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Problems 591-653

591. The point O is arbitrarily selected from the interior of the angle KAM. A line g is constructed through the point O, intersecting the ray AK at the point B and the ray AM at the point C. Prove that the value of the expression

$$\frac{1}{[AOB]} + \frac{1}{[AOC]}$$

does not depend on the choice of the line g. [*Note:* $[MNP]$ denotes the area of triangle MNP.]

592. The incircle of the triangle ABC is tangent to the sides BC, CA and AB at the respective points D, E and F. Points K from the line DF and L from the line EF are such that $AK \| BL \| DE$. Prove that:

(a) the points A, E, F and K are concyclic, and the points B, D, F and L are concyclic;

(b) the points C, K and L are collinear.

593. Consider all natural numbers M with the following properties:

(i) the four rightmost digits of M are 2008;

(ii) for some natural numbers $p > 1$ and $n > 1$, $M = p^n$.

Determine all numbers n for which such numbers M exist.

594. For each natural number N, denote by $S(N)$ the sum of the digits of N. Are there natural numbers N which satisfy the condition severally:

(a) $S(N) + S(N^2) = 2008$;

(b) $S(N) + S(N^2) = 2009$?

595. What are the dimensions of the greatest $n \times n$ square chessboard for which it is possible to arrange 111 coins on its cells so that the numbers of coins on any two adjacent cells (*i.e.* that share a side) differ by 1?

596. A 12×12 square array is composed of unit squares. Three squares are removed from one of its major diagonals. Is it possible to cover completely the remaining part of the array by 47 rectangular tiles of size 1×3 without overlapping any of them?

597. Find all pairs of natural numbers (x, y) that satisfy the equation

$$2x(xy - 2y - 3) = (x + y)(3x + y) \ .$$

598. Let $a_1, a_2, \cdots, a_n$ be a finite sequence of positive integers. If possible, select two indices j, k with $1 \leq j < k \leq n$ for which a_j does not divide a_k; replace a_j by the greatest common divisor of a_j and a_k, and replace a_k by the least common multiple of a_j and a_k. Prove that, if the process is repeated, it must eventually stop, and the final sequence does not depend on the choices made.

599. Determine the number of distinct solutions x with $0 \leq x \leq \pi$ for each of the following equations. Where feasible, give an explicit representation of the solution.

(a) $8 \cos x \cos 2x \cos 4x = 1$;

(b) $8 \cos x \cos 4x \cos 5x = 1$.

600. Let $0 < a < b$. Prove that, for any positive integer n,

$$\frac{b+a}{2} \le \sqrt[n]{\frac{b^{n+1}-a^{n+1}}{(b-a)(n+1)}} \le \sqrt[n]{\frac{a^n+b^n}{2}} \ .$$

601. A convex figure lies inside a given circle. The figure is seen from every point of the circumference of the circle at right angles (that is, the two rays drawn from the point and supporting the convex figure are perpendicular). Prove that the centre of the circle is a centre of symmetry of the figure.

602. Prove that, for each pair (m,n) of integers with $1 \le m \le n$,

$$\sum_{i=1}^{n} i(i-1)(i-2)\cdots(i-m+1) = \frac{(n+1)n(n-1)\cdots(n-m+1)}{m+1} \ .$$

(b) Suppose that $1 \le r \le n$; consider all subsets with r elements of the set $\{1,2,3,\cdots,n\}$. The elements of this subset are arranged in ascending order of magnitude. For $1 \le i \le r$, let t_i denote the ith smallest element in the subset, and let $T(n,r,i)$ denote the arithmetic mean of the elements t_i. Prove that

$$T(n,r,i) = i\left(\frac{n+1}{r+1}\right) \ .$$

603. For each of the following expressions severally, determine as many integer values of x as you can so that it is a perfect square. Indicate whether your list is complete or not.

(a) $1+x$;
(b) $1+x+x^2$;
(c) $1+x+x^2+x^3$;
(d) $1+x+x^2+x^3+x^4$;
(e) $1+x+x^2+x^3+x^4+x^5$.

604. $ABCD$ is a square with incircle Γ. Let l be a tangent to Γ, and let A', B', C', D' be points on l such that AA', BB', CC', DD' are all prependicular to l. Prove that $AA' \cdot CC' = BB' \cdot DD'$.

605. Prove that the number $299\cdots998200\cdots029$ can be written as the sum of three perfect squares of three consecutive numbers, where there are $n-1$ nines between the first 2 and the 8, and $n-1$ zeros between the last pair of twos.

606. Let $x_1 = 1$ and let $x_{n+1} = \sqrt{x_n + n^2}$ for each positive integer n. Prove that the sequence $\{x_n : n > 1\}$ consists solely of irrational numbers and calculate $\sum_{k=1}^{n} \lfloor x_k^2 \rfloor$, where $\lfloor x \rfloor$ is the largest integer that does not exceed x.

607. Solve the equation

$$\sin x \left(1 + \tan x \tan \frac{x}{2}\right) = 4 - \cot x \ .$$

608. Find all positive integers n for which n, n^2+1 and n^3+3 are simultaneously prime.

609. The first term of an arithmetic progression is 1 and the sum of the first nine terms is equal to 369. The first and ninth terms of the arithmetic progression coincide respectively with the first and ninth terms of a geometric progression. Find the sum of the first twenty terms of the geometric progression.

610. Solve the system of equations

$$\log_{10}(x^3 - x^2) = \log_5 y^2$$

$$\log_{10}(y^3 - y^2) = \log_5 z^2$$
$$\log_{10}(z^3 - z^2) = \log_5 x^2$$

where $x, y, z > 1$.

611. The triangle ABC is isosceles with $AB = AC$ and I and O are the respective centres of its inscribed and circumscribed circles. If D is a point on AC for which $ID \| AB$, prove that $CI \perp OD$.

612. $ABCD$ is a rectangle for which $AB > AD$. A rotation with centre A takes B to a point B' on CD; it takes C to C' and D to D'. Let P be the point of intersection of the lines CD and $C'D'$. Prove that $CB' = DP$.

613. Let ABC be a triangle and suppose that

$$\tan \frac{A}{2} = \frac{p}{u} \qquad \tan \frac{B}{2} = \frac{q}{v} \qquad \tan \frac{C}{2} = \frac{r}{w} \ ,$$

where p, q, r, u, v, w are positive integers and each fraction is written in lowest terms.

(a) Verify that $pqw + pvr + uqr = uvw$.

(b) Let f be the greatest common divisor of the pair $(vw - qr, qw + vr)$, g be the greatest common divisor of the pair $(uw - pr, pw + ur)$, and h be the greatest common divisor of the pair $(uv - pq, pv + qu)$. Prove that

$$fp = vw - qr \qquad fu = qw + vr$$
$$gq = uw - pr \qquad gv = pw + ur$$
$$hr = uv - pq \qquad hw = pv + qu \ .$$

(c) Prove that the sides of the triangle ABC are proportional to $fpu : gqv : hrw$.

614. Determine those values of the parameter a for which there exist at least one line that is tangent to the graph of the curve $y = x^3 - ax$ at one point and normal to the graph at another.

615. The function $f(x)$ is defined for real nonzero x, takes nonzero real values and satisfies the functional equation

$$f(x) + f(y) = f(xyf(x + y)) \ ,$$

whenever $xy(x + y) \neq 0$. Determine all possibilities for f.

616. Let T be a triangle in the plane whose vertices are lattice points (*i.e.*, both coordinates are integers), whose edges contain no lattice points in their interiors and whose interior contains exactly one lattice point. Must this lattice point in the interior be the centroid of the T?

617. Two circles are externally tangent at A and are internally tangent to a third circle Γ at points B and C. Suppose that D is the midpoint of the chord of Γ that passses through A and is tangent there to the two smaller given circles. Suppose, further, that the centres of the three circles are not collinear. Prove that A is the incentre of triangle BCD.

618. Let a, b, c, m be positive integers for which $abcm = 1 + a^2 + b^2 + c^2$. Show that $m = 4$, and that there are actually possibilities with this value of m.

619. Suppose that $n > 1$ and that S is the set of all polynomials of the form

$$z^n + a_{n-1}z^{n-1} + a_{n-2}z^{n-2} + \cdots + a_1 z + a_0 \ ,$$

whose coefficients are complex numbers. Determine the minimum value over all such polynomials of the maximum value of $|p(z)|$ when $|z| = 1$.

620. Let $a_1, a_2, \cdots, a_n$ be distinct integers. Prove that the polynomial

$$p(z) = (z - a_1)^2 (z - a_2)^2 \cdots (z - a_n)^2 + 1$$

cannot be written as the product of two nonconstant polynomials with integer coefficients.

621. Determine the locus of one focus of an ellipse reflected in a variable tangent to the ellipse.

622. Let I be the centre of the inscribed circle of a triangle ABC and let u, v, w be the respective lengths of IA, IB, IC. Let P be any point in the plane and p, q, r the respective lengths of PA, PB, PC. Prove that, with the sidelengths of the triangle given conventionally as a, b, c,

$$ap^2 + bq^2 + cr^2 = au^2 + bv^2 + cw^2 + (a + b + c)z^2 \ ,$$

where z is the length of IP.

623. Given the parameters a, b, c, solve the system

$$x + y + z = a + b + c;$$

$$x^2 + y^2 + x^2 = a^2 + b^2 + c^2;$$

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 3 \ .$$

624. Suppose that $x_i \geq 0$ and

$$\sum_{i=1}^{n} \frac{1}{1 + x_i} \leq 1 \ .$$

Prove that

$$\sum_{i=1}^{n} 2^{-x_i} \leq 1 \ .$$

625. Given an odd number of intervals, each of unit length, on the real line, let S be the set of numbers that are in an odd number of these intervals. Show that S is a finite union of disjoint intervals of total length not less than 1.

626. Let ABC be an isosceles triangle with $AB = AC$, and suppose that D is a point on the side BC with $BC > BD > DC$. Let BE and CF be diameters of the respective circumcircles of triangles ABD and ADC, and let P be the foot of the altitude from A to BC. Prove that $PD : AP = EF : BC$.

627. Let

$$f(x, y, z) = 2x^2 + 2y^2 - 2z^2 + \frac{7}{xy} + \frac{1}{z} \ .$$

There are three pairwise distinct numbers a, b, c for which

$$f(a, b, c) = f(b, c, a) = f(c, a, b) \ .$$

Determine $f(a, b, c)$. Determine three such numbers a, b, c.

628. Suppose that AP, BQ and CR are the altitudes of the acute triangle ABC, and that

$$9\overrightarrow{AP} + 4\overrightarrow{BQ} + 7\overrightarrow{CR} = \overrightarrow{O} \ .$$

Prove that one of the angles of triangle ABC is equal to $60°$.

629. Let $a > b > c > d > 0$ and $a + d = b + c$. Show that $ad < bc$.

(b) Let a, b, p, q, r, s be positive integers for which

$$\frac{p}{q} < \frac{a}{b} < \frac{r}{s}$$

and $qr - ps = 1$. Prove that $b \geq q + s$.

630. (a) Show that, if

$$\frac{\cos\alpha}{\cos\beta} + \frac{\sin\alpha}{\sin\beta} = -1 \quad ,$$

then

$$\frac{\cos^3\beta}{\cos\alpha} + \frac{\sin^3\beta}{\sin\alpha} = 1 \quad .$$

(b) Give an example of numbers α and β that satisfy the condition in (a) and check that both equations hold.

631. The sequence of functions $\{P_n\}$ satisfies the following relations:

$$P_1(x) = x \ , \qquad P_2(x) = x^3 \ ,$$

$$P_{n+1}(x) = \frac{P_n^3(x) - P_{n-1}(x)}{1 + P_n(x)P_{n-1}(x)} \quad , \qquad n = 1, 2, 3, \cdots .$$

Prove that all functions P_n are polynomials.

632. Let a, b, c, x, y, z be positive real numbers for which $a \leq b \leq c$, $x \leq y \leq z$, $a + b + c = x + y + z$, $abc = xyz$, and $c \leq z$, Prove that $a \leq x$.

633. Let ABC be a triangle with $BC = 2 \cdot AC - 2 \cdot AB$ and D be a point on the side BC. Prove that $\angle ABD = 2\angle ADB$ if and only if $BD = 3CD$.

634. Solve the following system for real values of x and y:

$$2^{x^2+y} + 2^{x+y^2} = 8$$

$$\sqrt{x} + \sqrt{y} = 2 \ .$$

635. Two unequal spheres in contact have a common tangent cone. The three surfaces divide space into various parts, only one of which is bounded by all three surfaces; it is "ring-shaped". Being given the radii r and R of the spheres with $r < R$, find the volume of the "ring-shaped" region in terms of r and R.

636. Let ABC be a triangle. Select points D, E, F outside of ΔABC such that ΔDBC, ΔEAC, ΔFAB are all isosceles with the equal sides meeting at these outside points and with $\angle D = \angle E = \angle F$. Prove that the lines AD, BE and CF all intersect in a common point.

637. Let n be a positive integer. Determine how many real numbers x with $1 \leq x < n$ satisfy

$$x^3 - \lfloor x^3 \rfloor = (x - \lfloor x \rfloor)^3 \ .$$

638. Let x and y be real numbers. Prove that

$$\max(0, -x) + \max(1, x, y) = \max(0, x - \max(1, y)) + \max(1, y, 1 - x, y - x)$$

where $\max(a,b)$ is the larger of the two numbers a and b.

639. (a) Let $ABCDE$ be a convex pentagon such that $AB = BC$ and $\angle BCD = \angle EAB = 90°$. Let X be a point inside the pentagon such that AX is perpendicular to BE and CX is perpendicular to BD. Show that BX is perpendicular to DE.

(b) Let N be a regular nonagon, *i.e.*, a regular polygon with nine edges, having O as the centre of its circumcircle, and let PQ and QR be adjacent edges of N. The midpoint of PQ is A and the midpoint of the radius perpendicular to QR is B. Determine the angle between AO and AB.

640. Suppose that $n \geq 2$ and that, for $1 \leq i \leq n$, we have that $x_i \geq -2$ and all the x_i are nonzero with the same sign. Prove that

$$(1 + x_1)(1 + x_2) \cdots (1 + x_n) > 1 + x_1 + x_2 + \cdots + x_n \quad ,$$

641. Observe that $x^2 + 5x + 6 = (x + 2)(x + 3)$ while $x^2 + 5x - 6 = (x + 6)(x - 1)$. Determine infinitely many coprime pairs (m, n) of positive integers for which both $x^2 + mx + n$ and $x^2 + mx - n$ can be factored as a product of linear polynomials with integer coefficients.

642. In a convex polyhedron, each vertex is the endpoint of exactly three edges and each face is a concyclic polygon. Prove that the polyhedron can be inscribed in a sphere.

643. Let n^2 distinct integers be arranged in an $n \times n$ square array ($n \geq 2$). Show that it is possible to select n numbers, one from each row and column, such that if the number selected from any row is greater than another number in this row, then this latter number is less than the number selected from its column.

644. Given a point P, a line $\mathfrak{L}$ and a circle $\mathfrak{C}$, construct with straightedge and compasses an equilateral triangle PQR with one vertex at P, another vertex Q on $\mathfrak{L}$ and the third vertex R on $\mathfrak{C}$.

645. Let $n \geq 3$ be a positive integer. Are there n positive integers $a_1, a_2, \cdots, a_n$ not all the same such that for each i with $3 \leq i \leq n$ we have

$$a_i + S_i = (a_i, S_i) + [a_i, S_i] \quad .$$

where $S_i = a_1 + a_2 + \cdots + a_i$, and where $(\cdot, \cdot)$ and $[\cdot, \cdot]$ represent the greatest common divisor and least common multiple respectively?

646. Let ABC be a triangle with incentre I. Let AI meet BC at L, and let X be the contact point of the incircle with the line BC. If D is the reflection of L in X on line BC , we construct B' and C' as the reflections of D with respect to the lines BI and CI, respectively. Show that the quadrailateral $BCC'B'$ is cyclic.

647. Find all continuous functions $f : \mathbf{R} \to \mathbf{R}$ such that

$$f(x + f(y)) = f(x) + y$$

for every $x, y \in \mathbf{R}$.

648. Prove that for every positive integer n, the integer $1 + 5^n + 5^{2n} + 5^{3n} + 5^{4n}$ is composite.

649. In the triangle ABC, $\angle BAC = 20°$ and $\angle ACB = 30°$. The point M is located in the interior of triangle ABC so that $\angle MAC = \angle MCA = 10°$. Determine $\angle BMC$.

650. Suppose that the nonzero real numbers satisfy

$$\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = \frac{1}{xyz} \ .$$

Determine the minimum value of

$$\frac{x^4}{x^2+y^2}+\frac{y^4}{y^2+z^2}+\frac{z^4}{z^2+x^2} \ .$$

651. Determine polynomials $a(t)$, $b(t)$, $c(t)$ with integer coefficients such that the equation $y^2+2y = x^3-x^2-x$ is satisfied by $(x,y) = (a(t)/c(t), b(t)/c(t))$.

652. (a) Let m be any positive integer greater than 2, such that $x^2 \equiv 1 \pmod m$ whenever the greatest common divisor of x and m is equal to 1. An example is $m = 12$. Suppose that n is a positive integer for which $n+1$ is a multiple of m. Prove that the sum of all of the divisors of n is divisible by m.

(b) Does the result in (a) hold when $m = 2$?

(c) Find all possible values of m that satisfy the condition in (a).

653. Let $f(1) = 1$ and $f(2) = 3$. Suppose that, for $n \geq 3$, $f(n) = \max\{f(r)+f(n-r) : 1 \leq r \leq n-1\}$. Determine necessary and sufficient conditions on the pair (a,b) that $f(a+b) = f(a)+f(b)$.

Solutions.

591. The point O is arbitrarily selected from the interior of the angle KAM. A line g is constructed through the point O, intersecting the ray AK at the point B and the ray AM at the point C. Prove that the value of the expression

$$\frac{1}{[AOB]}+\frac{1}{[AOC]}$$

does not depend on the choice of the line g. [*Note:* $[MNP]$ denotes the area of triangle MNP.]

Solution 1. Construct a line passing through the point O and parallel to AC. Let this line intersect the line AB at the point P. Taking note that two triangles having their bases on a line and their third vertex on a parallel line have areas in proportion to their bases, we obtain that

$$\begin{aligned}\frac{1}{[AOB]}+\frac{1}{[AOC]} &= \frac{[AOB]+[AOC]}{[AOB][AOC]} = \frac{[ABC]}{[AOB][AOC]} \\ &= \frac{[ABC]}{[AOB][AOC]}\cdot\frac{[APO]}{[APO]} = \frac{[ABC]}{[AOC]}\cdot\frac{[APO]}{[AOB]}\cdot\frac{1}{[APO]} = \frac{[ABC]}{[AOC]}\cdot\frac{|AP|}{|AB|}\cdot\frac{1}{[APO]} \\ &= \frac{[ABC]}{[AOC]}\cdot\frac{[APC]}{[ABC]}\cdot\frac{1}{[APO]} = \frac{[APC]}{[AOC]}\cdot\frac{1}{[APO]} = \frac{1}{[APO]} \ ,\end{aligned}$$

Since none of the points A, P, O depend on the position of the line g, the desired result follows.

Solution 2. Let $a = |AO|$, $b = |AB|$, $c = |AC|$, $\beta = \angle BAO$, $\gamma = \angle CAO$ and $\theta = \angle AOB$. The distance from O to AB is $a\sin\beta$ and from O to AC is $a\sin\gamma$. Therefore, $[AOB] = \frac{1}{2}ba\tan\beta$ and $[AOC] = \frac{1}{2}ca\tan\gamma$. Note that $\angle ABO = 180° - (\theta+\beta)$ and $\angle ACO = \theta - \gamma$, so that, by the Law of Sines,

$$b = \frac{a\sin\theta}{\sin(\theta+\beta)} \qquad \text{and} \qquad c = \frac{a\sin\theta}{\sin(\theta-\gamma)} \ .$$

Therefore

$$\begin{aligned}
\frac{1}{[AOB]} + \frac{1}{[AOC]} &= \frac{2}{ba\sin\beta} + \frac{2}{ca\sin\gamma} \\
&= \left(\frac{2}{a^2\sin\theta\sin\beta\sin\gamma}\right)(\sin(\theta+\beta)\sin\gamma + \sin(\theta-\gamma)\sin\beta) \\
&= \left(\frac{2}{a^2\sin\theta\sin\beta\sin\gamma}(\sin\theta\cos\beta\sin\gamma + \cos\theta sin\beta\sin\gamma + \sin\theta\cos\gamma\sin\beta - \cos\theta sin\gamma\sin\beta) \right. \\
&= \left(\frac{2}{a^2\sin\beta\sin\gamma}(\cos\beta\sin\gamma + \cos\gamma\sin\beta) = 2a^{-2}(\cot\beta + \cot\gamma\ , \right.
\end{aligned}$$

which does not depend on the variable quantities b, c and θ. The result follows.

592. The incircle of the triangle ABC is tangent to the sides BC, CA and AB at the respective points D, E and F. Points K from the line DF and L from the line EF are such that $AK \| BL \| DE$. Prove that:

(a) the points A, E, F and K are concyclic, and the points B, D, F and L are concyclic;

(b) the points C, K and L are collinear.

Solution. (a) Since AE is tanget to the circumcircle of triangle DEF and since $AK \| BL$,

$$\angle AEF = \angle EDF = \angle AKF\ ,$$

whence A, E, F, K are concyclic. Since BC is tangent to the circumcircle of triangle DEF and since $DE \| BL$,

$$\angle BDF = \angle FED = \angle LED = 180^\circ - \angle BLE = 180^\circ - \angle BLF\ ,$$

whence B, D, F, L are concyclic.

(b) Since $DE \| AK$, $AKEF$ is a concyclic quadrilateral and AB is tangent to circle DEF, we have that

$$\angle DEK = \angle EKA = \angle EFA = \angle EDK\ ,$$

whence $KD = KE$. Since $DE \| BL$, $BLFD$ is a concyclic quadrilateral and AB is tangent to circle DEF, we have that

$$\angle LDE = \angle BLD = \angle BFD = \angle LED\ ,$$

whence $LD = LE$. Since CD and CE are tangents to circle DEF, $CD = CE$. Therefore, all three points C, K, L lie on the right bisector of DE and so are collinear.

