v_n in another way. Suppose that the train consists of i white coaches, j blue coaches and k red coaches, so that $i+j+2k = n$. There are $(i+j+k)!$ ways of arranging the coaches in order; any permutation of the i white coaches among themselves, the j blue coaches among themselves and k red coaches among themselves does not change the train. Therefore

$$u_n = \sum \left\{ \frac{(i+j+k)!}{i!j!k!} : i,j,k \geq 0, i+j+2k = n \right\} .$$

Solution 2. Let $f(t) = \sum_{n=0}^{\infty} u_n t^n$. Then

$$\begin{aligned} f(t) &= u_0 + u_1 t + (2u_1 + u_0)t^2 + (2u_2 + u_1)t^3 + \cdots \\ &= u_0 + u_1 t + 2t(f(t) - u_0) + t^2 f(t) = u_0 + (u_1 - 2u_0)t + (2t + t^2)f(t) \end{aligned}$$

$$= 1 + (2t + t^2)f(t) ,$$

whence

$$\begin{aligned} f(t) &= \frac{1}{1 - 2t - t^2} = \frac{1}{1 - t - t - t^2} \\ &= \sum_{n=0}^{\infty} (t + t + t^2)^n = \sum_{n=0}^{\infty} t^n \left[\sum \left\{ \frac{(i+j+k)!}{i!j!k!} : i,j,k \geq 0, i+j+2k = n \right\} \right] . \end{aligned}$$

Solution 3. Let w_n be the sum in the problem. It is straightforward to check that $u_0 = w_0$ and $u_1 = w_1$. We show that, for $n \geq 1$, $w_{n+1} = 2w_n + w_{n-1}$ from which it follows by induction that $u_n = w_n$ for each n. By convention, let $(-1)! = \infty$. Then, for $i,j,k \geq 0$ and $i+j+2k = n+1$, we have that

$$\begin{aligned} \frac{(i+j+k)!}{i!j!k!} &= \frac{(i+j+k)(i+j+k-1)!}{i!j!k!} \\ &= \frac{(i+j+k-1)!}{(i-1)!j!k!} + \frac{(i+j+k-1)!}{i!(j-1)!k!} + \frac{(i+j+k-1)!}{i!j!(k-1)!} , \end{aligned}$$

whence

$$\begin{aligned} w_{n+1} = \sum &\left\{ \frac{(i+j+k-1)!}{(i-1)!j!k!} : i,j,k \geq 0, (i-1)+j+2k = n \right\} \\ &+ \sum \left\{ \frac{(i+j+k-1)!}{i!(j-1)!k!} : i,j,k \geq 0, i+(j-1)+2k = n \right\} \\ &+ \sum \left\{ \frac{(i+j+k-1)!}{i!j!(k-1)!} : i,j,k \geq 0, i+j+2(k-1) = n-1 \right\} \\ = w_n &+ w_n + w_{n-1} = 2w_n + w_{n-1} \end{aligned}$$

as desired.

681. Let $\mathbf{a}$ and $\mathbf{b}$, the latter nonzero, be vectors in $\mathbf{R}^3$. Determine the value of λ for which the vector equation

$$\mathbf{a} - (\mathbf{x} \times \mathbf{b}) = \lambda \mathbf{b}$$

is solvable, and then solve it.

Solution 1. If there is a solution, we must have $\mathbf{a} \cdot \mathbf{b} = \lambda |\mathbf{b}|^2$, so that $\lambda = (\mathbf{a} \cdot \mathbf{b})/|\mathbf{b}|^2$. On the other hand, suppose that λ has this value. Then

$$\begin{aligned} \mathbf{0} &= \mathbf{b} \times \mathbf{a} - \mathbf{b} \times (\mathbf{x} \times \mathbf{b}) \\ &= \mathbf{b} \times \mathbf{a} - [(\mathbf{b} \cdot \mathbf{b})\mathbf{x} - (\mathbf{b} \cdot \mathbf{x})\mathbf{b}] \end{aligned}$$

so that

$$\mathbf{b} \times \mathbf{a} = |\mathbf{b}|^2 \mathbf{x} - (\mathbf{b} \cdot \mathbf{x})\mathbf{b} \ .$$