593. Consider all natural numbers M with the following properties:

(i) the four rightmost digits of M are 2008;

(ii) for some natural numbers $p > 1$ and $n > 1$, $M = p^n$.

Determine all numbers n for which such numbers M exist.

Solution. Since, modulo 10, squares are congruent to one of 0, 1, 4, 6, 9, and p^n is square for even values of n, there are no even values of n for which such a number M exists.

Since $p^n \equiv 2008 \pmod{10^4}$ implies that $p^n \equiv 8 \pmod{16}$, we see that p must be even. When p is divisible by 4, then $p^n \equiv 0 \pmod{16}$ for $n \geq 2$, and when p is twice an odd number, $p^n \equiv 0 \pmod{16}$ for $n \geq 4$. Therefore the only possibility for M is that it be the cube of a number congruent to 2 (mod 4).

The condition that $p^3 \equiv 2008 \pmod{10^4}$ implies that $p^3 \equiv 8 \pmod{125}$. Since

$$p^3 - 8 = (p-2)(p^2 + 2p + 4) = (p-2)[(p+1)^2 + 3]\ ,$$

and since the second factor is never divisible by 5 (the squares, modulo 5, are 0, 1, 4), we must have that $p \equiv 2 \pmod{125}$. Putting this together with p being twice an odd number, we find that the smallest possibilities are equal to 502 and 1002.

We have that $502^3 = 126506008$ and $1002^3 = 1006012008$. Thus, such numbers M exist if and only $n = 3$.

594. For each natural number N, denote by $S(N)$ the sum of the digits of N. Are there natural numbers N which satisfy the condition severally:

(a) $S(N) + S(N^2) = 2008$;

(b) $S(N) + S(N^2) = 2009$?

Solution. We have that

$$S(N) + S(N^2) \equiv N + N^2 = N(N+1)$$

(mod 9). This number is congruent to either 0 or 2, modulo 3. In particular, it can never assume the value of 2008, which is congruent to 1, modulo 3.

For part (b), we try a number N of the form

$$N = 1 + 10^3 + 10^6 + \cdots + 10^{3r} \ ,$$

where $100 \leq r \leq 999$. Then $S(N) = r + 1$,

$$N^2 = 1 + 2 \cdot 10^3 + 3 \cdot 10^6 + \cdots + r \cdot 10^{r-1} + (r+1) \cdot 10^r + r \cdot 10^{r+1} + \cdots + 2 \cdot 10^{6r-1} + 10^{6r}$$

and, since each coefficient of a power of 10 has at most three digits and there is no carry to a digit arising from another power,

$$S(n^2) = 2\sum_{k=1}^{r} S(k) + S(r+1) = 2\sum_{k=1}^{99} S(k) + 2\sum_{k=101}^{r} S(k) + S(r+1) \ .$$

The numbers less than 100 have 200 digits in all (counting 0 as the first digit of single-digit numbers), each appearing equally often (20 times), so that

$$2\sum_{k=1}^{99} S(k) = 2[20(1 + 2 + \cdots + 9)] = 1800 \ .$$

Now let $r = 108$. Then $S(100) + S(101) + S(108) = 9 + 36 = 45$, so that, when $N = 1001001 \cdots 1001$ with 109 ones interspersed by double zeros,

$$S(N) + S(N^2) = 109 + 1800 + 90 + 10 = 2009 \ .$$

Therefore, the equation in (b) is solvable for some natural number N.

595. What are the dimensions of the greatest $n \times n$ square chessboard for which it is possible to arrange 111 coins on its cells so that the numbers of coins on any two adjacent cells (*i.e.* that share a side) differ by 1?

Solution. We begin by establishing some restrictions.The parity of the number of coins in any two adjacent cells differ, so that at least one of any pair of adjacent cells contains at least one coin. This ensures that the number of cells cannot exceed $2 \times 111 + 1 = 2 < 15^2$, so that $n \leq 14$. Since there are 111 cells, there must be an odd number of cells that contain an odd number of coins. Since in a 14×14 chessboard, there must be $98 = \frac{1}{2} \times 196$ cells with an odd number of coins, $n = 14$ is not possible.

We show that a 13×13 chessboard admits a suitable placement of coins. Begin by placing a single coin in every second cell so that each corner cell contains one coin. This uses up 85 coins. Now place two coins in each of thirteen of the remaining 84 vacant cells. We have placed $85 + 26 = 111$ coins in such a ways as to satisfy the condition.

Hence, a 13×13 chessboard is the largest that admits the desired placement.

596. A 12×12 square array is composed of unit squares. Three squares are removed from one of its major diagonals. Is it possible to cover completely the remaining part of the array by 47 rectangular tiles of size 1×3 without overlapping any of them?

Solution. Let the major diagonal in question go from upper left to lower right. Label the cells by letters A, B, C with A in the upper left corner, so that ABC appears in this cuyclic order across each row and ACB appears in this cyclic order down each column. There are thus 48 occurrences of each label, and each cell of the major diagonal is labelled with an A. Since each horizontal or vertical placement of 1×3 tiles must cover one cell with each label, any placement of any number of such tiles must cover equally many cells of each label. However, removing three cells down the major diagonal removes three cells of a single label and leaves of dearth of cells with label A. Therefore, a covering of the remaining 141 cells with 47 tiles is not possible.

597. Find all pairs of natural numbers (x, y) that satisfy the equation

$$2x(xy - 2y - 3) = (x + y)(3x + y) \ .$$

Solution. The given equation can be rewritten as a quadratic in y:

$$y^2 + (8x - 2x^2)y + (3x^2 + 6x) = 0 \ .$$

Its discriminant is equal to

$$(64x^2 - 32x^3 + 4x^4) - 4(3x^2 + 6x) = 4x(x^3 - 8x^2 + 13x - 6) = 4x(x - 6)(x - 1)^2 \ .$$

For there to be a solution in integers, it is necessary that this discriminant be a perfect square. This happens if and only of

$$z^2 = x(x - 6) = (x - 3)^2 - 9 \ ,$$

or

$$9 = (x - 3)^2 - z^2 = (x + z - 3)(x - z - 3) \ ,$$

for some integer z. Checking all the factorizations $9 = (-9) \times (-1) = (-3) \times (-3) = (-1) \times (-9) = 9 \times 1 = 3 \times 3 = 1 \times 9$, we find that $(x, z) = (-2, \pm 4), (0, 0), (8, \pm 4), (6, 0)$.

This leads to a complete solutions set in integers:

$$(x, y) = (-2, 0), (-2, -8), (-, 0), (8, 4), (8, 60), (6, 12) \ .$$

Therefore, the only solutions in natural numbers to the equation are

$$(x, y) = (6, 12), (8, 4), (8, 60) \ ,$$

all of which check out.

598. Let $a_1, a_2, \cdots, a_n$ be a finite sequence of positive integers. If possible, select two indices j, k with $1 \le j < k \le n$ for which a_j does not divide a_k; replace a_j by the greatest common divisor of a_j and

a_k, and replace a_k by the least common multiple of a_j and a_k. Prove that, if the process is repeated, it must eventually stop, and the final sequence does not depend on the choices made.

Solution. Let $\{p_i : 1 \le i \le m\}$ be the set of of all primes, listed in some order, dividing at least one of the a_i. All the terms of any sequence thereafter are divisible by only these primes. For each sequence obtained and for each prime p_i, define a vector with n components whose sth entry is the exponent of the highest power of p_i that divides the sth term of the sequence.

Suppose that $a_j = \prod_{s=1}^m p_s^{u_s}$ and $a_k = \prod_{s=1}^m p_s^{v_s}$ are two terms of one of the sequences. Then gcd $(a_j, a_k) = \prod_{s=1}^m p_s^{w_s}$ and lcm $(a_j, a_k) = \prod_{s=1}^m p_s^{z_s}$, where w_s is the minimum and z_s is the maximum of u_s and v_s for each s. The condition that a_j divides a_k is equivalent to $u_s \le v_s$ for each s.

Let us see what the effect of the operation on a sequence has on the m vectors associated with the sequence. If two elements, the jth and kth for which the jth does not divide the kth, then there is at least one vector for which the jth term is larger than the kth term. The operation just interchanges these terms. This reduces the number of pairs of components of the vector for which the earlier one exceeds the second.

Since there are only finitely many vectors (one for each prime) and each vector has only finitely many component pairs, the process must terminate after a finite number of operations. No moves are possible only when each vector is increasing. Since each move permutes the entries of each vectors, in the final stage we must obtain the unique rearrangement of each vector in which the components are increasing. The kth terms of the vectors give the exponents of the primes p_s that constitute the prime factorization of the kth term of the sequence at the end. The result follows.

599. Determine the number of distinct solutions x with $0 \le x \le \pi$ for each of the following equations. Where feasible, give an explicit representation of the solution.

(a) $8 \cos x \cos 2x \cos 4x = 1$;

(b) $8 \cos x \cos 4x \cos 5x = 1$.

Solution 1. (a) It is clear that no multiple of π satisfies the equation. So we must have that $\sin x \neq 0$. Multiply the equation by $\sin x$ to obtain

$$8 \sin x \cos x \cos 2x \cos 4x = 4 \sin 2x \cos 2x \cos 4x = 2 \sin 4x \cos 4x = \sin 8x \ .$$

Hence the given equation is equivalent to $\sin 8x = \sin x$ with $\sin x \neq 0$. Hence, we must have $x + 8x = (2k+1)\pi$, $8x = (2k)\pi + x$, since $0 \le x \le \pi$. These lead to $x = \pi/9$ ($20°$), $x = 2\pi/7$, $x = \pi/3$ ($60°$), $x = 4\pi/7$, $x = 5\pi/9$ ($100°$), $x = 6\pi/7$ ($120°$), $x = 7\pi/9$. Thus there are seven solutions to the equation.

(b) [Z. Liu] It can be checked that no multiple of π nor any odd multiple of $\pi/4$ satisfies the equation. The truth of the equation implies that

$$\begin{aligned}\sin 8x \cos 5x &= 2 \sin 4x \cos 4x \cos 5x = 4 \sin 2x \cos 2x \cos 4x \sin 5x \\ &= (\sin x \cos 2x)(8 \cos x \cos 4x \cos 5x) = \sin x \cos 2x \ .\end{aligned}$$

Using the product to sum conversion formula yields

$$\sin 13x + \sin 3x = \sin 3x - \sin x \ ,$$

whence $\sin 13x = \sin(-x)$. Therefore, either $12x = 13x + (-x)$ is an odd multiple of π or $14x = 13x - (-x)$ is an even multiple of π. However, $x = 0, \pi/4, \pi/2, 3\pi/4$ are extraneous solutions that do not satisfies the given equation. Therefore, there are ten solutions, namely

$$x = \frac{\pi}{12}, \frac{5\pi}{12}. \frac{7\pi}{12}, \frac{11\pi}{12}, \frac{\pi}{7}, \frac{2\pi}{7}, \frac{3\pi}{7}, \frac{4\pi}{7}, \frac{5\pi}{7}, \frac{6\pi}{7} \ .$$

Solution 2. (a) Let $t = \cos x$. Then $\cos 2x = 2t^2 - 1$ and $\cos 4x = 2(2t^2-1) - 1 = 8t^4 - 8t^2 + 1$, so that

$$\cos x \cos 2x \cos 4x = t(2t^2-1)(8t^4-8t^2+1) \; .$$

Let

$$\begin{aligned} f(t) &= 8t(2t^2-1)(8t^4-8t^2+1) - 1 \\ &= 128t^7 - 192t^5 + 80t^3 - 8t - 1 \\ &= (2t-1)(64t^6 + 32t^5 - 80t^4 - 40t^3 + 20t^2 + 10t + 1) \\ &= (2t-1)(8t^3+4t^2-4t-1)(8t^3-6t-1) \; . \end{aligned}$$

(The factor $(2t-1)$ can be found by noting that $x = \pi/3$, corresponding to $t = 1/2$, is an obvious solution to the equation given in the problem.)

Let $g(t) = 8t^3+4t^2-4t-1$ and $h(t) = 8t^3-6t-1$. Since $g(-1) = -9$, $h(-1) = -1$, $g(-\frac{1}{2}) = h(-\frac{1}{2}) = 1$, $g(0) = h(0) = -1$, $g(1) = 7$ and $h(1) = 1$, both of $g(t)$ and $h(t)$ have a root in each of the intervals $(-1, -\frac{1}{2})$, $(-\frac{1}{2}, 0)$ and $(0,1)$.

Since the only roots of $g(t) - h(t) = 4t^2 + 2t = 2t(2t+1)$ are $-\frac{1}{2}$ and 0, $g(t)$ and $h(t)$ do not have a root in common. Therefore , $f(t)$ has seven roots and these correspond to seven solutions of the given equation.

(b) We have that

$$\begin{aligned} 1 &= 8\cos x \cos 4x \cos 5x = 4\cos^2 4x + 4\cos 4x \cos 6x \\ &= (2\cos 8x + 2) + (2\cos 2x + 2\cos 10x) \; , \end{aligned}$$

so that

$$2\cos 2x + 2\cos 8x + 2\cos 10x + 1 = 0 \; .$$

Substituting $t = \cos 2x$ yields $\cos 4x = 2t^2 - 1$, $\cos 8x = 8t^2 - 4t^2 + 1$, $\cos 10x = 16t^5 - 20t^3 + 5t$, so that the equation becomes

$$0 = (4t^2-3)(8t^3+4t^2-4t-1) \; .$$

The polynomial $4t^2 - 3$ has two roots in the interval $[-1,1]$ corresponding to four values of x in the interval $[0,\pi]$. Let $f(t) = 8t^3 + 4t^2 - 4t - 1$. Since $f(-1) = -1$, $f(-\frac{1}{2}) = 1$, $f(0) = -1$, $f(1) = 7$, $f(t)$ has three real roots, once in each of the intervals $(-1,-\frac{1}{2})$, $(-\frac{1}{2},0)$, $(0,1)$, and each of these corresponds to two solution x in the interval $[0,\pi]$. Therefore, the equation in x has ten solutions in the interval.

Comments. (a) The seven solutions of the equation $\sin 8x = \sin x$ can be seen from a sketch of the graphs of the two functions on the same axes.

(b) Since $2\cos x \cos 5x = \cos 4x + \cos 6x$, the equation is equivalent to

$$4(\cos^2 4x + \cos 4x \cos 6x) = 1 \; .$$

Some solutions can be found by solving $\cos 6x = 0$ and $\cos^2 4x = \frac{1}{4}$. These are satisfied by $x = \pi/12$, $5\pi/12$, $7\pi/12$ and $11\pi/12$.

The trial, taking $\cos 4x = \frac{1}{2}$, is also reasonable, as it gives $x = \pi/12$. With this substitution, the left side become $4\cos\pi/12 \sin\pi 12 = 2\sin\pi/6 = 1$. The other multiples of $\pi/12$ can be handled in the same way.

When $t = \cos 2x$, there is another route to the equation in t to be analyzed. The equation, in the form, $1 = 4(\cos 4x)(\cos 4x + \cos 6x)$, is transformed to

$$1 = 4(2t^2-1)(2t^2-1+4t^3-3t) = 4(8t^5 + 4t^4 - 10t^2 - 4t^2 + 3t + 1) \; .$$

This simplifies to

$$\begin{aligned} 0 &= 32t^5 + 16t^4 - 40t^3 - 16t^2 + 12t + 3 \\ &= (4t^2-3)(8t^3+4t^2-4t-1) \; . \end{aligned}$$

Since $x = \pi/12$ is a solution, $t = \cos \pi/6 = \sqrt{3}/2$ satisfies the equation in t and accounts for the factor $4t^2 - 3$ on the right side of the equation.

600. Let $0 < a < b$. Prove that, for any positive integer n,

$$\frac{b+a}{2} \le \sqrt[n]{\frac{b^{n+1} - a^{n+1}}{(b-a)(n+1)}} \le \sqrt[n]{\frac{a^n + b^n}{2}} \ .$$

Solution 1. Dividing the inequality through by $(b+a)/2$ yields the equivalent inequality

$$1 \le \sqrt[n]{\frac{b'^{n+1} - a'^{n+1}}{(b'-a')(n+1)}} \le \sqrt[n]{\frac{a'^n + b'^n}{2}} \ ,$$

with $a' = (2a)/(b+a)$ and $b' = (2b)/(b+a)$. Note that $(a'+b')/2 = 1$. and we can write $b' = 1+u$ and $a' = 1-u$ with $0 < u < 1$. The central term becomes the nth root of

$$\begin{aligned} \frac{(1+u)^{n+1} - (1-u)^{n+1}}{2(n+1)u} &= \frac{2[(n+1)u + \binom{n+1}{3}u^3 + \binom{n+1}{5}u^5 + \cdots]}{2(n+1)u} \\ &= 1 + \frac{1}{3}\binom{n}{2}u^2 + \frac{1}{5}\binom{n}{4}u^4 + \cdots \end{aligned}$$

which clearly exceeds 1 and gives the left inequality. The right term become the nth roots of

$$\frac{1}{2}[(1+u)^n + (1-u)^n] = 1 + \binom{n}{2}u^2 + \binom{n}{4}u^4 + \cdots$$

and the right inequality is true.

Solution 2. The inequality

$$\sqrt[n]{\frac{b^{n+1} - a^{n+1}}{(b-a)(n+1)}} \le \sqrt[n]{\frac{a^n + b^n}{2}}$$

is equivalent to

$$0 \le (n+1)(a^n + b^n) - \frac{2(b^{n+1} - a^{n+1})}{b-a}.$$

The right side is equal to

$$\begin{aligned} &(n+1)(a^n + b^n) - 2(b^n + b^{n-1}a + b^{n-2}a^2 \cdots + b^2a^{n-2} + ba^{n-1} + a^n) \\ &= (a^n - b^n) + (a^n - b^{n-1}a) + (a^n - b^{n-2}a) + \cdots + (a^n - ba^{n-1}) + (a^n - a^n) \\ &\qquad + (b^n - b^n) + (b^n - b^{n-1}a) + \cdots + (b^n - ba^{n-1}) + (b^n - a^n) \\ &= (a^n - b^n) + a(a^{n-1} - b^{n-1}) + a^2(a^{n-2} - b^{n-2}) + \cdots + a^{n-1}(a-b) + 0 \\ &\qquad + 0 + b^{n-1}(b-a) + \cdots + b(b^{n-1} - a^{n-1}) + (b^n - a^n) \\ &= 0 + (b-a)(b^{n-1} - a^{n-1}) + (b^2 - a^2)(b^{n-2} - a^{n-2}) + \cdots + (b^{n-1} - a^{n-1})(b-a) \\ &> 0 \ . \end{aligned}$$

The left inequality

$$\frac{b+a}{2} \le \sqrt[n]{\frac{b^{n+1} - a^{n+1}}{(b-a)(n+1)}}$$

is equivalent to

$$\left(\frac{b+a}{2}\right)^n \le \frac{b^{n+1}-a^{n+1}}{(b-a)(n+1)} \quad .$$

Let $v = \frac{1}{2}(b-a)$ so that $b+a = 2(a+v)$. Then

$$\begin{aligned}
\frac{b^{n+1}-a^{n+1}}{(b-a)(n+1)} - \left(\frac{b+a}{2}\right)^n &= \frac{(a+2v)^{n+1}-a^{n+1}}{(b-a)(n+1)} - (a+v)^n \\
&= \frac{1}{2v(n+1)}\left(\sum_{k=1}^{n+1}\binom{n+1}{k}a^{n+1-k}(2v)^k\right) - (a+v)^n \\
&= \frac{1}{n+1}\left(\sum_{k=1}^{n+1}\binom{n+1}{k}a^{n-(k-1)}(2v)^{k-1}\right) - (a+v)^n \\
&= \frac{1}{n+1}\left(\sum_{k=0}^{n}\binom{n+1}{k+1}a^{n-k}(2v)^k\right) - \sum_{k=0}^{n}\binom{n}{k}a^{n-k}v^k \\
&= \left(\sum_{k=0}^{n}\frac{1}{k+1}\binom{n}{k}a^{n-k}(2v)^k - \sum_{k=0}^{n}\binom{n}{k}a^{n-k}v^k\right) \\
&= \sum_{k=0}^{n}\left(\frac{2^k}{k+1}-1\right)\binom{n}{k}a^{n-k}v^k \ge 0 \quad ,
\end{aligned}$$

since $2^k = (1+1)^k = 1 + k + \binom{k}{2} + \cdots \ge 1+k$ with equality if and only if $k=0$ or 1. The result follows.