A particular solution of this equation is

$$\mathbf{x} = \mathbf{u} \equiv \frac{\mathbf{b} \times \mathbf{a}}{|\mathbf{b}|^2} \ .$$

Let $\mathbf{x} = \mathbf{z}$ be any other solution. Then

$$\begin{aligned} |\mathbf{b}|^2(\mathbf{z} - \mathbf{u}) &= |\mathbf{b}|^2 \mathbf{z} - |\mathbf{b}|^2 \mathbf{u} \\ &= (\mathbf{b} \times \mathbf{a} + (\mathbf{b} \cdot \mathbf{z})\mathbf{b}) - (\mathbf{b} \times \mathbf{a} + (\mathbf{b} \cdot \mathbf{u})\mathbf{b}) \\ &= (\mathbf{b} \cdot \mathbf{z})\mathbf{b} \end{aligned}$$

so that $\mathbf{z} - \mathbf{u} = \mu \mathbf{b}$ for some scalar μ.

We check when this works. Let $\mathbf{x} = \mathbf{u} + \mu \mathbf{b}$ for some scalar μ. Then

$$\begin{aligned} \mathbf{a} - (\mathbf{x} \times \mathbf{b}) = \mathbf{a} - (\mathbf{u} \times \mathbf{b}) &= \mathbf{a} - \frac{(\mathbf{b} \times \mathbf{a}) \times \mathbf{b}}{|\mathbf{b}|^2} \\ &= \mathbf{a} + \frac{\mathbf{b} \times (\mathbf{b} \times \mathbf{a})}{|\mathbf{b}|^2} \\ &= \mathbf{a} + \frac{(\mathbf{b} \cdot \mathbf{a})\mathbf{b} - (\mathbf{b} \cdot \mathbf{b})\mathbf{a}}{|\mathbf{b}|^2} \\ &= \mathbf{a} + \left(\frac{\mathbf{b} \cdot \mathbf{a}}{|\mathbf{b}|^2} \right) \mathbf{b} - \mathbf{a} = \lambda \mathbf{b} \ , \end{aligned}$$

as desired. Hence, the solutions is

$$\mathbf{x} = \frac{\mathbf{b} \times \mathbf{a}}{|\mathbf{b}|^2} + \mu \mathbf{b} \ ,$$

where μ is an arbitrary scalar.

Solution 2. [B. Yahagni] Suppose, to begin with, that $\{\mathbf{a}, \mathbf{b}\}$ is linearly dependent. Then $\mathbf{a} = [(\mathbf{a} \cdot \mathbf{b})/|\mathbf{b}|^2]\mathbf{b}$. Since $(\mathbf{x} \times \mathbf{b}) \cdot \mathbf{b} = 0$ for all $\mathbf{x}$, the equation has no solutions except when $\lambda = (\mathbf{a} \cdot \mathbf{b})/|\mathbf{b}|^2$. In this case, it becomes $\mathbf{x} \times \mathbf{b} = \mathbf{0}$ and is satisfied by $\mathbf{x} = \mu \mathbf{b}$, where μ is any scalar.

Otherwise, $\{\mathbf{a}, \mathbf{b}, \mathbf{a} \times \mathbf{b}\}$ is linearly independent and constitutes a basis for $\mathbb{R}^3$. Let a solution be

$$\mathbf{x} = \alpha \mathbf{a} + \mu \mathbf{b} + \beta (\mathbf{a} \times \mathbf{b}) \ .$$

Then

$$\mathbf{x} \times \mathbf{b} = \alpha(\mathbf{a} \times \mathbf{b}) + \beta[(\mathbf{a} \times \mathbf{b}) \times \mathbf{b}] = \alpha(\mathbf{a} \times \mathbf{b}) + \beta(\mathbf{a} \cdot \mathbf{b})\mathbf{b} - \beta(\mathbf{b} \cdot \mathbf{b})\mathbf{a}$$

and the equation becomes

$$(1 + \beta|\mathbf{b}|^2)\mathbf{a} - \beta(\mathbf{a} \cdot \mathbf{b})\mathbf{b} - \alpha(\mathbf{a} \times \mathbf{b}) = \lambda\mathbf{b} \ .$$

Therefore $\alpha = 0$, μ is arbitrary, $\beta = -1/|\mathbf{b}|^2$ and $\lambda = -\beta(\mathbf{a} \cdot \mathbf{b}) = (\mathbf{a} \cdot \mathbf{b})/|\mathbf{b}|^2$.

Therefore, the existence of a solution requires that $\lambda = (\mathbf{a} \cdot \mathbf{b})/|\mathbf{b}|^2$ and the solution then is

$$\mathbf{x} = \mu\mathbf{b} - \frac{1}{|\mathbf{b}|^2}(\mathbf{a} \times \mathbf{b}) \ .$$

Solution 3. Writing the equation in vector components yields the system

$$b_3x_2 - b_2x_3 = a_1 - \lambda b_1 \ ;$$

$$-b_3x_1 + b_1x_3 = a_2 - \lambda b_2 \ ;$$

$$b_2x_1 - b_1x_2 = a_3 - \lambda b_3 \ .$$

The matrix of coefficients of the left side is of rank 2, so that the corresponding homogeneous system of equations has a single infinity of solutions. Multiplying the three equations by b_1, b_2 and b_3 respectively and adding yields

$$0 = a_1b_1 + a_2b_2 + a_3b_3 - \lambda(b_1^2 + b_2^2 + b_3^2) \ .$$

Thus, for a solution to exist, we require that

$$\lambda = \frac{a_1b_1 + a_2b_2 + a_3b_3}{b_1^2 + b_2^2 + b_3^2)}.$$

In addition, we learn that the corresponding homogeneous system is satisfied by

$$(x_1, x_2, x_3) = \mu(b_1, b_2, b_3)$$

where μ is an arbitrary scalar.

It remains to find a particular solution for the nonhomogeneous system. Multiplying the third equation by b_2 and subtracting the second multiplied by b_3, we obtain that

$$(b_2^2 + b_3^2)x_1 = b_1(b_2x_2 + b_3x_3) + (a_3b_2 - a_2b_3) \ .$$

Therefore, setting $b_1^2 + b_2^2 + b_3^2 = b^2$, we have that

$$b^2x_1 = b_1(b_1x_1 + b_2x_2 + b_3x_3) + (a_3b_2 - a_2b_3) \ .$$

Similarly

$$b^2x_2 = b_2(b_1x_1 + b_2x_2 + b_3x_3) + (a_1b_3 - a_3b_1) \ ,$$

$$b^2x_3 = b_3(b_1x_1 + b_2x_2 + b_3x_3) + (a_2b_1 - a_1b_2) \ .$$

Observing that $b_1x_1 + b_2x_2 + b_3x_3$ vanishes when

$$(x_1, x_2, x_3) = (a_3b_2 - a_2b_3, a_1b_3 - a_3b_1, a_2b_1 - a_1b_2) \ ,$$

we obtain a particular solution to the system:

$$(x_1, x_2, x_3) = b^{-2}(a_3b_2 - a_2b_3, a_1b_3 - a_3b_1, a_2b_1 - a_1b_2) \ .$$

Adding to this the general solution of the homogeneous system yields the solution of the nonhomogeneous system.

682. The plane is partitioned into n regions by three families of parallel lines. What is the least number of lines to ensure that $n \geq 2010$?