Solution 3. [D. Nicholson] (partial) Let $n \ge 2i+1$. Then

$$(b^{n-i}a^i + b^i a^{n-i}) - (b^{n-i-1}a^{i+1} + b^{i+1}a^{n-i-1}) = (b-a)a^i b^i (b^{n-2i-1} - a^{n-2i-1}) \ge 0 \; .$$

Hence, for $0 \le j \le \frac{1}{2}(n+1)$,

$$b^n + a^n \ge b^{n-1}a + ab^{n-1} \ge \cdots \ge b^{n-j}a^j + b^j a^{n-j} \quad .$$

When $n = 2k+1$,

$$b^n + b^{n-1}a + \cdots + ba^{n-1} + a^n = \sum_{i=0}^{k}(b^{n-i}a^i + b^i a^{n-i}) \le (k+1)(b^n + a^n) = \frac{n+1}{2}(b^n + a^n)$$

and when $n = 2k$, we use the Arithmetic-Geometric Means Inequality to obtain $b^k a^k \le \frac{1}{2}(a^{2k} + b^{2k})$, so that

$$b^n + b^{n-1}a + \cdots + ba^{n-1} + a^n = \sum_{i=0}^{k-1}(b^{n-i}a^i + b^i a^{n-i}) + b^k a^k \le k(b^n + a^n) + \frac{b^n + a^n}{2} = \frac{n+1}{2}(b^n + a^n) \; .$$

Hence

$$\frac{b^{n+1}-a^{n+1}}{(b-a)(n+1)} \le \frac{b^n + a^n}{2} \; .$$

Solution 4. [Y. Shen] Let $1 \le k \le n$ and $1 \le i \le k$. Then

$$(b^{k+1} + a^{k+1}) - (b^i a^{k+1-i} + a^i b^{k+1-i}) = (b^i - a^i)(b^{k+1-i} - a^{k+1-i}) \ge 0 \; .$$

Hence

$$k(b^{k+1} + a^{k+1}) \ge \sum_{i=1}^{k}(b^i a^{k+1-i} + a^i b^{k+1-i}) = 2\sum_{i=1}^{k} b^i a^{k+1-i} \quad .$$

This is equivalent to

$$\begin{aligned}(2k+2)\sum_{i=0}^{k+1} b^i a^{k+1-i} &= (2k+2)(b^{k+1}+a^{k+1}) + (2k+2)\sum_{i=1}^{k} b^i a^{k+1-i} \\ &\geq (k+2)(b^{k+1}+a^{k+1}) + (2k+4)\sum_{i=1}^{k} b^i a^{k+1-i} \\ &= (k+2)(b^{k+1} + 2\sum_{i=1}^{k} b^i a^{k+1-i} + a^{k+1}) \\ &= (k+2)(b+a)\sum_{i=0}^{k} b^i a^{k-i}\end{aligned}$$

which in turn is equivalent to

$$\frac{\sum_{i=0}^{k+1} b^i a^{k+1-i}}{k+2} \geq \frac{(b+a)(\sum_{i=0}^{k} b^i a^{k-i})}{2(k+1)} \quad .$$

We establish by induction that

$$\left(\frac{b+a}{2}\right)^n \leq \frac{1}{n+1}\sum_{i=0}^{n} b^i a^{n-i}$$

which will yield the left inequality. This holds for $n = 1$. Suppose that it holds for $n = k$. Then

$$\begin{aligned}\left(\frac{b+a}{2}\right) &= \left(\frac{b+a}{2}\right)\cdot\left(\frac{b+a}{2}\right)^k \\ &\leq \left(\frac{b+a}{2}\right)\cdot\left(\frac{1}{k+1}\right)\sum_{i=0}^{k} b^i a^{k-i} \leq \frac{1}{k+2}\sum_{i=0}^{k+1} b^i a^{k-i} \quad .\end{aligned}$$

As above, we have, for $k = n-1$,

$$(n-1)(b^n + a^n) \geq 2\sum_{i=1}^{n-1} b^i a^{n-i}$$

so that

$$(n+1)(b^n + a^n) \geq 2\sum_{i=0}^{n} b^i a^{n-i} = 2\left(\frac{b^{n+1} - a^{n+1}}{b-a}\right)$$

from which the right inequality follows.

Comment. The inequality

$$\frac{b+a}{2} \leq \sqrt[n]{\frac{b^{n+1} - a^{n+1}}{(b-a)(n+1)}}$$

is equivalent to

$$0 \leq \frac{2^n(b^{n+1} - a^{n+1})}{b-a} - (n+1)(b+a)^n \ .$$

When $n = 1$, the right side is equal to 0. When $n = 2$, it is equal to

$$4(b^2 + ba + a^2) - 3(b+a)^2 = (b-a)^2 > 0 \ .$$

When $n = 3$, we have

$$8(b^3 + b^2a + ba^2 + a^3) - 4(b + a)^3 = 4b^3 - 4b^2a - 4ba^2 + 4a^3 = 4(b^2 - a^2)(b - a) = 4(b + a)(b - a)^2 > 0 \ .$$

When $n = 4, 5$ and 6, the right side is, respectively,

$$(11b^2 + 18ab + 11a^2)(b - a)^2$$

$$(26b^3 + 54b^2a + 54ba^2 + 26a^3)(b - a)^2$$

$$(57b^4 + 136b^3a + 174b^2a^2 + 136ba^3 + 57a^4)(b - a)^2 \ .$$

There is a pattern here; can anyone express it in a general way that will yield the result, or at least show that the right side is the product of $(b - a)^2$ and a polynomial with positive coefficients?

601. A convex figure lies inside a given circle. The figure is seen from every point of the circumference of the circle at right angles (that is, the two rays drawn from the point and supporting the convex figure are perpendicular). Prove that the centre of the circle is a centre of symmetry of the figure.

Solution 1. Let the figure be denoted by $\mathfrak{F}$ and the circle by $\mathfrak{C}$, and let ρ be the central reflection through the centre of the circle. Suppose that m is any line of support for $\mathfrak{F}$ and that it intersects the circle in P and Q. Then there are lines p and q through P and Q respectively, perpendicular to m, which support $\mathfrak{F}$. Let p meet the circle in P and R, and q meet it in Q and S; let t be the line RS. Since $PQRS$ is concyclic with adjacent right angles, it is a rectangle, and t is a line of support of $\mathfrak{F}$. Since PS and RQ are both diameters of $\mathfrak{C}$, it follows that $S = \rho(P)$, $R = \rho(Q)$ and $t = \rho(m)$.

Hence, every line of support of $\mathfrak{F}$ is carried by ρ into a line of support of $\mathfrak{F}$. We note that $\mathfrak{F}$ must be on the same side of its line of support as the centre of the circle.

Suppose that $X \in \mathfrak{F}$. Let $Y = \rho(X)$. Suppose, if possible that $Y \not\in \mathfrak{F}$. Then there must be a disc containing Y that does not intersect $\mathfrak{F}$, so we can find a line m of support for $\mathfrak{F}$ such that $\mathfrak{F}$ is on one side and Y is strictly on the other side of m. Let $n = \rho(m)$. Then n is a line of support for $\mathfrak{F}$ which has $X = \rho(Y)$ on one side and $O = \rho(O)$ on the other. But this is not possible. Hence $Y \in \mathfrak{F}$ and so $\rho(\mathfrak{F}) \subseteq \mathfrak{F}$. Now $\rho \circ \rho$ is the identity mapping, so $\mathfrak{F} = \rho(\rho(\mathfrak{F})) \subseteq \rho(\mathfrak{F})$. It follows that $\mathfrak{F} = \rho(\mathfrak{F})$ and the result follows.

Solution 2. Let P be any point on the circle $\mathfrak{C}$. There are two perpendicular lines of support from P meeting the circle in Q and S. As in the first solution, we see that P is one vertex of a rectangle $PQRS$ each of whose sides supports $\mathfrak{F}$. Let $\mathfrak{G}$ be the intersection of all the rectangles as P ranges over the circumference of the circle $\mathfrak{C}$. Since each rectangle has central symmetry about the centre of $\mathfrak{C}$, the same is true of $\mathfrak{G}$. It is clear that $\mathfrak{F} \subseteq \mathfrak{G}$. It remains to show that $\mathfrak{G} \subseteq \mathfrak{F}$. Suppose a point X in $\mathfrak{G}$ does not belong to $\mathfrak{F}$. Then there is a line r of support to $\mathfrak{F}$ for which X and $\mathfrak{F}$ are on opposite sides. This line of support intersects $\mathfrak{C}$ at the endpoints of a chord which must be a side of a supporting rectangle for $\mathfrak{F}$. The point X lies outside this rectangle, and so must lie outside of $\mathfrak{G}$. The result follows.

Solution 3. [D. Arthur] If the result is false, then there is a line through the centre of the circle such that $OP > OQ$, where P is where the line meets the boundary of the figure on one side and Q is where it meets the boundary on the other. Let m be the line of support of the figure through Q. Then, as shown in Solution 1, its reflection t in the centre of the circle is also a line of support. But then P and O lie on opposite sides of t and we obtain a contradiction.

602. Prove that, for each pair (m, n) of integers with $1 \leq m \leq n$,

$$\sum_{i=1}^{n} i(i-1)(i-2)\cdots(i-m+1) = \frac{(n+1)n(n-1)\cdots(n-m+1)}{m+1} \ .$$

(b) Suppose that $1 \leq r \leq n$; consider all subsets with r elements of the set $\{1, 2, 3, \cdots, n\}$. The elements of this subset are arranged in ascending order of magnitude. For $1 \leq i \leq r$, let t_i denote the ith smallest element in the subset, and let $T(n, r, i)$ denote the arithmetic mean of the elements t_i. Prove that

$$T(n, r, i) = i\left(\frac{n+1}{r+1}\right) .$$

(a) *Solution 1.* $i(i-1)(i-2)\cdots(i-m+1) = \frac{[(i+1)-(i-m)]}{m+1} i(i-1)(i-2)\cdots(i-m+1)$

$$= \frac{(i+1)i(i-1)\cdots(i-m+1) - i(i-1)(i-2)\cdots(i-m+1)(i-m)}{m+1}$$

so that

$$\begin{aligned}\sum_{i=1}^{n} i(i-1)(i-2)\cdots(i-m+1) &= \sum_{i=2}^{n+1} \frac{i(i-1)\cdots(i-m)}{m+1} - \sum_{i=1}^{n} \frac{i(i-1)\cdots(i-m)}{m+1} \\ &= \frac{(n+1)n(n-1)\cdots(n-m+1)}{m+1} - 0 \\ &= \frac{(n+1)n(n-1)\cdots(n-m+1)}{m+1} .\end{aligned}$$

(a) *Solution 2.* [W. Choi] Recall the identity

$$\sum_{i=m}^{n} \binom{i}{m} = \binom{n+1}{m+1}$$

which is obvious for $n = m$ and can be established by induction for $n \geq m+1$. There is an alternative combinatorial argument. Consider the number $\binom{n+1}{m+1}$ of selecting $m+1$ numbers from the set $\{1, 2, 3, \cdots, n+1\}$. The largest number must be $i+1$ where $m \leq i \leq n$, and the number of $(m+1)-$sets for which the largest number is $i+1$ is $\binom{i}{m}$. Summing over all relevant i yields the result.

We have that

$$\begin{aligned}\sum_{i=1}^{m} i(i-1)\cdots(i-m+1) &= \sum_{i=m}^{n} \frac{i!}{(i-m)!} = m! \sum_{i=m}^{n} \binom{i}{m} = m!\binom{n+1}{m+1} \\ &= \frac{(n+1)!}{(m+1)(n-m)!} = \frac{(n+1)n(n-1)\cdots(n-m+1)}{m+1} .\end{aligned}$$

(a) *Solution 3.* [K. Yeats] Let $n = m + k$. Then

$$\sum_{i=1}^{n} i(i-1)(i-2)\cdots(i-m+1) = m! + \frac{(m+1)!}{1!} + \cdots + \frac{n!}{(n-m)!}$$

$$= \frac{1}{(m+1)k!}\left[(m+1)!k! + \frac{(m+1)!k!(m+1)}{1!} + \frac{(m+2)!k!(m+1)}{2!} + \cdots + n!(m+1)\right]$$

$$= \frac{(m+1)!}{(m+1)k!}\left[k! + \frac{k!}{1!}(m+1) + \frac{k!}{2!}(m+2)(m+1) + \cdots + n(n-1)\cdots(m+2)(m+1)\right] .$$

The quantity in square brackets has the form (with $q = 0$)

$$\frac{k!}{q!} + \frac{k!}{(q+1)!}(m+1) + \frac{k!}{(q+2)!}(m+q+2)(m+1) + \frac{k!}{(q+3)!}(m+q+3)(m+q+2)(m+1) + \cdots$$

$$+\frac{k!}{k!}n(n-1)\cdots(m+q+2)(m+1)$$

$$= (m+q+2)\left[\frac{k!}{(q+1)!}+\frac{k!}{(q+2)!}(m+1)+\frac{k!}{(q+3)!}(m+q+3)(m+1)+\cdots+\frac{k!}{k!}n\cdots(m+q+3)(m+1)\right].$$

Applying this repeatedly with $q = 0, 1, 2, \cdots, k-1$ leads to the expression for the left sum in the problem of

$$\frac{(m+k+1)!}{(m+1)k!}\left[\frac{k!}{k!}\right] = \frac{(n+1)!}{(m+1)(n-m)!} = \frac{(n+1)n(n-1)\cdots(n-m+1)}{m+1} \quad .$$

[A variant, due to D. Nicholson, uses an induction on r to prove that, for $m \leq r \leq n$,

$$\sum_{i=m}^{r} i(i-1)\cdots(i-m+1) = \frac{(r+1)!}{(r-m)!(m+1)} \quad .]$$

(a) *Solution 4.* For $1 \leq i \leq m-1$, $i(i-1)\cdots(i-m+1) = 0$. For $m \leq i \leq n$, $i(i-1)\cdots(i-m+1) = m!\binom{i}{m}$. Also,

$$\frac{(n+1)n\cdots(n-m+1)}{m+1} = m!\binom{n+1}{m+1}$$

so the statement is equivalent to

$$\sum_{m}^{n}\binom{i}{m} = \binom{n+1}{m+1} \quad .$$

This is clear for $n = m$. Suppose it holds for $n = k \geq m$. Then

$$\sum_{i=m}^{k+1}\binom{i}{m} = \binom{k+1}{m+1} + \binom{k+1}{m} = \binom{k+2}{m+1}$$

and the result follows by induction.

(a) *Solution 5.* Use induction on n. If $n = 1$, then $m = 1$ and both sides of the equation are equal to 1. Suppose that the result holds for $n = k$ and $1 \leq m \leq k$. Then, for $1 \leq m \leq k$,

$$\begin{aligned}\sum_{i=1}^{k+1} i(i-1)\cdots(i-m+1) &= \frac{(k+1)k(k-1)\cdots(k-m+1)}{m+1} + (k+1)k(k-1)\cdots(k-m+2) \\ &= \frac{(k+1)k(k-1)\cdots(k-m+2)}{m+1}[(k-m+1)+(m+1)] \\ &= \frac{(k+2)(k+1)k(k-1)\cdots(k-m+2)}{m+1}\end{aligned}$$

as desired. When $m = n = k+1$, all terms on the left have $k+1$ terms and so they vanish except for the one corresponding to $i = k+1$. This one is equal to $(k+1)!$ and so to the right side.

(b) *Solution 1.* For $1 \leq i \leq r \leq n$, let $S(n,r,i)$ be the sum of the elements t_i where $(t_1, t_2, \cdots, t_r)$ runs over r-tples with $1 \leq t_1 < t_2 < \cdots < t_r \leq n$. Then $S(n,r,i) = \binom{n}{r}T(n,r,i)$. For $1 \leq k \leq n$, $1 \leq i \leq r$, the number of ordered $r-$tples $(t_1, t_2, \cdots, t_r)$ with $t_i = k$ is $\binom{k-1}{i-1}\binom{n-k}{r-i}$ where $\binom{0}{0} = 1$ and $\binom{a}{b} = 0$ when $b > a$. Hence

$$\binom{n}{r} = \sum_{k=1}^{n}\binom{k-1}{i-1}\binom{n-k}{r-i} \quad .$$

Replacing n by $n+1$ and r by $r+1$ yields a reading

$$\binom{n+1}{r+1} = \sum_{k=1}^{n+1}\binom{k-1}{i-1}\binom{n+1-k}{r-(i-1)} \qquad \text{for} \quad 1 \leq i \leq r+1 \ .$$

Replacing $i-1$ by i yields

$$\binom{n+1}{r+1} = \sum_{k=1}^{n+1} \binom{k-1}{i}\binom{n+1-k}{r-i} \qquad \text{for} \ \ 0 \leq i \leq r \ .$$

When $1 \leq i \leq r$, the first term of the sum is 0, so that

$$\binom{n+1}{r+1} = \sum_{k=2}^{n+1} \binom{k-1}{i}\binom{n-(k-1)}{r-i} = \sum_{k=1}^{n} \binom{k}{i}\binom{n-k}{r-i} \quad .$$

Thus

$$S(n,r,i) = \sum_{k=1}^{n} k\binom{k-1}{i-1}\binom{n-k}{r-i} = i\sum_{k=1}^{n} \binom{k}{i}\binom{n-k}{r-i} = i\binom{n+1}{r+1}$$

so

$$T(n,r,i) = i\left(\frac{n+1}{r+1}\right) \quad .$$

(b) *Solution 2.* [Z. Liu] Define $S(n,r,i)$ for $1 \leq i \leq r \leq n$ as in Solution 1. We prove by induction that

$$S(n,r,i) = i\binom{n+1}{r+1}$$

from which

$$T(n,r,i) = i\left(\frac{n+1}{r+1}\right) \quad .$$

For each positive integer n, we have that $S(n,1,1) = 1+2+\cdots+n = \binom{n+1}{2}$ and $S(n,n,i) = i$. Suppose that $n \geq 2$, $r \geq 2$ and that $S(k,r,1) = \binom{k+1}{r+1}$ for $1 \leq k \leq n-1$. Of the $\binom{n}{r}$ $r-$tples from $\{1,2,\cdots,n\}$, $\binom{n-1}{r-1}$ of them have smallest element equal to 1, and $\binom{n-1}{r}$ of them have smallest element exceeding 1. The latter set of $r-$tples can be put into one-one corrrespondence with $r-$tples of $\{1,2,\cdots,n-1\}$ by subtracting one from each entry. Therefore the sum of the first (smallest) elements of the latter $r-$tples is $\binom{n-1}{r} + S(n-1,r,1)$. Hence

$$S(n,r,1) = \binom{n-1}{r-1} + \binom{n-1}{r} + S(n-1,r,1) = \binom{n}{r} + \binom{n}{r+1} = \binom{n+1}{r+1} \ .$$

Suppose as an induction hypothesis that

$$S(m,s,j) = j\binom{m+1}{s+1}$$

for $1 \leq j \leq s \leq n-1$. This holds for $n=2$. Let $r \geq 2$ and $1 \leq i \leq r \leq n-1$. Consider the ordered $r-$subsets of $\{1,2,\cdots,n\}$. There are $\binom{n-1}{r-1}$ of them that begin with 1; making use of the one-one correspondence between these and $(r-1)-$ subsets of $\{1,2\cdots,n-1\}$ obtained by subtracting 1 from each entry beyond the first, we have that the sum of the ith elements of these is

$$\binom{n-1}{r-1} + S(n-1,r-1,i-1) = \binom{n-1}{r-1} + (i-1)\binom{n}{r} \ .$$

There are $\binom{n-1}{r}$ of the ordered subsets that do not begin with 1; making use of the one-one correspondence between these subsets and the $r-$subsets of $\{1,2,\cdots,n-1\}$ obtained by subtracting 1 from each entry, we find that the sum of the ith elements is

$$\binom{n-1}{r} + S(n-1,r,i) = \binom{n-1}{r} + i\binom{n}{r+1} \ .$$

Hence the sum of the ith elements of all these $r-$subsets is

$$S(n,r,i) = \left[\binom{n-1}{r-1} + \binom{n-1}{r} - \binom{n}{r}\right] + i\left[\binom{n}{r} + \binom{n}{r+1}\right] = 0 + i\binom{n+1}{r+1} .$$

Putting all these elements together yields the result.

(b) *Solution 3.* When $r = 1$, we have that

$$T(n,1,1) = \frac{1+2+\cdots+n}{n} = \frac{n+1}{2} .$$

When $r = 2$, the subsets are $\{1,2\}, \{1,3\}, \cdots, \{1,n\}, \{2,3\}, \{2,4\}, \cdots \{2,n\}, \cdots, \{n-1,n\}$, so that

$$\begin{aligned} T(n,2,1) &= \frac{1\times(n-1) + 2\times(n-2) + \cdots + (n-1)\times 1}{\binom{n}{2}} \\ &= \frac{[(n-1)+(n-2)+\cdots+1] + [(n-2)+(n-3)+\cdots+1] + \cdots + 1}{\binom{n}{2}} \\ &= \frac{\sum_{j=1}^{n-1}[1+2+\cdots+(n-j)]}{n(n-1)/2} = \frac{\sum_{j=1}^{n-1}(n-j+1)(n-j)/2}{n(n-1)/2} \\ &= \frac{(1/6)(n+1)n(n-1)}{(1/2)n(n-1)} = \frac{n+1}{3} , \end{aligned}$$

and

$$\begin{aligned} T(n,2,2) &= \frac{(n-1)\times n + (n-2)\times(n-1) + \cdots + 1\times 2}{\binom{n}{2}} \\ &= \frac{(n+1)n(n-1)/3}{n(n-1)/2} = 2\left(\frac{n+1}{3}\right) . \end{aligned}$$

Thus, the result holds for $n = 1, 2$ and all i, r with $1 \le i \le r \le n$, and for all n and $1 \le i \le r \le 2$. Suppose as an induction hypothesis, we have established the result up to $n-1$ and all appropriate r and i, and for n and $1 \le i \le r-1$. The $r-$element subsets of $\{1,2,\cdots,n\}$ have $\binom{n-1}{r}$ instances without n and $\binom{n-1}{r-1}$ instances with n.

Let $1 \le i \le r-1$. Then

$$\begin{aligned} T(n,r,i) &= \frac{\binom{n-1}{r}T(n-1,r,i) + \binom{n-1}{r-1}T(n-1,r-1,i)}{\binom{n}{r}} \\ &= \frac{i[\binom{n-1}{r}\frac{n}{r+1} + \binom{n-1}{r-1}\frac{n}{r}]}{\binom{n}{r}} = \frac{i[\binom{n}{r+1} + \binom{n}{r}]}{\binom{n}{r}} \\ &= i\frac{\binom{n+1}{r+1}}{\binom{n}{r}} = i\left(\frac{n+1}{r+1}\right) \end{aligned}$$

Also

$$\begin{aligned} T(n,r,r) &= \frac{\binom{n-1}{r}T(n-1,r,r) + \binom{n-1}{r-1}n}{\binom{n}{r}} \\ &= \frac{\binom{n-1}{r}\frac{rn}{r+1} + \binom{n-1}{r-1}\frac{rn}{r}}{\binom{n}{r}} = \frac{r[\binom{n}{r+1} + \binom{n}{r}]}{\binom{n}{r}} = r\left(\frac{n+1}{r+1}\right) . \end{aligned}$$

(b) *Solution 4.* For $1 \le i \le r \le n$, let $S(n,r,i)$ be the sum of the elements t_i where $(t_1, t_2, \cdots, t_r)$ runs over r-tples with $1 \le t_1 < t_2 < \cdots < t_r \le n$. Then $S(n,r,i) = \binom{n}{r}T(n,r,i)$. We observe first that

$$S(n,r,i) = S(n-1,r-1,i) + S(n-2,r-1,i) + \cdots + S(r-1,r-1,i)$$

for $1 \leq i \leq r-1$. This is true, since, for each j with $1 \leq j \leq n-r+1$, $S(n-j, r-1, i)$ adds the t_i over all $r-$tples for which $t_r = n-j+1$.

Now $S(n,1,1) = 1+2+\cdots+n = \frac{1}{2}(n+1)n$ and $S(n,2,1) = \frac{1}{2}n(n-1)+\cdots+1 = \frac{1}{3!}(n+1)n(n-1)$. As an induction hypothesis, suppose that $S(n,r-1,1) = \frac{1}{r!}(n+1)n(n-1)\cdots(n-r+2)$. Then

$$\begin{aligned}
S(n,r,1) &= \sum_{k=r-1}^{n-1} S(k,r-1,1) \\
&= \frac{1}{r!}\sum_{k=r-1}^{n-1}(k+1)k(k-1)\cdots(k-r+2) = \frac{1}{r!}\sum_{k=1}^{n} k(k-1)\cdots(k-r+1) \\
&= \frac{1}{(r+1)!}(n+1)n(n-1)\cdots(n-r+1) = \binom{n+1}{r+1}\frac{n!}{r!(n-r)!} = \binom{n+1}{r+1}\binom{n}{r} .
\end{aligned}$$

Thus, for each r with $1 \leq r \leq n$, $S(n,r,1) = \binom{n}{r}(n+1)/(r+1)$ so that $T(n,r,1) = (n+1)/(r+1)$.

Let $n \geq 2$. Suppose that for $1 \leq k \leq n-1$ and $1 \leq i \leq r \leq k$, it has been established that $S(k,r,i) = iS(k,r,1)$. Then for $1 \leq i \leq r \leq n$,

$$\begin{aligned}
S(n,r,i) &= S(n-1,r-1,i) + S(n-2,r-1,i) + \cdots + S(r-1,r-1,i) \\
&= i[S(n-1,r-1,1) + S(n-2,r-1,1) + \cdots + S(r-1,r-1,1) = iS(n,r,1) .
\end{aligned}$$

Dividing by $\binom{n}{r}$ yields

$$T(n,r,i) = iT(n,r,1) = i\binom{n+1}{r+1} .$$

Comments. (1) There is a one-one correspondence

$$(t_1, t_2, \cdots, t_r) \longleftrightarrow (n+1-t_r, n+1-t_{r-1}, \cdots, n+1-t_r)$$

of the set of suitable $r-$tples to itself, it follows that

$$\begin{aligned}
S(n,r,r) &= \binom{n}{r}(n+1) - S(n,r,1) = \binom{n}{r}(n+1)\left[1 - \frac{1}{r+1}\right] \\
&= \frac{r(n+1)}{r+1}\binom{n}{r} = rS(n,r,1)
\end{aligned}$$

from which $T(n,r,r) = r(n+1)/(r+1) = rT(n,r,1)$.

(2) To illustrate another method for getting and using the recursion, we prove first that $T(n,r,2) = 2T(n,r,1)$ for $2 \leq r \leq n$. Consider the case $r=2$. For $1 \leq t_1 < t_2 \leq n$, $(t_1,t_2) \leftrightarrow (t_2-t_1, t_2)$ defines a one-one correspondence between suitable pairs. Since $t_2 = t_1 + (t_2 - t_1)$, it follows from this correspondence that $S(n,2,2) = 2S(n,2,1)$. Dividing by $\binom{n}{2}$ yields $T(n,2,2) = 2T(n,2,1)$.

Suppose that $r \geq 2$. For each positive integer j with $1 \leq j \leq n-r+1$, we define a one-one correspondence between $r-$tples $(t_1, t_2, \cdots, t_r)$ with $1 \leq t_1 < t_2 < \cdots < t_r \leq n$ and $t_3 - t_2 = j$ and $(r-1)-$tples $(s_1, s_2, s_3, \cdots, s_r) = (t_1, t_2, t_4 - j, \cdots, t_r - j)$ with $1 \leq s_1 = t_1 < s_2 = t_2 < s_3 = t_4 - j < \cdots < s_r = t_r - j \leq n-j$. The sum of the elements t_2 over all $r-$tples with $t_3 - t_2 = j$ is equal to the sum of t_2 over all the $(r-1)-$tples. Hence

$$S(n,r,2) = S(n-1,r-1,2) + S(n-2,r-1,2) + \cdots + S(r-1,r-1,2) .$$

More generally, for $1 \leq j \leq n-r-1$, there is a one-one correspondence between $r-$tples $(t_1, t_2, \cdots, t_r)$ with $t_{i+1} - t_i = j$ and $(r-1)-$tples $(s_1, s_2, \cdots, s_{r-1}) = (t_1, \cdots, t_i, t_{i+2} - j, \cdots, t_r - j)$ with $1 \leq s_1 = t_1 < \cdots < s_i = t_i < s_{i+1} = t_{i+2} - j < \cdots < s_{r-1} = t_r - j \leq n - j$. We now use induction on r. We have that

$$S(n,r,i) = S(n-1,r-1,i) + S(n-2,r-1,i) + \cdots + S(r-1,r-1,i) \quad .$$

(b) *Solution 5.* [Y. Shen] We establish that

$$\sum_{k=i}^{i+(n-r)} \binom{k}{i}\binom{n-k}{r-i} = \binom{n+1}{r+1} \quad .$$

Consider the $(r+1)-$element sets where $t_{i+1} = k+1$ and $t_{r+1} \leq n+1$. We must have $i \leq k \leq n-(r-i)$ and there are $\binom{k}{i}\binom{n-k}{r-i}$ ways of selecting $t_1, \cdots, t_i$ and $t_{i+2}, \cdots, t_{r+1}$. The desired equation follows from a counting argument over all possibilities for t_{i+1}.

In a similar way, we note that $t_i = k$ for $\binom{k-1}{i-1}\binom{n-k}{r-i}$ sets $\{t_1, \cdots, t_r\}$ chosen from $\{1, \cdots, n\}$, where $1 \leq k \leq n-r+1$. Observe that

$$\binom{k-1}{i-1}\binom{n-k}{r-i} = \frac{i}{k}\binom{k}{i}\binom{n-k}{r-i} \; .$$

Then

$$\begin{aligned} T(n,r,i) &= \frac{\sum_{k=i}^{n-r+1} k\binom{k-1}{i-1}\binom{n-k}{r-i}}{\binom{n}{r}} \\ &= \frac{i\sum_{k=i}^{n-r+i} \binom{k}{i}\binom{n-k}{r-i}}{\binom{n}{r}} \\ &= \frac{\binom{n+1}{r+1}}{\binom{n}{r}} = i\left(\frac{n+1}{r+1}\right) \quad . \end{aligned}$$

(b) *Solution 6.* [Christopher So] Note that

$$\sum_{k=i}^{n-r+i} \binom{k}{i}\binom{n-k}{r-i}$$

is the coefficient of $x^i y^{r-i}$ in the polynomial

$$\sum_{k=i}^{n-r+i} (1+x)^k (1+y)^{n-k}$$

or in

$$\begin{aligned} \sum_{k=0}^{n+1} (1+x)^k (1+y)^{n-k} &= \frac{(1+y)^{n+1} - (1+x)^{n+1}}{y-x} \\ &= \frac{\sum_{j=0}^{n+1} \binom{n+1}{j}(y^j - x^j)}{y-x} \; . \end{aligned}$$

Now the only summand which involves terms of degree r corresponds to $j = r+1$, so that the coefficient of $x^i y^{r-1}$ in the sum is the coefficient in the single term

$$\binom{n+1}{r+1}\frac{y^{r+1} - x^{r+1}}{y-x}$$

namely, $\binom{n+1}{r+1}$. We can now complete the argument as in the fourth solution.

(b) *Comment.* Let r and n be fixed values and consider i to be variable. The $\binom{n}{r}$ $r-$term sets contain altogether $r\binom{n}{r}$ numbers, each number occurring equally often: $\frac{r}{n}\binom{n}{r}$ times. The sum of all the elements in the set is

$$S(n,r,1)+S(n,r,2)+\cdots+S(n,r,r)=\frac{r}{n}\binom{n}{r}(1+2+\cdots+n)=\frac{r(n+1)}{2}\binom{n}{r}$$

where $S(n,r,i)$ is the sum of the elements t_i over the $\binom{n}{r}$ subsets. The ordered $r-$element subsets $(t_1,t_2,\cdots,t_r)$ can be mapped one-one to themselves by

$$(t_1,t_2,\cdots,t_r)\longleftrightarrow(n+1-t_r,n+1-t_{r-1},\cdots,n+1-t_1)\ .$$

From this, we see that, for $1\le r$,

$$S(n,r,r+1-i)=\binom{n}{r}(n+1)-S(n,r,i)$$

so that

$$S(n,r,1)+S(n,r,r)=S(n,r,2)+S(n,r,r-1)=\cdots=S(n,r,i)+S(n,r,r+1-i)=\cdots=\binom{n}{r}(n+1)\ .$$

This is not enough to imply that $S(n,r,i)$ is an arithmetic progression in i, but along with this fact would give a quick solution to the problem.

603. For each of the following expressions severally, determine as many integer values of x as you can so that it is a perfect square. Indicate whether your list is complete or not.

(a) $1+x$;

(b) $1+x+x^2$;

(c) $1+x+x^2+x^3$;

(d) $1+x+x^2+x^3+x^4$;

(e) $1+x+x^2+x^3+x^4+x^5$.

Solution. (a) $1+x$ is a square when $x=u^2-1$ for some integer u (or when x is the product of two integers $u-1$ and $u+1$ that differ by 2).

(b) *Solution 1.* Suppose that $x^2+x+1=u^2$. Then $(2x+1)^2+3=4x^2+4x+4=4u^2=(2u)^2$, whence

$$3=(2u)^2-(2x+1)^2=(2u+2x+1)(2u-2x-1)\ .$$

The factors on the right must be ±3 and ±1 in some order, and this leads to the possibilities $(x,u)=(-1,\pm1),(0,\pm1)$.

(b) *Solution 2.* If $x>0$, then $x^2<x^2+x+1<(x+1)^2$, so that x^2+x+1 cannot be square. If $x<-1$, then $x^2>x^2+x+1>(x+1)^2$ and x^2+x+1 cannot be square. This leaves only the possibilities $x=0,-1$.

(b) *Solution 3.* For given u, consider the quadratic equation

$$x^2+x+1=u^2\ .$$

Its discriminant is $1-4(1-u^2)=4u^2-3$. It will have integer solutions only if $4u^2-3=v^2$ for some integer v, *i.e.*, $(v+2u)(v-2u)=-3$. The only possibilities are $(u,v)=(\pm1,\pm1),(\pm1,\mp1)$.

(b) *Solution 4.* [J. Chui] If $f(x) = 1 + x + x^2$, then $f(x) = f(-(x+1))$, so we need deal only with nonnegative values of x. We have that $f(0) = f(-1) = 1$ is square. Let $x \geq 1$ and suppose that $1 + x + x^2 = u^2$ for some integer u. Then $(1+x)^2 - u^2 = x > 0$ so that $1 + x > u$. This implies that $x \geq u$, whence $x^2 \geq u^2 = x^2 + x + 1$, a contradiction. Thus the only possibilities are $x = 0, -1$.

(b) *Solution 5.* [A. Birka] Suppose that $x^2 + x + 1 = u^2$ with $u \geq 0$. This is equivalent to $x = (1+x)^2 - u^2 = (1+x+u)(1+x-u)$, so that $1+x+u$ and $1+x-u$ both divide x. If $x \geq 1$, then $1+x+u$ exceeds x and so cannot divide x. If $x \leq 0$, then $(-x)+u-1$ divides x, which is impossible unless $u = 1$ or $u = 0$. Only $u = 1$ is viable, and this leads to $x = 0, -1$.

(c) *Solution.* $1 + x + x^2 + x^3 = (1+x^2)(1+x)$. Let d be a common prime divisor of $1+x$ and $1+x^2$. Then d must also divide $x(x-1) = (1+x^2) - (1+x)$. Since $\gcd(x, x+1) = 1$, d must divide $x - 1$ and so divide $2 = (x+1) - (x-1)$. Hence, the only common prime divisor of $1 + x^2$ and $1 + x$ is 2.

Suppose $1 + x + x^2 + x^3 = (1+x^2)(1+x)$ is square. Then there are only two possibilities:

$$\text{(i)} \quad 1 + x^2 = u^2 \quad \text{and} \quad 1 + x = v^2 \quad \text{for integers } u \text{ and } v\ ;$$

$$\text{(ii)} \quad 1 + x^2 = 2r^2 \quad \text{and} \quad 1 + x = 2s^2 \quad \text{for integers } r \text{ and } s\ .$$

Ad (i): $1 = u^2 - x^2 = (u-x)(u+x) \Leftrightarrow (x, u) = (0, \pm 1)$.
Ad (ii): We have $x^2 - 2r^2 = -1$ which has solutions

$$(x, r) = (-1, 1), (1, 1), (7, 5), (41, 29), \cdots \quad .$$

The complete set of solutions of $x^2 - 2r^2 = \pm 1$ in positive integers is given by $\{(x_n, r_n) : n = 1, 2, \cdots\}$, where $x_n + r_n\sqrt{2} = (1+\sqrt{2})^n$, with odd values of n yielding solutions of $x^2 - 2r^2 = -1$. We need to select values of x for which $x + 1 = 2s^2$ for some s. $x = -1, 1, 7$ work, yielding

$$1 - 1 + (-1)^2 + (-1)^3 = 0$$

$$1 + 1 + 1^2 + 1^3 = 2^2$$

$$1 + 7 + 7^2 + 7^3 = 8 \times 50 = 20^2\ .$$

There may be other solutions.

(d) *Solution 1.* Let $f(x) = x^4 + x^3 + x^2 + x + 1 = (x^5 - 1)/(x - 1)$, with the quotient form for $x \neq 1$. We have that $f(0) = f(-1) = 1^2$ and $f(3) = (243-1)/2 = 11^2$. Also $f(1) = 5$ and $f(2) = 31$. Suppose that $x \geq 4$. Then $x(x-2) > 3$, so that $x^2 > 2x + 3$. Hence

$$\begin{aligned}(2x^2 + x + 1)^2 &= 4x^4 + 4x^3 + 5x^2 + 2x + 1 \\ &> 4x^4 + 4x^3 + 4x^2 + 4x + 4 = 4f(x)\end{aligned}$$

and

$$\begin{aligned}4f(x) &= (4x^4 + 4x^3 + x^2) + (3x^2 + 4x + 4) \\ &= (2x^2 + x)^2 + (3x^2 + 4x + 4) > (2x^2 + x)^2\ .\end{aligned}$$

Thus, $4f(x)$ lies between the consecutive squares $(2x^2 + x)^2$ and $(2x^2 + x + 1)^2$ and so cannot be square. Hence $f(x)$ cannot be square.

Similarly, if $x \leq -2$, then $x(x-2) > 3$ and $3x^2 + 4x + 4 > 0$, and we again find that $4f(x)$ lies between the consecutive squares $(2x^2 + x)^2$ and $(2x^2 + x + 1)^2$. Hence $f(x)$ is square if and only if $x = -1, 0, 3$.

(d) *Solution 2.* [M. Boase] For $x > 3$,

$$\left(x^2 + \frac{x}{2}\right)^2 < x^4 + x^3 + x^2 + x + 1 < \left(x^2 + \frac{x+1}{2}\right)^2$$

so that, lying between two half integers, $x^4 + x^3 + x^2 + x + 1$ is not square. Suppose $x = -y$ is less than -1. Since $y - 1 < \frac{3}{4}y^2$ and $y^2 + 2y - 3 = (y + 3)(y - 1) > 0$,

$$\left(y^2 - \frac{y}{2}\right)^2 < 1 - y + y^2 - y^3 + y^4 < \left(y^2 - \frac{y-1}{2}\right)^2$$

so again the middle term is not square. The cases $x = -1, 0, 1, 2, 3$ can be checked directly.

(e) *Solution 1.* Let

$$\begin{aligned} g(x) &= x^5 + x^4 + x^3 + x^2 + x + 1 = (x + 1)(x^4 + x^2 + 1) \\ &= (x + 1)[(x^2 + 1)^2 - x^2] = (x + 1)(x^2 + x + 1)(x^2 - x + 1) \ . \end{aligned}$$

Observe that $g(x) < 0$ for $x \leq -2$, so $g(x)$ cannot be square in this case. Let us analyze common divisors of the three factors of $g(x)$.

Suppose that p is a prime divisor of $x + 1$. Then

$$x^2 + x + 1 = x(x + 1) + 1 \equiv 1 \mod p$$

and

$$x^2 - x + 1 = x(x + 1) - 2(x + 1) + 3 \equiv 3 \mod p \ .$$

Hence $\gcd(x + 1, x^2 + x + 1) = 1$ and $\gcd(x + 1, x^2 - x + 1)$ is either 1 or 3.

Suppose q is prime and $x^2 + x + 1 \equiv 0 \pmod q$. Then $x(x + 1) \equiv -1 \pmod q$, and $x^2 - x + 1 \equiv -2x \pmod q$. Since $x^2 + x + 1$ is odd, $q \neq 2$, then $x^2 - x + 1 \not\equiv 0 \pmod q$. Hence $\gcd(x^2 + x + 1, x^2 - x + 1) = 1$.

As we have seen from (b), $x^2 + x + 1$ is square if and only if $x = -1$ or 0. Indeed $g(-1) = 0^2$ and $g(0) = 1^2$. Otherwise, $x^2 + x + 1$ cannot be square. But $\gcd(x^2 + x + 1, (x + 1)(x^2 - x + 1)) = 1$, so $g(x)$ cannot be a square either. Hence $x^5 + x^4 + x^3 + x^2 + x + 1$ is square if and only if $x = -1$ or 0.

(e) *Solution 2.* [M. Boase] Observe that $x^5 + x^4 + \cdots + 1 = (x^3 + 1)(x^2 + x + 1)$. Since $x^3 + 1 = (x^2 + x + 1)(x - 1) + 2$, the greatest common divisor of $x^3 + 1$ and $x^2 + x + 1$ must divide 2. But $x^2 + x + 1 = x(x + 1) + 1$ is always odd, so the greatest common divisor must be 1. Hence $x^2 + x + 1$ and $x + 1$ must both be square. Hence x must be either -1 or 0.

604. $ABCD$ is a square with incircle Γ. Let l be a tangent to Γ, and let A', B', C', D' be points on l such that AA', BB', CC', DD' are all prependicular to l. Prove that $AA' \cdot CC' = BB' \cdot DD'$.

Solution 1. Let Γ be the circle of equation $x^2 + y^2 = 1$ and let l be the line of equation $y = -1$. The points of the square must lie on the circle of equation $x^2 + y^2 = 2$. Let them be

$$A \sim (\sqrt{2}\cos\theta, \sqrt{2}\sin\theta)$$

$$B \sim (-\sqrt{2}\sin\theta, \sqrt{2}\cos\theta)$$

$$C \sim (-\sqrt{2}\cos\theta, -\sqrt{2}\sin\theta)$$

$$D \sim (\sqrt{2}\sin\theta, -\sqrt{2}\cos\theta)$$

for some angle θ with $-\pi/4 \leq \theta \leq \pi/4$. Observe that $1/\sqrt{2} \leq \cos\theta \leq 1$ and that $-1/\sqrt{2} \leq \sin\theta \leq 1/\sqrt{2}$.

Then $A' \sim (\sqrt{2}\cos\theta, -1)$, $B' \sim (-\sqrt{2}\sin\theta, -1)$, $C' \sim (-\sqrt{2}\cos\theta, -1)$ and $D' \sim (\sqrt{2}\sin\theta, -1)$, so that $AA' = 1 + \sqrt{2}\sin\theta$, $BB' = 1 + \sqrt{2}\cos\theta$, $CC' = 1 - \sqrt{2}\sin\theta$ and $DD' = 1 - \sqrt{2}\cos\theta$. Hence

$$\begin{aligned} AA' \cdot CC' - BB' \cdot DD' &= (1 + \sqrt{2}\sin\theta)(1 - \sqrt{2}\sin\theta) - (1 + \sqrt{2}\cos\theta)|1 - \sqrt{2}\cos\theta| \\ &= (1 + \sqrt{2}\sin\theta)(1 - \sqrt{2}\sin\theta) + (1 + \sqrt{2}\cos\theta)(1 - \sqrt{2}\cos\theta) \\ &= 1 - 2\sin^2\theta + 1 - 2\cos^2\theta = 0 \ . \end{aligned}$$

Solution 2. One can proceed as in the first solution, taking the four points on the larger circle at the intersection with the perpendicular lines $y = mx$ and $y = -x/m$. The points are

$$A \sim \left(\frac{\sqrt{2}}{\sqrt{m^2+1}}, \frac{m\sqrt{2}}{\sqrt{m^2+1}}\right) \qquad B \sim \left(\frac{-m\sqrt{2}}{\sqrt{m^2+1}}, \frac{\sqrt{2}}{\sqrt{m^2+1}}\right)$$

$$C \sim \left(\frac{\sqrt{2}}{\sqrt{m^2+1}}, \frac{-m\sqrt{2}}{\sqrt{m^2+1}}\right) \qquad D \sim \left(\frac{m\sqrt{2}}{\sqrt{m^2+1}}, \frac{-\sqrt{2}}{\sqrt{m^2+1}}\right) \ .$$

In this case, the products turn out to be equal to $|(m^2-1)/(m^2+1)|$.