Solution. Suppose that there are x, y and z lines in the three families. Assume that no point is common to three distinct lines. The $x+y$ lines of the first two families partition the plane into $(x+1)(y+1)$ regions. Let λ be one of the lines of the third family. It is cut into $x+y+1$ parts by the lines in the first two families, so the number of regions is increased by $x+y+1$. Since this happens z times, the number of regions that the plane is partitioned into by the three families of

$$n = (x+1)(y+1) + z(x+y+1) = (x+y+z) + (xy+yz+zx) + 1 \ .$$

Let $u = x+y+z$ and $v = xy+yz+zx$. Then (by the Cauchy-Schwarz Inequality for example), $v \leq x^2+y^2+z^2$, so that $u^2 = x^2+y^2+z^2+2v \geq 3v$. Therefore, $n \leq u + \frac{1}{3}u^2 + 1$. This takes the value 2002 when $u = 76$. However, when $(x,y,z) = (26,26,25)$, then $u = 77$, $v = 1976$ and $n = 2044$. Therefore, we need at least 77 lines, but a suitably chosen set of 77 lines will suffice.

683. Let $f(x)$ be a quadratic polynomial. Prove that there exist quadratic polynomials $g(x)$ and $h(x)$ for which

$$f(x)f(x+1) = g(h(x)) \ ,$$

Solution 1. [A. Remorov] Let $f(x) = a(x-r)(x-s)$. Then

$$\begin{aligned} f(x)f(x+1) &= a^2(x-r)(x-s+1)(x-r+1)(x-s) \\ &= a^2(x^2+x-rx-sx+rs-r)(x^2+x-rx-sx+rs-s) \\ &= a^2[(x^2-(r+s-1)x+rs)-r][(x^2-(r+s-1)x+rs)-s] \\ &= g(h(x)) \ , \end{aligned}$$

where $g(x) = a^2(x-r)(x-s) = af(x)$ and $h(x) = x^2-(r+s-1)x+rs$.

Solution 2. Let $f(x) = ax^2+bx+c$, $g(x) = px^2+qx+r$ and $h(x) = ux^2+vx+w$. Then

$$f(x)f(x+1) = a^2x^4 + 2a(a+b)x^3 + (a^2+b^2+3ab+2ac)x^2 + (b+2c)(a+b)x + c(a+b-c)$$

$$\begin{aligned} g(h(x)) &= p(ux^2+vx+w)^2 + q(ux+vx+w) + r \\ &= pu^2x^4 + 2puvx^3 + (2puw+pv^2+qu)x^2 + (2pvw+qv)x + (pw^2+qw+r) \ . \end{aligned}$$

Equating coefficients, we find that $pu^2 = a^2$, $puv = a(a+b)$, $2puw+pv^2+qu = a^2+b^2+3ab+2ac$, $(b+2c)(a+b) = (2pw+q)v$ and $c(a+b+c) = pw^2+qw+r$. We need to find just one solution of this system. Let $p=1$ and $u=a$. Then $v = a+b$ and $b+2c = 2pw+q$ from the second and fourth equations. This yields the third equation automatically. Let $q = b$ and $w = c$. Then from the fifth equation, we find that $r = ac$.

Thus, when $f(x) = ax^2+bx+c$, we can take $g(x) = x^2+bx+ac$ and $h(x) = ax^2+(a+b)x+c$.

Solution 3. [S. Wang] Suppose that

$$f(x) = a(x+h)^2 + k = a(t-(1/2))^2 + k \ ,$$

where $t = x+h+\frac{1}{2}$. Then $f(x+1) = a(x+1+h)^2 + k = a(t+(1/2))^2 + k$, so that

$$\begin{aligned} f(x)f(x+1) &= a^2(t^2-(1/4))^2 + 2ak(t^2+(1/4)) + k^2 \\ &= a^2t^4 + \left(-\frac{a^2}{2} + 2ak\right)t^2 + \left(\frac{a^2}{16} + \frac{ak}{2} + k^2\right) \ . \end{aligned}$$

Thus, we can achieve the desired representation with $h(x) = t^2 = x^2 + (2h+1)x + \frac{1}{4}$ and $g(x) = a^2x^2 + (\frac{-a^2}{2} + 2ak)x + (\frac{a^2}{16} + \frac{ak}{2} + k^2)$.