Solution 3. [A. Birka] Let the circle have equation $x^2 + y^2 = 1$ and the square have vertices $A \sim (1, 1)$, $B \sim (-1, 1)$, $C \sim (-1, -1)$, $D \sim (1, -1)$. Suppose, wolog, that the line l is tangent to the circle at $P(t, \sqrt{1-t^2})$ with $0 < t < 1$ and intersects CB produced in Y and AD in X. The line l has equation $tx + \sqrt{1-t^2}y = 1$ and so the coordinates of X are $(1, u)$ and of Y are $(-1, 1/u)$ where $u = (1-t)/\sqrt{1-t^2}$. Now $YB : YC = (1-u) : (1+u) = AX : XD$. Since $\Delta YBB'$ is similar to $\Delta YCC'$ and $\Delta XAA'$ is similar to $\Delta XDD'$.

$$BB' : CC' = YB : YC = AX : XD = AA' : DD' \ ,$$

and the result follows.

Comment. If the circle has equation $x^2 + y^2 = r^2$, the square has vertices $(\pm r, \pm r)$ and the line through a point (a, b) on the circle has equation $ax + by = r^2$, then the distance product is $2ab$.

605. Prove that the number $299 \cdots 998200 \cdots 029$ can be written as the sum of three perfect squares of three consecutive numbers, where there are $n-1$ nines between the first 2 and the 8, and $n-1$ zeros between the last pair of twos.

Solution. Let $a-1$, a, $a+1$ be the three consecutive numbers. The sum of their square is $3a^2 + 2$; setting this equal to the given number yields

$$\begin{aligned} a^2 &= 9 \cdot 10^{2n+1} + \cdots + 9 \cdot 10^{n+3} + 9 \cdot 10^{n+2} + 4 \cdot 10^{n+1} + 9 \\ &= (10^n - 1)10^{n+2} + 4 \cdot 10^{n+1} + 9 = 10^{2n+2} - 6 \cdot 10^{n+1} + 9 \\ &= (10^{n+1} - 3)^2 \ , \end{aligned}$$

so that $a = 10^{n+1} - 3$.

606. Let $x_1 = 1$ and let $x_{n+1} = \sqrt{x_n + n^2}$ for each positive integer n. Prove that the sequence $\{x_n : n > 1\}$ consists solely of irrational numbers and calculate $\sum_{k=1}^{n} \lfloor x_k^2 \rfloor$, where $\lfloor x \rfloor$ is the largest integer that does not exceed x.

Solution. We prove that x_n is nonrational as well as positive for $n \geq 2$. Note that x_2 is nonrational. Suppose that $n \geq 2$ and that x_{n+1} were rational; then $x_n = x_{n+1}^2 - n^2$ would also be rational; repeating this would lead to x_2 being rational and a contradiction.

Observe that, for any positive integer $n \geq 2$,

$$x_n = \sqrt{x_{n-1} + (n-1)^2} > n - 1 \ .$$

We prove by induction that $x_n < n$. This is true for $n = 2$. If $x_{n-1} < n-1$, then

$$x_n^2 = x_{n-1} + (n-1)^2 < (n-1)n < n^2 \ ,$$

and the desired result follows. Thus, for each $n \geq 2$, $\lfloor x_n \rfloor = n - 1$, For $n \geq 3$,

$$\lfloor x_n^2 \rfloor = \lfloor x_{n-1} + (n-1)^2 \rfloor = (n-2) + (n-1)^2 = n^2 - n - 1 = n(n-1) - 1 \ .$$

Therefore

$$\begin{aligned}
\sum_{k=1}^{n} \lfloor x_k^2 \rfloor &= \lfloor x_1^2 \rfloor + \lfloor x_2^2 \rfloor + \sum_{k=3}^{n} \lfloor x_k^2 \rfloor \\
&= 3 + \left[(\sum_{k=3}^{n} k(k-1) \right] - (n-2) \\
&= 5 - n + \frac{1}{3} \sum_{k=3}^{n} [(k+1)k(k-1) - k(k-1)(k-2)] \\
&= 5 - n + \frac{1}{3}[(n+1)n(n-1) - 6] = 3 - n + \frac{1}{3}(n^3 - n) \\
&= \frac{1}{3}(n^3 - 4n + 9) \ ,
\end{aligned}$$

607. Solve the equation

$$\sin x \left(1 + \tan x \tan \frac{x}{2} \right) = 4 - \cot x \ .$$

Solution. For the equation to be defined, x cannot be a multiple of π, so that $\sin x \neq 0$. Rearranging the terms of the equation and manipulating yields that

$$\begin{aligned}
4 &= \cot x + \sin x \left(\frac{\cos x \cos \frac{x}{2} + \sin x \sin \frac{x}{2}}{\cos x \cos \frac{x}{2}} \right) \\
&= \cot x + \sin x \left(\frac{\cos(x - (x/2))}{\cos x \cos(x/2)} \right) \\
&= \frac{\cos x}{\sin x} + \frac{\sin x}{\cos x} \\
&= \frac{\cos^2 x + \sin^2 x}{\sin x \cos x} = \frac{2}{\sin 2x} \ ,
\end{aligned}$$

whence $\sin 2x = \frac{1}{2}$. Therefore $x = (-1)^k \frac{\pi}{12} + \frac{k\pi}{2}$, where k is an integer.

608. Find all positive integers n for which n, $n^2 + 1$ and $n^3 + 3$ are simultaneously prime.

Solution. If $n = 2$, then the numbers are 2, 5 and 11 and all are prime. Otherwise, n must be odd. But in this case, the other two numbers are even exceeding 2 and so nonprime. Therefore $n = 2$ is the only possibility.

609. The first term of an arithmetic progression is 1 and the sum of the first nine terms is equal to 369. The first and ninth terms of the arithmetic progression coincide respectively with the first and ninth terms of a geometric progression. Find the sum of the first twenty terms of the geometric progression.

Solution. The sum of the first nine terms of an arithmetic progression is equal to 9/2 the sum of the first and ninth terms, from which it is seen that the ninth term is 81. Let r be the common ratio of the geometric progression whose first term is 1 and whose ninth term is 81. Then $r^8 = 81$, whence $r = \pm\sqrt{3}$. The sum of the first twenty terms of the geometric progression is $\frac{1}{2}(3^{10} - 1)(\pm\sqrt{3} + 1)$.

610. Solve the system of equations

$$\log_{10}(x^3 - x^2) = \log_5 y^2$$

$$\log_{10}(y^3 - y^2) = \log_5 z^2$$
$$\log_{10}(z^3 - z^2) = \log_5 x^2$$

where $x, y, z > 1$.

Solution. For $x > 1$, let

$$f(x) = 5^{\log_{10}(x^3 - x^2)} .$$

The three equations are $f(x) = y^2$, $f(y) = z^2$ and $f(z) = x^2$. Since $x^3 - x^2 = x^2(x-1)$ is increasing, f is an increasing function. If, say, $x < y$, then $y < z$ and $z < x$, yielding a contradiction. Thus, we can only have that $x = y = z$ and so

$$\log_{10}(x^3 - x^2) = \log_5 x^2 .$$

Let $2t = \log_5 x^2$ so that $t > 0$, $x^2 = 5^{2t}$ and so $x = 5^t$. Therefore

$$5^{3t} - 5^{2t} = 10^{2t} \Longrightarrow 5^t - 1 = 4^t \Longrightarrow 5^t - 4^t = 1 .$$

Since $5^t - 4^t = 4^t[(5/4)^t - 1]$ is an increasing function of t, we see that the equation for t has a unique solution, namely $t = 1$. Therefore $x = 5$.

611. The triangle ABC is isosceles with $AB = AC$ and I and O are the respective centres of its inscribed and circumscribed circles. If D is a point on AC for which $ID \| AB$, prove that $CI \perp OD$.

Solution. Since ABC is isosceles, the points A, O, I lie on the right bisector of BC. Let AO meet BC at P, DI meet BC at E, DO meet BC at F and CI meet DF at Q.

Suppose that angle A is less than 60°. Then O lies between I and A, and Q lies within triangle APB. Since $DE \| AB$ and O is the centre of the circumcircle of ABC, we have that

$$\angle CDI = \angle BAC = \angle COI ,$$

so that $CIOD$ is concyclic. Therefore

$$\begin{aligned}\angle CQD &= 180^\circ - (\angle QOI + \angle QIO) = 180^\circ - (\angle ICD + \angle PIC) \\ &= 180^\circ - (\angle ICP + \angle PIC) = 90^\circ .\end{aligned}$$

Suppose that angle A exceeds 60°. Then I lies between O and A, and Q lies on the same side of AP as C. Since

$$\angle IDC + \angle IOC = \angle BAC + \angle AOC = 180^\circ ,$$

the quadrilateral $IOCD$ is concyclic. Therefore

$$\begin{aligned}\angle CQD &= 180^\circ - (\angle DCQ + \angle QDC) = 180^\circ - (\angle QCP + \angle ODC) \\ &= 180^\circ - (\angle QCP + \angle OIC) = 180^\circ - (\angle ICP + \angle PIC) = 90^\circ .\end{aligned}$$

Finally, if $\angle A = 60^\circ$, then I and O coincide so that $DF = DE \| AB$ and the result is clear.

612. $ABCD$ is a rectangle for which $AB > AD$. A rotation with centre A takes B to a point B' on CD; it takes C to C' and D to D'. Let P be the point of intersection of the lines CD and $C'D'$. Prove that $CB' = DP$.

Solution 1. [N. Lvov; K. Zhou] Since $\angle CB'P = 90^\circ - \angle DB'A = \angle DAB'$ and $AD = BC = B'C'$, triangles $AB'D$ and $B'PC$ are congruent (ASA). Therefore

$$\begin{aligned}DP &= B'P - B'D = AB' - B'D \\ &= AB - B'D = CD - B'D = CB' .\end{aligned}$$

Solution 2. Let the respective lengths of AB and BC be a and b respectively, and suppose that the rotation about A is through the angle 2α. Then $\angle CBB' = \alpha$ and we find that

$$\begin{aligned} a &= b(\tan\alpha + \cot 2\alpha) \\ &= b\left(\frac{\sin\alpha}{\cos\alpha} + \frac{\cos 2\alpha}{\sin 2\alpha}\right) \\ &= b\left(\frac{2\sin^2\alpha + 1 - 2\sin^2\alpha}{\sin 2\alpha}\right) \\ &= b\left(\frac{1}{\sin 2\alpha}\right) . \end{aligned}$$

Since $|B'C'| = b$ and $\angle C'B'P = 90° - 2\alpha$, then $\angle B'PC' = 2\alpha$. Thus $\sin 2\alpha = |B'C'|/|B'P|$, so that $|B'P| = b/\sin 2\alpha = a = |CD|$. The result follows.

Solution 3. [A. Dhawan] The circle with centre A and radius $|AD|$ passes though D and D'; the tangent through P are PD and PD' and so

$$\angle DAP = \frac{1}{2}\angle D'AD = \frac{1}{2}\angle B'AB .$$

Also, we have that

$$\angle B'BC = 90° - \angle B'BA = \frac{1}{2}(180° - \angle B'BA - \angle BB'A) = \frac{1}{2}\angle B'AB ,$$

so that $\angle PAD = \angle B'BC$. Since also $\angle PDA = 90° = \angle B'CB$ and $DA = CB$, triangles PDA and $B'CB$ are congruent (ASA). Therefore $PD = B'C$.

613. Let ABC be a triangle and suppose that

$$\tan\frac{A}{2} = \frac{p}{u} \qquad \tan\frac{B}{2} = \frac{q}{v} \qquad \tan\frac{C}{2} = \frac{r}{w} ,$$

where p, q, r, u, v, w are positive integers and each fraction is written in lowest terms.

(a) Verify that $pqw + pvr + uqr = uvw$.

(b) Let f be the greatest common divisor of the pair $(vw - qr, qw + vr)$, g be the greatest common divisor of the pair $(uw - pr, pw + ur)$, and h be the greatest common divisor of the pair $(uv - pq, pv + qu)$. Prove that

$$\begin{aligned} fp &= vw - qr & \qquad fu &= qw + vr \\ gq &= uw - pr & \qquad gv &= pw + ur \\ hr &= uv - pq & \qquad hw &= pv + qu . \end{aligned}$$

(c) Prove that the sides of the triangle ABC are proportional to $fpu : gqv : hrw$.

Solution 1. Since $A/2$ and $B/2 + C/2$ are complementary, $\cot(A/2) = \tan(B/2 + C/2)$, whence

$$\frac{u}{p} = \frac{qw + vr}{vw - qr} .$$

Parts (a) and (b) follow immediately.

The sides of the triangle are proportional to $\sin A : \sin B : \sin C$. Now

$$\sin A = \frac{2\tan\frac{A}{2}}{\sec^2\frac{A}{2}} = \frac{2pu}{p^2 + u^2} = \frac{2fpu}{f(p^2 + u^2)} ;$$

$$\sin B = \frac{2\tan\frac{B}{2}}{\sec^2\frac{B}{2}} = \frac{2qv}{q^2+v^2} = \frac{2gqv}{g(q^2+v^2)} \ ;$$

$$\sin C = \frac{2\tan\frac{C}{2}}{\sec^2\frac{C}{2}} = \frac{2rw}{r^2+w^2} = \frac{2hrw}{h(r^2+w^2)} \ .$$

From (b), we have that

$$f^2(p^2+u^2) = (q^2+v^2)(r^2+w^2)$$

so that

$$f^2(p^2+u^2)^2 = (p^2+u^2)(q^2+v^2)(r^2+w^2) \ .$$

Similar equations hold for g and h. We find that

$$f(p^2+u^2) = g(q^2+v^2) = h(r^2+w^2) \ .$$

Hence $\sin A : \sin B : \sin C = fpu : gqv : hrw$ as desired.

Solution 2. (a) and (b) can be obtained as above. For (c), let x, y, z be the respective distances from A, B, C to the adjacent tangency points of the incircle of triangle ABC. Then $\tan A/2 = r/x$, $\tan B/2 = r/y$ and $\tan C/2 = r/z$. Also $a = y+z$, $b = z+x$ and $c = x+y$. It follows that

$$\begin{aligned} a : b : c &= y+z : z+x : x+y \\ &= \left(\frac{1}{\tan B/2} + \frac{1}{\tan C/2}\right) : \left(\frac{1}{\tan C/2} + \frac{1}{\tan A/2}\right) : \left(\frac{1}{\tan A/2} + \frac{1}{\tan B/2}\right) \\ &= \left(\frac{v}{q} + \frac{w}{r}\right) : \left(\frac{w}{r} + \frac{u}{p}\right) : \left(\frac{u}{p} + \frac{v}{q}\right) \\ &= p(qw+vr) : q(pw+ru) : r(pv+qu) = fpu : gqv : hrw \ . \end{aligned}$$

614. Determine those values of the parameter a for which there exist at least one line that is tangent to the graph of the curve $y = x^3 - ax$ at one point and normal to the graph at another.

Solution. The tangent at $(u, u^3 - au)$ has equation $y = (3u^2 - a)x - 2u^3$. This line intersects the curve again at the point whose abscissa is $-2u$ and whose tangent has slope $12u^2 - a$. The condition that the first tangent be normal at the second point is

$$(12u^2 - a)(3u^2 - a) = -1$$

or

$$36u^4 - 15u^2a + (a^2+1) = 0 \ .$$

The discriminant of this quadratic in u^2 is

$$225a^2 - 144(a^2+1) = 9(3a-4)(3a+4) \ .$$

The quadratic has positive real roots for u^2 if and only if $|a| \geq 4/3$.

615. The function $f(x)$ is defined for real nonzero x, takes nonzero real values and satisfies the functional equation

$$f(x) + f(y) = f(xyf(x+y)) \ ,$$

whenever $xy(x+y) \neq 0$. Determine all possibilities for f.

Solution. [J. Rickards] The functional equation is satisfied by $f(x) = 1/x$. More generally, suppose, if possible, that there exists a number a for which $f(a) = 1/b$ with $b \neq a$. Then

$$f(b) + f(a-b) = f(b(a-b)f(a)) = f(a-b) \ ,$$

whence $f(b) = 0$. But this contradicts the condition on f. Therefore there is no such a and $f(x) = 1/x$ is the unique solution.

616. Let T be a triangle in the plane whose vertices are lattice points (*i.e.*, both coordinates are integers), whose edges contain no lattice points in their interiors and whose interior contains exactly one lattice point. Must this lattice point in the interior be the centroid of the T?

Solution 1. [M. Valkov] Let ABC be the triangle and let X be the single lattice point within its interior. Using Pick's Theorem that the area of a lattice triangle is $(1/2)b + i - 1$, where b is the number of lattice points on the boundary and i the number in the interior, we find that $[ABC] = 3/2$ and $[ABX] = [BCX] = [CAX] = 1/2$. Let the line through X parallel to BC meet AB at Y and AC at Z. This is line is one-third of the distance from BC as A. Let AX meet BC at P. Then $YX : BP = AX : AP = 2 : 3$.

Since X is one-third the distance from AB as C, we have that $YX : BC = 1 : 3$, whence $2BP = BC$ and X is on the median from A. Similarly, X is on the other two medians and so is the centroid of the triangle.

Solution 2. [J. Schneider; J. Rickards] The answer is "yes". Without loss of generality, we can assume that the three points are $(0, 0)$, (a, b) and (u, v). The area of the triangle can be computed in two ways, by Pick's Theorem ($\frac{1}{2}b + i - 1$ where b is the number of lattice points on the boundary and i the number of lattice points in the interior of a polygon whose vertices are at lattice points) and directly using the formula for the area of a triangle with given vertices. This yields the equation

$$\frac{3}{2} = \frac{1}{2}|av - bu| \ ,$$

whence we deduce that $av - bu \equiv 0 \pmod 3$.

Since there is no lattice point in the interior of the sides of the triangle, it follows that, modulo 3, $a \equiv b \equiv 0$, $u \equiv v \equiv 0$ and $a \equiv u \& b \equiv v$ are each individually impossible. If $(a, b) \equiv (0, \pm 1)$, then $u \equiv 0$ and $v \equiv \mp 1$; thus, modulo 3, $a + u \equiv b + v \equiv 0$ and the centroid $(\frac{1}{3}(a + u), \frac{1}{3}(b + v))$ is a lattice point. Since the centroid lies inside the triangle and there is exactly one lattice point inside the triangle, the interior point must be the centroid. A similar analysis can be made if none of the coordinates a, b, u, v are divisible by 3. Thus, in all cases, the interior point is the centroid of the triangle.

617. Two circles are externally tangent at A and are internally tangent to a third circle Γ at points B and C. Suppose that D is the midpoint of the chord of Γ that passses through A and is tangent there to the two smaller given circles. Suppose, further, that the centres of the three circles are not collinear. Prove that A is the incentre of triangle BCD.

Solution 1. Let G denote the centre of the circle with points B and A on the circumference and H the centre of the circle with the points C and A on the circumference. Wolog, we assume that the former circle is the larger. Suppose that O is the centre of the circle Γ. The points G, A and H are collinear, as are B, G, O and C, H, O. Let the chord of Γ tangent to the smaller circles meet the circumference of Γ at J and K.

We have the OD and GH are both perpendicular to JK so that $GH \| OD$. Let BA and OD intersect at F. Since $BG = GA$ and triangles BGA and BOF are similar, $BO = OF$ and F lies on Γ. Similarly, the point E where CA and OD intersect lies on Γ. Since $\angle ABE = \angle FBE = 90° = \angle ADE$, the points B, E, D, A are concyclic. Therefore $\angle CBF = \angle CEF = \angle AED = \angle ABD$ and so A lies on the bisector of angle CBD. Similarly, A lies on the bisector of angle DCB. If follows that A is the incentre of triangle BCD.

Solution 2. Use the same notation as in the previous solution. Wolog, let the circle with centre G be at least as large as the circle with centre H. Suppose that the tangents to the circle Γ at B and C meet at the point L, and that LB and LA' are the tangents from L to the circle with centre G. Then $LC = LB = LA'$. There is a unique circle Δ that is tangent to LC and LA' at the points C and A'. This circle is tangent also to the circle Γ and the circle with centre G. Therefore, this circle must be the same circle with centre H, so that LA, LB and LC are each tangent to two of the three circles. Therefore, $LA = LB = LC$.

Observe that, because of subtended right angles, each of the quadrilaterals $LBOC$, $LDOB$, $LODC$ is concyclic. We have that

$$\angle LDC = \angle LOC = \angle LBC = \angle LCB = \angle LOB = \angle LDB \ ,$$

with the result that A lies on the bisector of angle BDC.

Let $\angle ABO = \beta$ and $\angle ACO = \gamma$. Then $\angle ACL = 90° - \gamma$, so that $\angle DLC = 2\gamma$. Similarly, $\angle DLB = 2\beta$. Therefore

$$\angle BLC = 2(\beta + \gamma) \Longrightarrow \angle BCL = 90° - \beta - \gamma \Longrightarrow \angle BCA = \angle ACL - \angle BCL = \beta \ .$$

Because $LODC$ is concyclic,

$$\angle OCD = \angle OLD = \angle OLC - \angle DLC = (\beta + \gamma) - 2\gamma = \beta - \gamma \ .$$

Hence

$$\angle ACD = \angle ACO + \angle OCD = \gamma + (\beta - \gamma) = \beta = \angle BCA$$

and A lies on the bisector of angle BCA. Therefore A is the incentre of triangle BCD.

618. Let a, b, c, m be positive integers for which $abcm = 1 + a^2 + b^2 + c^2$. Show that $m = 4$, and that there are actually possibilities with this value of m.

Solution. [J. Schneider] If any of a, b, c are even, then so is $abcm$. If a, b, c are all odd, then the right side of the equation is even and $abcm$ is even. Thus, $abcm$ must be even and an even number of a, b, c are even. If two of a, b, c are even, then the left side is congruent to 0 modulo 4 while the right is congruent to 2. Hence, it follows that all of a, b, c are odd. Therefore the right side is congruent to 4 modulo 8, and so m must be an odd multiple of 8.

If $m = 4$, then we have infinitely many solutions. One solution is $(m, a, b, c) = (4, 1, 1, 1)$. Suppose that we are given a solution $(m, a, b, c) = (4, 1, u, v)$. Then the equation is equivalent to $v^2 - 4uv + (2 - u^2) = 0$, *i.e.* v is a root of the quadratic equation

$$x^2 - 4ux + (2 - u^2) = 0 \ .$$

The second root $4u - v$ of this quadratic equation also yields a solution: $(m, a, b, c) = (4, 1, u, 4u - v)$. In this way, we can find an infinite sequence of solutions of the form $(m, a, b, c) = (4, 1, u_n, u_{n+1})$ where $u_1 = u_2 = 1$ and $u_{n+1} = 4u_n - u_{n-1}$.

Now suppose that $m \geq 12$. The equation can be rewritten

$$\frac{a}{bc} + \frac{b}{ac} + \frac{c}{ab} + \frac{1}{abc} = m \ .$$

Wolog, let $a \leq b \leq c$. Then only the term c/ab is not less than 1, and we must have $c \geq 9ab$. Since

$$2 < (81a^2 - 1)(81b^2 - 1) = 81(81a^2b^2 - a^2 - b^2) + 1 \leq 81(c^2 - a^2 - b^2) + 1 \ ,$$

whence $c^2 > a^2 + b^2 + 1$.

Suppose that the given equation is solvable and that (m, a, b, c) is that solution which minimizes the sum $a + b + c$ for the given m. Since (m, a, b, x) satisfies the equation if and only if

$$x^2 - mbcx + (a^2 + b^2 + 1) = 0 \ ,$$

and since c is one root of this equation, the other root yields the solution $(m, a, b, (a^2 + b^2 + 1)/c)$. However, the last entry of this is less than c and yields a solution with a smaller sum. Thus, we have a contradiction. Therefore there are no solutions with $m > 4$.

619. Suppose that $n > 1$ and that S is the set of all polynomials of the form

$$z^n + a_{n-1}z^{n-1} + a_{n-2}z^{n-2} + \cdots + a_1 z + a_0 \ ,$$

whose coefficients are complex numbers. Determine the minimum value over all such polynomials of the maximum value of $|p(z)|$ when $|z| = 1$.

Solution. [J. Schneider] For each value of n, the minimum is equal to 1. This minimum is attained for the polynomial z^n whose absolute value is equal to 1 when $|z| = 1$.

Let $q(z) = a_0 z^n + a_1 z^{n-1} + \cdots + a_{n-1}z + 1$, so that $p(z) = z^n q(1/z)$. Hence $|p(z)| = |q(1/z)|$, when $|z| = 1$. Thus, the existence of z with $|z| = 1$ for which $|p(z)| \geq 1$ is equivalent to the existence of z with $|z| = 1$ for which $|q(z)| \geq 1$.

Let ζ be a primitive $(n+1)$th root of unity (*i.e.*, $\zeta = \cos(2\pi/(n+1)) + i\sin(2\pi/(n+1))$, say). Then the set of $(n+1)$ roots of unity consists of 1 and $\zeta_k = \zeta^k$ (for $1 \leq a \leq n$). Observe that for $1 \leq k, i \leq n$,

$$1 + \zeta_1^i + \zeta_2^i + \cdots + \zeta_n^i = 1 + (\zeta^i)^1 + (\zeta^i)^2 + \cdots + (\zeta^i)^n = \frac{(\zeta_i)^{n+1} - 1}{\zeta_i - 1} = 0 \ .$$

Therefore

$$q(1) + q(\zeta_1) + q(\zeta_2) + \cdots + q(\zeta_n) = a_0(1 + \zeta_1^n + \cdots + \zeta_n^n) + \cdots + a_{n-1}(1 + \zeta_1 + \cdots + \zeta_n) + (n+1) = n + 1 \ .$$

However, then

$$n + 1 = |q(1) + q(\zeta_1) + \cdots + q(\zeta_n)| \leq |q(1)| + |q(\zeta_1)| + \cdots + |q(\zeta_n)| \ ,$$

so that at least one of the values in the right member is not less than 1. The desired result follows.

620. Let $a_1, a_2, \cdots, a_n$ be distinct integers. Prove that the polynomial

$$p(z) = (z - a_1)^2(z - a_2)^2 \cdots (z - a_n)^2 + 1$$

cannot be written as the product of two nonconstant polynomials with integer coefficients.

Solution. Suppose, if possible that $p(z) = q(z)r(z)$, where $q(z)$ and $r(z)$ are two polynomials of positive degree with integer coefficients. Then, for each a_i, $q(a_i)$ and $r(a_i)$ are integers whose product is 1; therefore they can be only 1 or -1. Since the polynomial $p(z)$ is positive for real z, neither of the polynomials $q(z)$ nor $r(z)$ can vanish for any real value of z; therefore, the sign of each is constant for real z. By multiplying both by -1 if necessary, we may assume that both polynomials q and r are always positive for real z. Hence $q(a_i) = r(a_i) = 1$ for $1 \leq i \leq n$. Thus, each of the polynomial $q(z) - 1$ and $r(z) - 1$ has n distinct zeros a_i and so have degree not less than n. Since the degree of $p(z)$ is exactly $2n$, the degrees of $q(z)$ and $r(z)$ must be exactly n. Therefore

$$q(z) = r(z) = 1 + (z - a_1)(z - a_2)(z - a_3) \cdots (z - a_n) \ .$$

Therefore

$$(z - a_1)^2(z - a_2)^2(z - a_3)^2 \cdots (z - a_n)^2 + 1 = q(z)^2 \ ,$$

whence

$$1 = [q(z) - (z - a_1)(z - a_2) \cdots (z - a_n)][q(z) + (z - a_1)(z - a_2) \cdots (z - a_n)] \ .$$

But this is impossible as the second factor on the right has positive degree. The desired result follows.

621. Determine the locus of one focus of an ellipse reflected in a variable tangent to the ellipse.

Solution. Let the foci of the ellipse be F and G, and let P be an arbitrary point on the ellipse. Suppose that H is the reflected image of F in the tangent through P. We note that

$$|HP| + |GP| = |FP| + |GP$$

is constant. Also, if X is an arbitrary point on the tangent on the same side of P as FH and Y is a point on the tangent on the opposite side, then $\angle HPX = \angle FPX = \angle GPY = 180° - \angle GPX$, so that G, P, H are collinear. Therefore H lies on the circle with centre G and radius $|GP| + |FP|$.

Conversely, let K be any point on this circle. Since the ellipse is contained in the interior of the circle, the segment GK intersects the ellipse at a point P. We have that

$$|PK| = |GK| - |GP| = |FP| \ .$$

Let XY be the tangent to the ellipse at P with X on the same side of P as KF and Y on the opposite side. Then

$$\angle KPX = \angle GPY = \angle FPX \ ,$$

from which it follows that K is the reflection of F in the tangent XY.

Comment. To show that the locus is the prescribed circle, you need to show, not only that each point on the locus lies on the circle, but also that each point on the circle satisfies the locus.

622. Let I be the centre of the inscribed circle of a triangle ABC and let u, v, w be the respective lengths of IA, IB, IC. Let P be any point in the plane and p, q, r the respective lengths of PA, PB, PC. Prove that, with the sidelengths of the triangle given conventionally as a, b, c,

$$ap^2 + bq^2 + cr^2 = au^2 + bv^2 + cw^2 + (a + b + c)z^2 \ ,$$

where z is the length of IP.

Solution 1. [R. Cheng] The equation can be rearranged to read

$$a(p^2 - u^2 - z^2) + b(q^2 - v^2 - z^2) + c(r^2 - w^2 - z^2) = 0 \ .$$

By the Law of Cosines applied to triangle API, we have that

$$p^2 - u^2 - z^2 = 2uz \cos \angle PIA = \overrightarrow{IA} \cdot \overrightarrow{IP} \ .$$

Similar relations can be obtained for triangles PIB and PIC, and so the equation to be derived is

$$a\overrightarrow{IA} \cdot \overrightarrow{IP} + b\overrightarrow{IB} \cdot \overrightarrow{IP} + c\overrightarrow{IC} \cdot \overrightarrow{IP} = 0 \ .$$

Since this has to be derived for all points P, we need to show that

$$(a\overrightarrow{IA} + b\overrightarrow{IB} + c\overrightarrow{IC}) = \overrightarrow{O} \ .$$

We show that $a\overrightarrow{IA} + b\overrightarrow{IB}$ is collinear with $\overrightarrow{IC}$. Construct points X and Y on the line CI so that AX and BY are both perpendicular to CI. We show that $a|AX| = b|BY|$. Select M on AB so that $IM \perp AB$. Then, from the Law of Sines, $AI : IB = \sin(B/2) : \sin(A/2)$ and $AX : BY = \sin(B/2)\cos(B/2) : \sin(A/2)\cos(A/2) = \sin B : \sin A$, from which $a|AX| = b|BY|$. Thus, $a\overrightarrow{AX} + b\overrightarrow{BY}$ has zero component in the direction orthogonal to CI and so $a\overrightarrow{IA}+b\overrightarrow{IB}$ is collinear with $\overrightarrow{IC}$. Repeat this for the other two vectors to find that $a\overrightarrow{IA}+b\overrightarrow{IB}+c\overrightarrow{IC}=0$ is collinear with each of its summands, and therefore must be zero.

Solution 2. [N. Lvov] Let $\mathbf{p}=\overrightarrow{AP}$, $\mathbf{q}=\overrightarrow{BP}$, $\mathbf{r}=\overrightarrow{CP}$ and $\mathbf{z}=\overrightarrow{IP}$. Let

$$\mathbf{u}=\frac{b\mathbf{c}-c\mathbf{b}}{a+b+c}\ .$$

This is a vector that points into the triangle from vertex A. Suppose that Q is the tip of this vector, so that $\mathbf{u}=\overrightarrow{AQ}$. The distance of Q from side AC is equal to

$$\frac{2[AQC]}{b}=\frac{|\mathbf{u}\times\mathbf{v}|}{b}=\frac{|\mathbf{b}\times\mathbf{c}|}{a+b+c}=\frac{2[ABC]}{a+b+c}\ ,$$

which is the inradius of triangle ABC. Similarly, the distance of Q from side AB is equal to the inradius. Therefore, Q must be the incentre of the triangle. A similar analysis can be made for the other two vertices of the triangle and we find that

$$\mathbf{u}=\frac{b\mathbf{c}-c\mathbf{b}}{a+b+c}=\overrightarrow{AI}\ ;$$

$$\mathbf{v}\equiv\frac{c\mathbf{a}-a\mathbf{c}}{a+b+c}=\overrightarrow{BI}\ ;$$

and

$$\mathbf{w}=\frac{a\mathbf{b}-b\mathbf{a}}{a+b+c}=\overrightarrow{CI}\ .$$

Since $a\mathbf{u}+b\mathbf{b}+c\mathbf{c}=\mathbf{0}$,

$$a(\mathbf{p}+\mathbf{u})+b(\mathbf{q}+\mathbf{v})+c(\mathbf{c}+\mathbf{r})=a(\mathbf{p}-\mathbf{u})+b(\mathbf{q}-\mathbf{v})+c(\mathbf{c}-\mathbf{r})\ .$$

Taking the dot product of this equation with the vector $\mathbf{z}=\mathbf{p}-\mathbf{u}=\mathbf{q}-\mathbf{v}=\mathbf{r}-\mathbf{w}$ leads to

$$(ap^2+bq^2+cr^2)-(au^2+bv^2+cr^2)=(a+b+c)z^2\ ,$$

as desired.

623. Given the parameters a,b,c, solve the system

$$x+y+z=a+b+c;$$

$$x^2+y^2+x^2=a^2+b^2+c^2;$$

$$\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=3\ .$$

Solution. [N. Lvov, J. Schneider] The first and third equations represent two planes in space that intersect in a line; the second represents a sphere, which the line intersects in at most two points. Therefore there are at most two solutions to the equation. One is $(x,y,z)=(a,b,c)$. The second is equal to

$$(x,y,z)=(a[1-k(b-c)],b[1-k(c-a)],c[1-k(a-b)])$$

where

$$\begin{aligned}k=&\frac{2[a^2(b-c)+b^2(c-a)+c^2(a-b)]}{a^2(b-c)^2+b^2(c-a)^2+c^2(a-b)^2}\\ &=\frac{(a-b)(b-c)(c-a)}{a^2b^2+b^2c^2+c^2a^2-abc(a+b+c)}\ .\end{aligned}$$

Comment. This satisfies the linear equations regardless of the value of k, and substitution into the quadratic equation will establish the appropriate value of k.

624. Suppose that $x_i \geq 0$ and

$$\sum_{i=1}^{n} \frac{1}{1+x_i} \leq 1 \ .$$

Prove that

$$\sum_{i=1}^{n} 2^{-x_i} \leq 1 \ .$$

Solution. [J. Schneider] Let $f(x) = x2^{1/x}$. Since $f'(x) = (1 - (\log 2/x))2^{1/x} < 0$ for $0 < x < \log 2$, it follows that $f(x)$ decreases on the interval $(0, \frac{1}{2}]$.

The function 2^{x-1} is convex, so that the graphs of $y = x$ and $y = 2^{x-1}$ intersect in at most two points. Since they intersect at $x = 1$ and $x = 2$, it follows that $x > 2^{x-1}$ when $1 < x < 2$ and $x < 2^{x-1}$ when $x > 2$.

It suffices to prove the problem under the condition that $\sum(1+x_i)^{-1} = 1$, for if $\sum(1+x_i)^{-1} < 1$, then we can select $X > 0$ so that $(1+X)^{-1} + \sum(1+x_i)^{-1} = 1$ and obtain $2^{-X} + \sum 2^{-x_i} \leq 1$, from which the desired result would follow.

Let $y_i = (1+x_i)^{-1}$ so that $\sum y_i = 1$. Suppose, to begin with that $y_i \leq \frac{1}{2}$ for each i. Then, since $f(y_i) \geq f(\frac{1}{2} = 2$, it follows that

$$2^{-x_i} = 2^{(1-(1/y_i))} = \frac{2}{2^{1/y_i}} \leq y_i$$

so that $\sum_{i=1}^{n} 2^{-x_i} \leq \sum_{i=1}^{n} y_i = 1$ as desired.

The remaining case is that at least one y_i exceeds $\frac{1}{2}$. There can be at most one such y_i, so we may suppose that $y_1, y_2, \cdots, y_{n-1} \leq \frac{1}{2} < y_n$.

Suppose that $g(x) = 2^{(1-(1/x))}$. We show that

$$g(y_1) + g(y_2) + \cdots + g(y_{n-1}) \leq g(y_1 + y_2 + \cdots + y_{n-1}) \ .$$

Suppose that $Y = y_1 + y_2 + \cdots + y_{n-1}$; note that $Y < \frac{1}{2}$. Then

$$\begin{aligned} g(y_1) + g(y_2) + \cdots + g(y_n) &= 2\left[\frac{y_1}{f(y_1)} + \frac{y_2}{f(y_2)} + \cdots + \frac{y_{n-1}}{f(y_{n-1})}\right] \\ &\leq 2\left[\frac{y_1}{f(Y)} + \frac{y_2}{f(Y)} + \cdots + \frac{y_{n-1}}{f(Y)}\right] \\ &\leq \frac{2Y}{f(Y)} = g(Y) = g(y_1 + y_2 + \cdots + y_{n-1}) \ . \end{aligned}$$

We need to show that $\sum_{i=1}^{n} g(y_i) \leq 1$ when $\sum_{i=1}^{n} y_i = 1$. This can be achieved by showing that $g(Y) + g(1-Y) \leq 1$; this amounts to

$$\frac{1}{2^{\frac{1-Y}{Y}}} + \frac{1}{2^{\frac{Y}{1-Y}}} \leq\leq 1,$$

for $0 < Y < 1$. Let $z = (1-Y)/Y$. Then we need to show that

$$\frac{1}{2^z} + \frac{1}{2^{1/z}} \leq 1$$

for $z > 0$. Since the left side takes the same value at z and $1/z$, it is enough to establish this for $z \geq 1$.

When $z \geq 2$, we can use the fact that $2^{z-1} \geq 2$ and Bernoulli's inequality to obtain

$$\left(1 - \frac{1}{2^z}\right)^z \geq 1 - \frac{z}{2^z} \geq 1 - \frac{1}{2} = \frac{1}{2} \ ,$$

from which $1 - 2^{-z} \geq 2^{-1/z}$ as desired.

Suppose that $1 \leq z \leq 2$. Let $h(z) = 2^{-z} + 2^{-1/z}$. Then $h(1) = 1$. We show that $h(z)$ decreases for $z \geq 1$.

$$h'(z) = -\log 2 \cdot 2^{-z} + \log 2 \cdot z^{-2} 2^{-1/z} \ .$$

Since $1 \leq z \leq 2$, we have that $z \geq 2^{z-1}$, so that $z^2 \geq 2^{2z-2}$. However

$$(2z-2) - \left(z - \frac{1}{z}\right) = \left(z + \frac{1}{z}\right) - 2 \geq 0$$

so that $2z - 2 \geq z - (1/z)$. Therefore $z^2 \geq 2^{z - \frac{1}{z}}$ and so

$$h'(z) \leq -\log 2 \cdot 2^{-z} + \log 2 \cdot 2^{-1/z} \cdot 2^{\frac{1}{z} - z} = \log 2(-2^{-z} + 2^{-z}) = 0 \ .$$

Thus, $h(z)$ decreases on $[1,2]$ and so $h(z) \leq 1$ there. This completes the solution.

625. Given an odd number of intervals, each of unit length, on the real line, let S be the set of numbers that are in an odd number of these intervals. Show that S is a finite union of disjoint intervals of total length not less than 1.

Solution. The result holds when there is one interval. Suppose that n is an odd number greater than 1 and, as an induction hypothesis, that the result holds for any odd number of intervals fewer than n. Since all of the intervals have the same length, they can be linearly ordered from left to right. Let Z be the rightmost interval and Y the next to rightmost interval. Let T be the union of all the intervals but Y and Z, and S' the set of points that belong to an odd number of the intervals making up T. By the induction hypothesis, S' is the union of a finite number of disjoint intervals not less than 1.

S contains the entire interval $Z \backslash Y$, as points here are contained only in Z; $S \cap (Y \cap Z) = S' \cap (Y \cap Z)$, as we are adding evenly many intervals to the collection making up T for the points in $Y \cap Z$. Thus, the only points that lie in S' but not in S must lie within $Y \backslash Z$. Note that these points deleted from S' consitute a union of intervals, since they are obtained by intersecting intervals. Since Y and Z have equal length, $|Y \backslash Z| = |Z \backslash Y|$ and so we augment S' by an interval that exceeds the length of the intervals of S' deleted. Therefore, the total length of the intervals making up S is at least 1.

626. Let ABC be an isosceles triangle with $AB = AC$, and suppose that D is a point on the side BC with $BC > BD > DC$. Let BE and CF be diameters of the respective circumcircles of triangles ABD and ADC, and let P be the foot of the altitude from A to BC. Prove that $PD : AP = EF : BC$.

Solution 1. Since angles BDE and CDF are both right, E and F both lie on the perpendicular to BC through D. Since $ABDE$ and $ADCF$ are concyclic,

$$\angle AEF = \angle ABD = \angle ABC = \angle ACB = \angle ACD = \angle AFD = \angle AFE \ .$$

Therefore triangles AEF and ABC are similar. Thus AEF is isosceles and its altitude through A is perpendicular to DEF and parallel to BC, so that it is equal to PD. Therefore, from the similarity, $PD : AP = EF : BC$, as desired.

Solution 2. Since the chord AD subtends the same angle ($\angle ABC = \angle ACB$) in circles ABD and ACD, these circles must have equal diameters. The rotation with centre A that takes B to C takes the circle ABD to a circle with chord AC of equal diameter. The angle subtended at D by AB on the circumcircle of ABD

is the supplement of the angle subtended at D by AC on the circumcircle of ACD. Therefore, this image circle must be the circle ACD. Therefore the diameter BE is carried to the diameter CF, and E is carried to F. Hence $AE = AF$ and $\angle BAC = \angle EAF$. Thus, triangles ABC and AEF are similar.

Now consider the composite of a rotation about A through a right angle followed by a dilatation of factor $|AE|/|AB|$. This transformation take B to E and C to F, and therefore the altitude AP to the altitude AM of triangle AEF which is therefore parallel to BC. Since D lies on the circumcircle of ABD with diameter BE, $\angle BDE = 90°$. Similarly, $\angle CDF = 90°$. Hence $AMDP$ is a rectangle and $AM = PD$. The result follows from the similarity of triangles ABC and AEF.

627. Let

$$f(x,y,z) = 2x^2 + 2y^2 - 2z^2 + \frac{7}{xy} + \frac{1}{z} \ .$$

There are three pairwise distinct numbers a, b, c for which

$$f(a,b,c) = f(b,c,a) = f(c,a,b) \ .$$

Determine $f(a,b,c)$. Determine three such numbers a, b, c.

Solution. Suppose that a, b, c are pairwise distinct and $f(a,b,c) = f(b,c,a) = f(c,a,b)$. Then

$$2a^2 + 2b^2 - 2c^2 + \frac{7}{ab} + \frac{1}{c} = 2b^2 + 2c^2 - 2a^2 + \frac{7}{bc} + \frac{1}{a}$$

so that

$$4(a^2 - c^2) = \left(\frac{1}{a} - \frac{1}{c}\right)\left(1 - \frac{7}{b}\right) = \frac{1}{abc}(c-a)(b-7) \ .$$

Therefore $4abc(a+c) = 7 - b$. Similarly, $4abc(b+a) = 7 - c$. Subtracting these equations yields that $4abc(c-b) = c - b$ so that $4abc = 1$. It follows that $a + b + c = 7$.