Solution 4. [V. Krakovna] Let $f(x) = ax^2 + bx + c = au(x)$ where $u(x) = x^2 + dx + e$, where $b = ad$ and $c = ae$. If we can find functions $v(x)$ and $w(x)$ for which $u(x)u(x+1) = v(w(x))$, then $f(x)f(x+1) = a^2v(w(x))$, and we can take $h(x) = w(x)$ and $g(x) = a^2v(x)$.

Define $p(t) = u(x+t)$, so that $p(t)$ is a monic quadratic in t. Then, noting that $p''(t) = u''(x+t) = 2$, we have that

$$p(t) = u(x+t) = u(x) + u'(x)t + \frac{u''(x)}{2}t^2 = t^2 + u'(x)t + u(x) ,$$

from which we find that

$$\begin{aligned} u(x)u(x+1) = p(0)p(1) &= u(x)[u(x) + u'(x) + 1] \\ &= u(x)^2 + u'(x)u(x) + u(x) = p(u(x)) = u(x + u(x)) . \end{aligned}$$

Thus, $u(x)u(x+1) = v(w(x))$ where $w(x) = x + u(x)$ and $v(x) = u(x)$. Therefore, we get the desired representation with

$$h(x) = x + u(x) = x^2 + \left(1 + \frac{b}{a}\right)x + \frac{c}{a}$$

and

$$g(x) = a^2v(x) = a^2u(x) = af(x) = a^2x^2 + abx + ac .$$

Solution 5. [Generalization by J. Rickards.] The following statement is true: *Let the quartic polynomial $f(x)$ have roots r_1, r_2, r_3, r_4 (not necessarily distinct). Then $f(x)$ can be expressed in the form $g(h(x)$ for quadratic polynomials $g(x)$ and $h(x)$ if and only if the sum of two of r_1, r_2, r_3, r_4 is equal to the sum of the other two.*

Wolog, suppose that $r_1 + r_2 = r_3 + r_4$. Let the leading coefficient of $f(x)$ be a. Define $h(x) = (x - r_1)(x - r_2)$ and $g(x) = ax(x - r_3^2 + r_1r_3 + r_2r_3 - r_1r_2)$. Then

$$\begin{aligned} g(h(x)) &= a(x - r_1)(x - r_2)[(x - r_1)(x - r_2) - r_3^2 + r_1r_3 + r_2r_3 - r_1r_2 \\ &= a(x - r_1)(x - r_2)[x^2 - (r_1 + r_2)x - r_3^2 + r_1r_3 + r_2r_3) \\ &= a(x - r_1)(x - r_2)[x^2 - (r_3 + r_4)x + r_3(r_1 + r_2 - r_3)] \\ &= a(x - r_1)(x - r_2)(x^2 - (r_3 + r_4)x + r_3r_4 \\ &= a(x - r_1)(x - r_2)(x - r_3)(x - r_4) \end{aligned}$$

as required.

Conversely, assume that we are given quadratic polynomials $g(x) = b(x - r_5)(x - r_6)$ and $h(x)$ and that c is the leading coefficient of $h(x)$. Let $f(x) = g(h(x))$.

Suppose that

$$h(x) - r_5 = c(x - r_1)(x - r_2)$$

and that

$$h(x) - r_6 = c(x - r_3)(x - r_4) .$$

Then

$$f(x) = g(h(x)) = bc^2(x - r_1)(x - r_2)(x - r_3)(x - r_4) .$$

We have that

$$h(x) = c(x - r_1)(x - r_2) + r_5 = cx62 - c(r_1 + r_2)x + cr_1r_2 + r_5$$

and

$$h(x) = c9x - r_3)(x - r_4) + r_6 = cx^2 - c(r_3 + r_4)x + cr_3r_4 + r_6 \ ,$$

whereupon it follows that $r_1 + r_2 = r_3 + r_4$ and the desired result follows.

Comment. The second solution can also be obtained by looking at special cases, such as when $a = 1$ or $b = 0$, getting the answer and then making a conjecture.