Therefore

$$\begin{aligned} f(a,b,c) &= 2(a^2+b^2) - 2c^2 + 28c + 4ab \\ &= 2(a+b)^2 - 2c^2 + 28c = 2(7-c)^2 - 2c^2 + 28c \\ &= 98 - 28c + 2c^2 - 2c^2 + 28c = 98 \ . \end{aligned}$$

We can find such triples by picking any nonzero value of c and solving the quadratic equation $t^2 - (7 - c)t + (1/4c) = 0$ for a and b. For example, taking $c = 1$ yields the triple

$$(a,b,c) = \left(\frac{6+\sqrt{35}}{2}, \frac{6-\sqrt{35}}{2}, 1\right) \ .$$

628. Suppose that AP, BQ and CR are the altitudes of the acute triangle ABC, and that

$$9\overrightarrow{AP} + 4\overrightarrow{BQ} + 7\overrightarrow{CR} = \overrightarrow{O} \ .$$

Prove that one of the angles of triangle ABC is equal to $60°$.

Solution 1. [H. Spink] Since the sum of the three vectors $9\overrightarrow{AP}$, $4\overrightarrow{BQ}$, $7\overrightarrow{CR}$ is zero, there is a triangle whose sides have lengths $9|AP|$, $4|BQ|$, $7|CR|$ and are parallel to the corresponding vectors.

Where H is the orthocentre, we have that

$$\angle BHP = 90° - \angle QBC = \angle ACB$$

so that the angle between the vectors $\overrightarrow{AP}$ and $\overrightarrow{BQ}$ is equal to angle ACB. Similarly, the angle between vectors $\overrightarrow{BQ}$ and $\overrightarrow{CR}$ is equal to angle BAC. It follows that the triangle formed by the vectors is similar to triangle ABC and

$$|AB| : 7|CR| = |BC| : 9|AP| = |CA| : 4|BQ| \ .$$

Since twice the area of the triangle ABC is equal to

$$|AB| \times |CR| = |BC| \times |AP| = |CA| \times |BQ| \ ,$$

we have that (with conventional notation for side lengths)

$$\frac{c^2}{7} = \frac{a^2}{9} = \frac{b^2}{4}$$

so that $a : b : c = 3 : 2 : \sqrt{7}$.

If one angle of the triangle is equal to $60°$ we would expect it to be neither the largest nor the smallest. Accordingly, we compute the cosine of angle ACB, namely

$$\frac{a^2 + b^2 - c^2}{2ab} = \frac{9 + 4 - 7}{2 \times 3 \times 2} = \frac{6}{12} = \frac{1}{2} \ .$$

Therefore $\angle ACB = 60°$.

Solution 2. Let the angles of the triangle be $\alpha = \angle BAC$, $\beta = \angle CBA$ and $\gamma = \angle ACB$; let p, q, r be the respective magnitudes of vectors $\overrightarrow{AP}$, $\overrightarrow{BQ}$, $\overrightarrow{CR}$. Taking the dot product of the vector equation with $\overrightarrow{BC}$ and noting that $\angle QBC = 90 - \gamma$ and $\angle BCR = 90 - \beta$, we find that $4q \sin\gamma = 7r \sin\beta$. Similarly, $9p \sin\gamma = 7r \sin\alpha$ and $9p \sin\beta = 4q \sin\alpha$. Using the conventional notation for the sides of the triangle, we have that

$$a : b : c = \sin\alpha : \sin\beta : \sin\gamma = 9p : 4q : 7r \ .$$

However, we also have that twice the area of triangle ABC is equal to $ap = bq = cr$, so that $a : b : c = (1/p) : (1/q) : (1/r)$. Therefore $9p^2 = 4q^2 = 7r^2 = k$, for some constant k. Therefore

$$\begin{aligned} \cos\angle ACB &= \frac{a^2 + b^2 - c^2}{2ab} = \frac{81p^2 + 16q^2 - 49r^2}{72pq} \\ &= \frac{9k + 4k - 7k}{12k} = \frac{1}{2} \ , \end{aligned}$$

from which it follows that $\angle C = 60°$.

Solution 3. [C. Deng] Observe that

$$|BQ| = |BC| \cos\angle QBC = |BC| \angle \sin ACB \ ,$$

$$|CR| = |BC| \cos\angle RCB = |BC| \sin\angle ABC \ .$$

Resolving in the direction of $\overrightarrow{BC}$, we find from the given equation that

$$4|BC| \cos^2\angle QBC = 4|BQ| \cos\angle QBC = 7|CR| \cos\angle RCB = 7|BC| \cos^2\angle RCB$$

$$\Longrightarrow 4\sin^2\angle ACB = 7\sin^2\angle ABC \ .$$

By the Law of Sines, $AC : AB = \sin\angle ABC : \sin\angle ACB = 2 : \sqrt{7}$. Similarly $AC : BC = 2 : 3$, so that $CA : AB : BC = 2 : \sqrt{7} : 3$. The cosine of angle ACB is equal to $(4+9-7)/12 = 1/2$, so that $\angle ACB = 60°$.

629. (a) Let $a > b > c > d > 0$ and $a + d = b + c$. Show that $ad < bc$.

(b) Let a, b, p, q, r, s be positive integers for which

$$\frac{p}{q} < \frac{a}{b} < \frac{r}{s}$$

and $qr - ps = 1$. Prove that $b \geq q + s$.

(a) *Solution 1.* Since $c = a + d - b$, we have that

$$bc - ad = b(a + d - b) - ad = (a - b)b - (a - b)d = (a - b)(b - d) > 0 \quad .$$

Solution 2. Let $a + d = b + c = u$. Then

$$bc - ad = b(u - b) - (u - d)d = u(b - d) - (b^2 - d^2) = (b - d)(u - b - d) \ .$$

Now $u = b + c > b + d$, so that $u - b - d > 0$ as well as $b - d > 0$. Hence $bc - ad > 0$ as desired.

Solution 3. Let $x = a - b > 0$. Since $a - b = c - d$, we have that $a = b + x$ and $d = c - x$. Hence

$$bc - ad = bc - (b + x)(c - x) = bx - cx + x^2 = x^2 + x(b - c) > 0 \ .$$

Solution 4. Since $\sqrt{a} > \sqrt{b} > \sqrt{c} > \sqrt{d}$, $\sqrt{a} - \sqrt{d} > \sqrt{b} - \sqrt{c}$. Squaring and using $a + d = b + c$ yields $2\sqrt{bc} > 2\sqrt{ad}$, whence the result.

(b) *Solution.* Since all variables represent integers,

$$aq - bp > 0, br - as > 0 \Longrightarrow aq - bp \geq 1, br - as \geq 1 \ .$$

Therefore

$$b = b(qr - ps) = q(br - as) + s(aq - bp) \geq q + s \ .$$

630. (a) Show that, if

$$\frac{\cos\alpha}{\cos\beta} + \frac{\sin\alpha}{\sin\beta} = -1 \quad ,$$

then

$$\frac{\cos^3\beta}{\cos\alpha} + \frac{\sin^3\beta}{\sin\alpha} = 1 \quad .$$

(b) Give an example of numbers α and β that satisfy the condition in (a) and check that both equations hold.

(a) *Solution 1.* Let

$$\lambda = \frac{\cos\beta}{\cos\alpha} \qquad \text{and} \qquad \mu = \frac{\sin\beta}{\sin\alpha} \ .$$

Since $\lambda^{-1} + \mu^{-1} = -1$, we have that $\lambda + \mu = -\lambda\mu$. Now

$$1 = \cos^2\beta + \sin^2\beta = \lambda^2\cos^2\alpha + \mu^2\sin^2\alpha = \lambda^2 + (\mu^2 - \lambda^2)\sin^2\alpha = \lambda^2 - (\mu - \lambda)\lambda\mu\sin^2\alpha \ .$$

Hence

$$\begin{aligned}
\frac{\cos^3\beta}{\cos\alpha} + \frac{\sin^3\beta}{\sin\alpha} &= \lambda^3\cos^2\alpha + \mu^3\sin^2\alpha \\
&= \lambda(\lambda^2\cos^2\alpha + \mu^2\sin^2\alpha) + (\mu - \lambda)\mu^2\sin^2\alpha \\
&= \lambda + (\mu - \lambda)\mu^2\sin^2\alpha \\
&= \frac{1}{\lambda}[\lambda^2 + (\lambda^2 - 1)\mu] \\
&= \frac{1}{\lambda}[\lambda^2 + \lambda^2\mu + \lambda + \lambda\mu \\
&= \lambda + \lambda\mu + 1 + \mu = 1 \quad .
\end{aligned}$$

Solution 2. [M. Boase]

$$\frac{\cos\alpha}{\cos\beta} + \frac{\sin\alpha}{\sin\beta} = -1 \Longrightarrow$$

$$\sin(\alpha+\beta) + \sin\beta\cos\beta = 0 \ . \qquad (*)$$

Therefore

$$\begin{aligned}
\frac{\cos^3\beta}{\cos\alpha} + \frac{\sin^3\beta}{\sin\alpha} &= \frac{\cos\beta(1-\sin^2\beta)}{\cos\alpha} + \frac{\sin\beta(1-\cos^2\beta)}{\sin\alpha} \\
&= \frac{\cos\beta}{\cos\alpha} + \frac{\sin\beta}{\sin\alpha} - \sin\beta\cos\beta\left(\frac{\sin\beta}{\cos\alpha} + \frac{\cos\beta}{\sin\alpha}\right) \\
&= \frac{\sin(\alpha+\beta)}{\cos\alpha\sin\alpha} - \frac{\cos\beta\sin\beta(\cos(\alpha-\beta))}{\cos\alpha\sin\alpha} \\
&= \frac{-2\sin\beta\cos\beta + 2\sin(\alpha+\beta)\cos(\alpha-\beta)}{2\sin\alpha\cos\alpha} \qquad \text{using } \ (*) \\
&= \frac{-2\sin\beta\cos\beta + [\sin 2\alpha + \sin 2\beta]}{\sin 2\alpha} = 1
\end{aligned}$$

since $2\sin\beta\cos\beta = \sin 2\beta$.

Solution 3. [A. Birka] Let $\cos\alpha = x$ and $\cos\beta = y$. Then

$$\frac{\sin\alpha}{\sin\beta} = \pm\sqrt{\frac{1-x^2}{1-y^2}} \quad .$$

Since

$$\frac{x}{y} + 1 = \mp\sqrt{\frac{1-x^2}{1-y^2}} \quad .$$

then

$$(x^2 + 2xy + y^2)(1-y^2) = y^2(1-x^2) \quad ,$$

whence

$$x^2 + 2xy = 2xy^3 + y^4 \quad .$$

Thus,

$$\begin{aligned}
\frac{\cos^3\beta}{\cos\alpha} + \frac{\sin^3\beta}{\sin\alpha} &= \frac{y^3}{x} \pm (1-y^2)\sqrt{\frac{1-y^2}{1-x^2}} \\
&= \frac{y^3}{x} - \frac{(1-y^2)y}{x+y} = \frac{y^4 + 2xy^3 - xy}{x(x+y)} \\
&= \frac{x^2 + xy}{x(x+y)} = 1 \quad .
\end{aligned}$$

Solution 4. [J. Chui] Note that the given equation implies that $\sin 2\beta = -2\sin(\alpha+\beta)$ and that the numerator of

$$\frac{\cos\alpha}{\cos\beta} + \frac{\sin\alpha}{\sin\beta} + \frac{\cos^3\beta}{\cos\alpha} + \frac{\sin^3\beta}{\sin\alpha}$$

is one quarter of

$$\begin{aligned}
&4[\cos^2\alpha\sin\alpha\sin\beta + \sin^2\alpha\cos\alpha\cos\beta + \cos^4\beta\sin\alpha\sin\beta + \sin^4\beta\cos\alpha\cos\beta] \\
&\quad = 4[\cos^2\alpha\sin\alpha\sin\beta + \sin^2\alpha\cos\alpha\cos\beta + (\cos^2\beta - \cos^2\beta\sin^2\beta)\sin\alpha\sin\beta \\
&\qquad\qquad + (\sin^2\beta - \sin^2\beta\cos^2\beta)\cos\alpha\cos\beta] \\
&\quad = (4\cos^2\alpha + 4\cos^2\beta - \sin^2 2\beta)\sin\alpha\sin\beta + (4\sin^2\alpha + 4\sin^2\beta - \sin^2 2\beta)\cos\alpha\cos\beta \\
&\quad = 2\sin 2\alpha\cos\alpha\sin\beta + 2\sin 2\beta\cos\beta\sin\alpha + 2\sin 2\alpha\sin\alpha\cos\beta + 2\sin 2\beta\cos\alpha\sin\beta \\
&\qquad\qquad - \sin^2 2\beta(\cos\alpha\cos\beta + \sin\alpha\sin\beta) \\
&\quad = 2(\sin 2\alpha + \sin 2\beta)\sin(\alpha+\beta) - \sin^2 2\beta\cos(\alpha-\beta) \\
&\quad = 2\sin(\alpha+\beta)[\sin 2\alpha + \sin 2\beta - 2\sin(\alpha+\beta)\cos(\alpha-\beta)] = 0 \quad ,
\end{aligned}$$

since

$$\sin 2\alpha + \sin 2\beta = \sin(\overline{\alpha+\beta} + \overline{\alpha-\beta}) + \sin(\overline{\alpha+\beta} - \overline{\alpha-\beta}) \ .$$

Solution 5. [A. Tang] From the given equation, we have that

$$\sin(\alpha+\beta) = -\sin\beta\cos\beta \quad ,$$

$$\frac{\cos\beta}{\cos\alpha} = \frac{-\sin\beta}{\sin\alpha+\sin\beta} \quad ,$$

and

$$\frac{\sin\beta}{\sin\alpha} = \frac{-\cos\beta}{\cos\alpha+\cos\beta} \quad .$$

Hence

$$\begin{aligned}\frac{\cos^3\beta}{\cos\alpha} + \frac{\sin^3\beta}{\sin\alpha} &= \cos^2\beta\left[\frac{-\sin\beta}{\sin\alpha+\sin\beta}\right] + \sin^2\beta\left[\frac{-\cos\beta}{\cos\alpha+\cos\beta}\right]\\ &= -\frac{\sin\beta\cos\beta[\cos\alpha\cos\beta+\sin\alpha\sin\beta+1]}{4\sin\frac{1}{2}(\alpha+\beta)\cos\frac{1}{2}(\alpha-\beta)\cos\frac{1}{2}(\alpha+\beta)\cos\frac{1}{2}(\alpha-\beta)}\\ &= \frac{\sin(\alpha+\beta)[\cos(\alpha-\beta)+1]}{[2\sin\frac{1}{2}(\alpha+\beta)\cos\frac{1}{2}(\alpha+\beta)][2\cos^2\frac{1}{2}(\alpha-\beta)]} = 1 \quad .\end{aligned}$$

Solution 6. [D. Arthur] The given equations yield $2\sin(\alpha+\beta) = -\sin 2\beta$, $\cos\alpha\sin\beta = -\cos\beta(\sin\alpha + \sin\beta)$ and $\sin\alpha\cos\beta = -\sin\beta(\cos\alpha+\cos\beta)$. Hence

$$\begin{aligned}\frac{\cos^3\beta}{\cos\alpha} + \frac{\sin^3\beta}{\sin\alpha} &= \frac{\cos^2\beta(\cos\beta\sin\alpha) + \sin^2\beta(\sin\beta\cos\alpha)}{\cos\alpha\sin\alpha}\\ &= \frac{-\cos^2\beta\sin\beta(\cos\alpha+\cos\beta) - \sin^2\beta\cos\beta(\sin\alpha+\sin\beta)}{\cos\alpha\sin\alpha}\\ &= \frac{-\cos\beta\sin\beta(\cos\alpha\cos\beta+\cos^2\beta+\sin\alpha\sin\beta+\sin^2\beta)}{\cos\alpha\sin\alpha}\\ &= \frac{-\sin 2\beta(1+\cos(\alpha-\beta))}{\sin 2\alpha}\\ &= \frac{-\sin 2\beta + 2\sin(\alpha+\beta)\cos(\alpha-\beta)}{\sin 2\alpha}\\ &= \frac{-\sin 2\beta+\sin 2\alpha+\sin 2\beta}{\sin 2\alpha} = 1 \quad .\end{aligned}$$

Solution 7. [C. Deng] Let $\sin\beta = x$, $\cos\beta = y$, and $(\sin\alpha)/(\sin\beta) = c$. Thus, $(\cos\alpha)/(\cos\beta) = -1-c$. We have that

$$x^2 + y^2 = 1$$

and

$$(cx)^2 + (-1-c)y)^2 = 1 \ .$$

Solving the system yields that

$$x^2 = \frac{c^2+2c}{1+2c} \ , \quad y^2 = \frac{1-c^2}{1+2c} \ .$$

Therefore,

$$\begin{aligned}\frac{\sin^3\beta}{\sin\alpha} + \frac{\cos^3\beta}{\cos\alpha} &= \frac{x^2}{c} + \frac{y^2}{-1-c} = \frac{c^2+2c}{c(2c+1)} + \frac{1-c^2}{(-c-1)(2c+1)}\\ &= \frac{c+2}{2c+1} + \frac{c-1}{2c+1} = 1 \ .\end{aligned}$$

(b) Solution. The given equation is equivalent to $2\sin(\alpha+\beta)+\sin 2\beta = 0$. Try $\beta = -45°$ so that $\sin(\alpha - 45°) = \frac{1}{2}$. We take $\alpha = 75°$. Now

$$\sin 75° = \sin(45° + 30°) = \frac{1}{\sqrt{2}}\left(\frac{\sqrt{3}+1}{2}\right)$$

and

$$\cos 75° = \cos(45° + 30°) = \frac{1}{\sqrt{2}}\left(\frac{\sqrt{3}-1}{2}\right) \quad .$$

It is straightforward to check that both equations hold.

631. The sequence of functions $\{P_n\}$ satisfies the following relations:

$$P_1(x) = x \ , \qquad P_2(x) = x^3 \ ,$$

$$P_{n+1}(x) = \frac{P_n^3(x) - P_{n-1}(x)}{1 + P_n(x)P_{n-1}(x)} \quad , \qquad n = 1, 2, 3, \cdots .$$

Prove that all functions P_n are polynomials.

Solution 1. Taking $x = 1, 2, 3, \cdots$ yields the respective sequences

$$\{1, 1, 0, -1, -1, 0, \cdots\} \ , \quad \{2, 8, 30, 112, 418, 1560, \cdots\} \ , \quad \{3, 27, 240, 2133, \cdots\} \ .$$

In each case, we find that

$$P_{n+1}(x) = x^2 P_n(x) - P_{n-1}(x) \tag{1}$$

for $n = 2, 3, \cdots$. If we can establish (1) in general, it will follow that all the functions P_n are polynomials.

From the definition of P_n, we find that

$$P_{n+1} + P_{n-1} = P_n(P_n^2 - P_{n+1}P_{n-1}) \quad .$$

Therefore, it suffices to establish that $P_n^2 - P_{n+1}P_{n-1} = x^2$ for each n. Now, for $n \geq 2$,

$$\begin{aligned}
[P_{n+1}^2 - P_{n+2}P_n] - [P_n^2 - P_{n+1}P_{n-1}] &= P_{n+1}(P_{n+1} + P_{n-1}) - P_n(P_{n+2} + P_n) \\
&= P_{n+1}P_n(P_n^2 - P_{n+1}P_{n-1}) - P_nP_{n+1}(P_{n+1}^2 - P_{n+2}P_n) \\
&= -P_{n+1}P_n[(P_{n+1}^2 - P_{n+2}P_n) - (P_n^2 - P_{n+1}P_{n-1})] \quad ,
\end{aligned}$$

so that either $P_{n+1}(x)P_n(x) + 1 \equiv 0$ or $P_{n+1}^2 - P_{n+2}P_n = P_n^2 - P_{n+1}P_{n-1}$. The first identity is precluded by the case $x = 1$, where it is false. Hence

$$P_{n+1}^2 - P_{n+2}P_n = P_n^2 - P_{n+1}P_{n-1}$$

for $n = 2, 3, \cdots$. Since $P_2^2(x) - P_3(x)P_1(x) = x^2$, the result follows.

Solution 2. [By inspection, we make the conjecture that $P_n(x) = x^2P_{n-1}(x) - P_{n-2}$. Rather than prove this directly from the rather awkward condition on P_n, we go through the back door.] Define the sequence $\{Q_n\}$ for $n = 0, 1, 2, \cdots$ by

$$Q_0(x) = 0 \ , \qquad Q_1(x) = x \ , \qquad Q_{n+1} = x^2Q_n(x) - Q_{n-1}(x)$$

for $n \geq 1$. It is clear that $Q_n(x)$ is a polynomial of degree $2n-1$ for $n = 1, 2, \cdots$. We show that $P_n(x) = Q_n(x)$ for each n.

Lemma: $Q_n^2(x) - Q_{n+1}Q_{n-1} = x^2$ for $n \geq 1$.

Proof: This result holds for $n = 1$. Assume that it holds for $n = k - 1 \geq 1$. Then

$$\begin{aligned} Q_k^2(x) - Q_{k+1}(x)Q_{k-1}(x) &= Q_k^2(x) - (x^2 Q_k(x) - Q_{k-1}(x))Q_{k-1}(x) \\ &= Q_k(x)(Q_k(x) - x^2 Q_{k-1}(x)) + Q_{k-1}^2(x) \\ &= -Q_k(x)Q_{k-2}(x) + Q_{k-1}^2(x) = x^2 \ . \qquad \spadesuit \end{aligned}$$

From the lemma, we find that

$$\begin{aligned} &Q_{n+1}(x) + Q_{n-1}(x) + Q_{n+1}(x)Q_n(x)Q_{n-1}(x) \\ &\quad = x^2 Q_n(x) + Q_{n+1}(x)Q_n(x)Q_{n-1}(x) = Q_n(x)(x^2 + Q_{n+1}(x)Q_{n-1}(x)) = Q_n^3(x) \end{aligned}$$

$$\Longrightarrow Q_{n+1}(x) = \frac{Q_n^3(x) - Q_{n-1}(x)}{1 + Q_n(x)Q_{n-1}(x)} \qquad (n = 1, 2, \cdots) \ .$$

We know that $Q_1(x) = P_1(x)$ and $Q_2(x) = P_2(x)$. Suppose that $Q_n(x) = P_n(x)$ for $n = 1, 2, \cdots, k$. Then

$$Q_{k+1}(x) = \frac{Q_k^3(x) - Q_{k-1}(x)}{1 + Q_k(x)Q_{k-1}(x)} = \frac{P_k^3(x) - P_{k-1}(x)}{1 + P_k(x)P_{k-1}(x)} = P_{k+1}(x)$$

from the definition of P_{k+1}. The result follows.

Comment: It can also be established that $P_{n+1}^2 + P_n^2 = (1 + P_n P_{n+1})x^2$ for each $n \geq 0$.

Solution 3. [I. Panayotov] First note that the sequence $\{P_n(x)\}$ is defined for all values of x, *i.e.*, the denominator $1 + P_{n-1}(x)P_n(x)$ never vanishes for n and x. Suppose otherwise, and let n be the least number for which there exists u for which $1 + P_{n-1}(u)P_n(u) = 0$. Then $n \geq 3$ and

$$-1 = P_{n-1}(u)P_n(u) = \frac{P_{n-1}(u)^4 - P_{n-1}(u)P_{n-2}(u)}{1 + P_{n-1}(u)P_{n-2}(u)}$$

which implies that $P_{n-1}(u)^4 = -1$, a contradiction.

We now prove by induction that $P_{n+1} = x^2 P_n - P_{n-1}$. Suppose that $P_k = x^2 P_{k-1} - P_{k-2}$ for $3 \leq k \leq n$, so that in particular we know that P_k is a polynomial for $1 \leq k \leq n$. Substituting for P_k yields

$$P_{k-1}^3(x) = P_{k-1}(x)[x^2 + x^2 P_{k-1}(x)P_{k-2}(x) - P_{k-2}^2(x)]$$

for all x. If $P_{k-1}(x) \neq 0$, then

$$P_{k-1}^2(x) = x^2 + x^2 P_{k-1}(x)P_{k-2}(x) - P_{k-2}^2(x) \ .$$

Both sides of this equation are polynomials and so continuous functions of x. Since the roots of P_{k-1} constitute a finite discreet set, this equation holds when x is one of the roots as well. Now

$$\begin{aligned} P_{n+1} &= \frac{P_n^3 - P_{n-1}}{1 + P_n P_{n-1}} = \frac{P_n(x^2 P_{n-1} - P_{n-2})^2 - P_{n-1}}{1 + P_n P_{n-1}} \\ &= \frac{P_n(x^4 P_{n-1}^2 - x^2 P_{n-1}P_{n-2} + x^2 - P_{n-1}^2) - P_{n-1}}{1 + P_n P_{n-1}} \\ &= \frac{P_n(x^2 P_n P_{n-1} + x^2 - P_{n-1}^2) - P_{n-1}}{1 + P_n P_{n-1}} \qquad \text{since} \quad x^2 P_{n-1} - P_{n-2} = P_n \\ &= \frac{(x^2 P_n - P_{n-1})(1 + P_n P_{n-1})}{1 + P_n P_{n-1}} = x^2 P_n - P_{n-1} \quad . \end{aligned}$$

The result follows.

632. Let a, b, c, x, y, z be positive real numbers for which $a \leq b \leq c$, $x \leq y \leq z$, $a + b + c = x + y + z$, $abc = xyz$, and $c \leq z$, Prove that $a \leq x$.

Solution. Let

$$p(t) = (t - a)(t - b)(t - c) = t^3 - (a + b + c)t^2 + (ab + bc + ca)t - abc$$

and

$$q(t) = (t - x)(t - y)(t - z) = t^3 - (x + y + z)t^2 + (xy + yz + zx)t - xyz \ .$$

Then $p(t) - q(t) = (ab + bc + ca - xy - yz - zx)t$ never changes sign for positive values of t. Since $p(t) > 0$ for $t > c$, we have that $p(z) - q(z) = p(z) \geq 0$, so that $p(t) \geq q(t)$ for all $t > 0$.

Hence, for $0 < t < a$, we have that $q(t) \leq p(t) < 0$, from which it follows that $q(t)$ has no root less than a. Hence $x \geq a$ as desired.

633. Let ABC be a triangle with $BC = 2 \cdot AC - 2 \cdot AB$ and D be a point on the side BC. Prove that $\angle ABD = 2\angle ADB$ if and only if $BD = 3CD$.

Solution 1. [A. Murali] Let $\angle ADB = \theta$, $|AB| = c$, $|CA| = b$, $|AD| = d$, $|CD| = x$, $|BD| = y$. Assume that $\angle ABD = 2\angle ADB$. By the Law of Sines applied to triangle ABD,

$$\frac{d}{\sin 2\theta} = \frac{c}{\sin \theta} \Longrightarrow d = 2c \cos \theta \ .$$

By the Law of Cosines in triangle ABD,

$$4c^2 \cos^2 \theta = d^2 = c^2 + y^2 - 2cy \cos 2\theta \ ,$$

from which

$$\begin{aligned} 0 &= y^2 - (2c \cos 2\theta)y + c^2(1 - 4\cos^2 \theta) \\ &= y^2 - (2c \cos 2\theta)y - c^2(2 \cos 2\theta + 1) \\ &= [y + c][y - c(2 \cos 2\theta + 1)] \ . \end{aligned}$$

Hence $y = (2 \cos 2\theta + 1)c$.

By the Law of Cosines in triangle ACD,

$$b^2 = d^2 + x^2 + 2xd \cos \theta \Longrightarrow 0 = 4[x^2 + (2d \cos \theta)x + (d^2 - b^2)] \ .$$

Since $x + y = 2(b - c)$, then

$$2b = x + y + 2c = x + (2 \cos 2\theta + 3)c \ .$$

Now $2d \cos \theta = 4c \cos^2 \theta = 2c \cos 2\theta + 2c$ and

$$4d^2 - 4b^2 = 16c^2 \cos^2 \theta - x^2 - 2c(2 \cos 2\theta + 3)x - (2 \cos 2\theta + 3)^2 c^2 \ ,$$

whence

$$\begin{aligned} 0 &= 4x^2 + (8 \cos 2\theta + 8)cx + 16c^2 \cos^2 \theta - x^2 - (4 \cos 2\theta + 6)cx - (4 \cos^2 2\theta + 12 \cos 2\theta + 9)c^2 \\ &= 3x^2 + (4 \cos 2\theta + 2)cx + [(8 \cos 2\theta + 8) - (4 \cos^2 2\theta + 12 \cos 2\theta + 9)]c^2 \\ &= 3x^2 + (4 \cos 2\theta + 2)cx - [4 \cos^2 2\theta + 4 \cos 2\theta + 1]c^2 \\ &= 3x^2 + (4 \cos 2\theta + 2)cx - (2 \cos 2\theta + 1)^2 c^2 \\ &= [3x - (2 \cos 2\theta + 1)c][x + (2 \cos 2\theta + 1)c] = [3x - y][x + y] = a(3x - y) \ . \end{aligned}$$

Hence $y = 3x$.

For the converse, let $y = 3x$, $\angle ADB = \theta$ and $\angle ABD = \beta$. By hypothesis, $|BC| = 4x = 2(b-c)$. By the Law of Cosines on triangle ABC, $b^2 = c^2 + 16x^2 - 8cx\cos\beta$, so that

$$\begin{aligned}\cos\beta &= \frac{16x^2 + c^2 - b^2}{8cx} = \frac{4(b-c)^2 + (c^2 - b^2)}{4c(b-c)} \\ &= \frac{4(b-c) - (c+b)}{4c} = \frac{3b-5c}{4c} \ .\end{aligned}$$

By Stewart's Theorem, $b^2(3x) + c^2(x) = 4x[d^2 + (3x)x]$, so that

$$\begin{aligned}d^2 &= \frac{3b^2 + c^2 - 12x^2}{4} = \frac{3b^2 + c^2 - 3(b-c)^2}{4} \\ &= \frac{6bc - 2c^2}{4} = \frac{(3b-c)c}{2} \ .\end{aligned}$$

From triangle ABD, we have that $c^2 = d^2 + 9x^2 - 6dx\cos\theta$, so that

$$\begin{aligned}\cos\theta &= \frac{9x^2 + d^2 - c^2}{6dx} = \frac{(3x-c)(3x+c) + d^2}{6dx} \\ &= \frac{(6x-2c)(6x+2c) + 4d^2}{24dx} = \frac{(3b-5c)(3b-c) + 2(3b-c)c}{12d(b-c)} \\ &= \frac{(3b-c)(3b-3c)}{12d(b-c)} = \frac{3b-c}{4d} \ .\end{aligned}$$

Therefore,

$$\begin{aligned}\cos 2\theta &= 2\cos^2\theta - 1 = \frac{2(3b-c)^2}{16d^2} - 1 \\ &= \frac{2(3b-c)^2 - 8(3b-c)c}{8(3b-c)c} = \frac{2(3b-c) - 8c}{8c} = \frac{3b-5c}{4c} = \cos\beta \ .\end{aligned}$$

Thus, either $2\theta = \beta$ or $2\theta = 2\pi - \beta$. But the latter case is excluded, since it would imply that β and θ are two angles of a triangle for which $\beta + \theta = 2\pi - \theta = \pi + \beta/2 > \pi$.

Solution 2. Case (i): Suppose that $\angle B$ is acute. Let $AH \perp BC$ and E lie on CH such that $AE = AB$.

$AC^2 - CH^2 = AB^2 - BH^2$ implies that

$$AC^2 - AB^2 = CH^2 - BH^2 = (CH - BH)(CH + BH) = (CH - HE)BC = CE \cdot BC = CE[2(AC - AB)] \ .$$

Hence $AC + AB = 2CE$. Also $AC - AB = \frac{1}{2}BC$. Therefore $2AB + \frac{1}{2}BC = 2CE$.

Suppose that $\angle ABD = 2\angle ADB$. Then $\angle AEB = 2\angle ADB \Rightarrow \Delta ADE$ is isosceles. Hence

$$AB = AE = DE \Rightarrow 2DE + \frac{1}{2}BC = 2CE \Rightarrow BC = 4(CE - DE) = 4CD \Rightarrow BD = 3CD \ .$$

Conversely, suppose that $BD = 3CD$. Then

$$BC = 4CD \Rightarrow \frac{1}{4}BC = CE - DE \ .$$

From the above,

$$\begin{gathered}AB = CE - \frac{1}{4}BC = DE \Rightarrow AE = DE \\ \Rightarrow \angle ABD = \angle AEB = 2\angle ADB \ .\end{gathered}$$

Case (ii): Suppose $\angle B = 90°$. Then

$$AC^2 - AB^2 = BC^2 = 2(AC - AB) \cdot BC \Rightarrow AC + AB = 2BC$$

$$\Rightarrow \frac{1}{2}BC + AB + AB = 2BC \Rightarrow AB = \frac{3}{4}BC \quad .$$

$$\angle ABD = 2\angle ADB \Rightarrow \angle ADB = 45° = \angle BAD \Rightarrow AB = BD$$

$$\Rightarrow BD = \frac{3}{4}BC \Rightarrow BD = 3CD \ .$$

$$BD = 3CD \Rightarrow BD = \frac{3}{4}BC = AB \Rightarrow \angle ADB = \angle BAD = 45° = \frac{1}{2}\angle ABD \quad .$$

Case (iii): Suppose $\angle B$ exceeds $90°$. Let $AH \perp BC$ and E be on CH produced such that $AE = AB$. Then

$$AC^2 - CH^2 = AB^2 - BH^2 \Rightarrow (AC - AB)(AC + AB) = CH^2 - BH^2 = (CH - BH)(CH + BH) = CB \cdot CE$$

$$\Rightarrow AC + AB = 2CE \ .$$

Also

$$AC - AB = \frac{1}{2}BC \Rightarrow 2AB + \frac{1}{2}BC = 2CE \Rightarrow AB + \frac{1}{4}BC = CE \quad .$$

Let $\angle ABD = 2\angle ADB$. Then

$$180° - \angle ABE = 2\angle ADB \Rightarrow \angle AEB + 2\angle ADE = \angle ABE + 2\angle ADB = 180° \ .$$

Also

$$\angle AEB + \angle EAD + \angle ADE = 180° \Rightarrow \angle EAD = \angle ADE \Rightarrow AE = ED \quad .$$

Hence

$$AB = ED \Rightarrow 2ED + \frac{1}{2}BC = 2CE \Rightarrow BC = 4(CE - DE) = 4CD \Rightarrow BD = 3CD \quad .$$

Conversely, suppose that $BD = 3CD$. Then $BC = 4CD$ and $ED = CE - CD = CE - \frac{1}{4}BC = AB$ so that $ED = AE$ and $\angle EAD = \angle ADE$. Therefore

$$\angle ABD = 180° - \angle AED = \angle EAD + \angle ADE = 2\angle ADE = 2\angle ADB \quad .$$

Solution 3. [R. Hoshino] Let $\angle ABD = 2\theta$. By the Law of Cosines, with the usual conventions for a, b, c,

$$\begin{aligned} 1 - 2\sin^2\theta = \cos 2\theta &= \frac{c^2 + 4(b-c)^2 - b^2}{4c(b-c)} \\ &= \frac{b-c}{c} - \frac{b+c}{4c} = \frac{3b-5c}{4c} \qquad (\text{since} \ \ b \neq c) \\ &\Rightarrow 3(b-c) = 6c - 8c\sin^2\theta \\ &\Rightarrow \frac{3(b-c)}{2}\sin\theta = c(3\sin\theta - 4\sin^3\theta) = c\sin 3\theta \\ &\Rightarrow \frac{\sin\theta}{c} = \frac{2\sin 3\theta}{3(b-c)} \quad . \ (*) \end{aligned}$$

Suppose now that D is selected so that $\angle ADB = \theta$. Then, by the Law of Sines,

$$\frac{\sin\theta}{c} = \frac{\sin(180° - 3\theta)}{x} = \frac{\sin 3\theta}{x}$$

where $x = |BD|$. Comparison with (*) yields $x = \frac{1}{2}(3(b-c))$ so $4BD = 3BC \Rightarrow BD = 3CD$ as desired.

On the other hand, suppose D is selected so that $BD = 3CD$. Then $BD = \frac{3}{2}(b-c)$. Let $\angle ADB = \phi$. Then

$$\frac{\sin\phi}{c} = \frac{\sin(180^\circ - \phi - 2\theta)}{\frac{3}{2}(b-c)} = \frac{\sin(\phi + 2\theta)}{\frac{3}{2}(b-c)} \ .$$

Hence

$$\begin{aligned}\frac{\sin(\phi+2\theta)}{\sin\phi} = \frac{\sin 3\theta}{\sin\theta} &\Rightarrow \sin\theta\sin(\phi+2\theta) = \sin 3\theta \sin\phi \\ &\Rightarrow \frac{1}{2}[\cos(\theta+\phi) - \cos(3\theta+\phi)] = \frac{1}{2}[\cos(3\theta-\phi) - \cos(3\theta+\phi)] \\ &\Rightarrow \cos(\theta+\phi) = \cos(3\theta-\phi) \\ &\Rightarrow \theta+\phi = \pm(3\theta-\phi) \quad \text{or} \quad \theta+\phi+3\theta-\phi = 360^\circ \ .\end{aligned}$$

The only viable possibility is $\theta + \phi = 3\theta - \phi \Rightarrow \theta = \phi$ as desired.

Solution 4. [J. Chui] First, recall *Stewart's Theorem.* Let XYZ be a triangle with sides x, y, z respectively opposite XYZ. Let W be a point on YZ so that $|XW| = u$, $|YW| = v$ and $|ZW| = w$. Then $x(u^2 + vw) = vy^2 + wz^2$. This is an immediate consequence of the Law of Cosines. Let $\theta = \angle YWX$. Then $z^2 = u^2 + v^2 - 2uv\cos\theta$ and $y^2 = u^2 + w^2 + 2uw\cos\theta$. Multiply these equations by u and v respectively, add and use $x = v + w$ to obtain the result.

Now to the problem. Suppose $BD = 3CD$. Let $|AC| = 2b$, $|AB| = 2c$, so that $|BC| = 4(b-c)$, $|BD| = 3(b-c)$ and $|CD| = b-c$. If $|AD| = d$, then an application of Stewart's Theorem yields $d^2 = 2c(3b-c)$. Applying the Law of Cosines to ΔABC and ΔABD respectively yields

$$\cos\angle ABC = \frac{3b-5c}{4c} \quad \text{and} \quad \cos\angle ADB = \frac{3b-c}{2\sqrt{2c(3b-c)}} \ .$$

Then $\cos 2\angle ADB = (3b-5c)/4c$. Hence, either $2\angle ADB = \angle ABC$ or $\angle ABC + 2\angle ADB = 360^\circ$. In the latter case, $\angle ABC + \angle ADB = 360^\circ - \angle ADB > 180^\circ$, which is false. Hence $\angle ABC = 2\angle ADB$.

On the other hand, let $2\angle ADB = \angle ABC$. If D' is a point on BC with $BD' = 3CD'$, the $2\angle AD'B = \angle ABC = 2\angle ADB$, so that $D = D'$. The result follows.

Solution 5. Let $|AB| = a$, $|AC| = a + 2$, $|BD| = 3$, $|CD| = 1$, $\angle ABD = 2\theta$, $\angle ADB = \phi$. Then $(a+2)^2 = a^2 + 16 - 8a\cos 2\theta$, whence $a = 3(1 + 2\cos 2\theta)^{-1}$ (so $0 < \theta < 60^\circ$). By the Law of Sines,

$$\frac{\sin(2\theta+\phi)}{3} = \frac{(1+2\cos 2\theta)\sin\phi}{3}$$

so that

$$\begin{aligned}0 &= \sin\phi + 2\sin\phi\cos 2\theta - \sin(2\theta+\phi) \\ &= \sin\phi + \sin\phi\cos 2\theta - \sin 2\theta\cos\phi \\ &= \sin\phi + \sin(\phi - 2\theta) = 2\sin(\phi-\theta)\cos\theta \quad .\end{aligned}$$

Since $0 \leq |\phi - \theta| < 180^\circ$, we find that $\phi = \theta$ as desired. The converse can be obtained as in the third solution.

Solution 6. [A. Birka] First, note that, when $BD = 3CD$, we must have $\angle ADB < 90^\circ$, since $AC > AB$ and D is on the same side of the altitude from A as C. Also, when $\angle ABD = 2\angle ADB$, $\angle ADB < 90^\circ$. Thus, we can assume that $\angle ADB$ is acute throughout.

We can select positive numbers u, v and w so that $|BC| = v + w$, $|AC| = u + w$ and $|AB| = u + v$. By hypothesis, $v + w = 2(w - v)$, so that $w = 3v$.

Suppose that $BD = 3CD$. Then $BC = 4CD$, whence $|CD| = v$. Hence $|BD| = 3v$. By the Law of Cosines,

$$(u + 3v)^2 = (u + v)^2 + (4v)^2 - 8v(u + v) \cos B$$

so that

$$\cos B = \frac{8v^2 - 4uv}{8v(u + v)} = \frac{2v - u}{2(u + v)} \ .$$

Hence

$$|AD|^2 = (u + v)^2 + (3v)^2 - 6v(u + v) \cos B = u^2 + 5uv + 4v^2 = (u + 4v)(u + v) \ .$$

Since $\sin^2 \angle ABD = 1 - \cos^2 B = [3u(u + 4v)]/[4(u + v)^2]$, and, by the Law of Sines,

$$\frac{\sin^2 \angle ADB}{\sin^2 \angle ABD} = \frac{u + v}{u + 4v} \ ,$$

we have that

$$\sin^2 \angle ADB = \frac{3u}{4(u + v)} \qquad \text{and} \qquad \cos^2 \angle ADB = \frac{u + 4v}{4(u + v)} \ .$$

Thus $\sin^2 \angle ABD = 4 \sin^2 \angle ADB \cos^2 \angle ADB$ so that either $\angle ABD = 2\angle ADB$ or $\angle ABD + 2\angle ADB = 180°$. The latter case would yield $\angle ADB = \angle BAD$, so that $AB = BD$. This would make ΔABC a $3 - 4 - 5$ right triangle and ΔABD an isosceles right triangle, whence $90° = \angle ABD = 2\angle ADB$. The converse can be shown as in the previous solutions. The result follows.

634. Solve the following system for real values of x and y:

$$2^{x^2+y} + 2^{x+y^2} = 8$$

$$\sqrt{x} + \sqrt{y} = 2 \ .$$

Preliminary comments. With the surds in the second equation, we must restrict ourselves to nonnegative values of x. Because of the complexity of the expressions, it is probably impossible to eliminate one of the variables and solve for the other. Let us make a few preliminary observations:

(i) $(x, y) = (1, 1)$ is an obvious solution;

(ii) Both equations are symmetric in x and y;

(iii) Taking $f(x, y) = 2^{x^2+y} + 2^{x+y^2}$ and $g(x, y) = \sqrt{x} + \sqrt{y}$, we have that $f(0, y) = 2^y + 2^{y^2}$ and $g(0, y) = \sqrt{y}$; thus, $f(0, y) = 8 \Rightarrow 1 < y < 2$ and $g(0, y) = 2 \Leftrightarrow y = 4$. The graphs of $f(x, y) = 8$ and $g(x, y) = 2$ should be sketched.

This suggests that $f(x, y) = 8 \Rightarrow x + y \le 2$ and $g(x, y) = 2 \Rightarrow x + y \ge 2$ with equality for both $\Leftrightarrow (x, y) = (1, 1)$. Hence we look for a relationship among $f(x, y)$, $g(x, y)$ and $x + y$.

Solution 1.

$$(\sqrt{x} + \sqrt{y})^2 = x + 2\sqrt{xy} + y \le x + (x + y) + y = 2(x + y)$$

by the Arithmetic-Geometric Means Inequality. Hence

$$\sqrt{x} + \sqrt{y} \le \sqrt{2(x + y)} \ .$$

Also, by the same AGM inequality,

$$2^{x^2+y} + 2^{x+y^2} \ge 2\sqrt{2^{x^2+y+x+y^2}} \ .$$

Now, using the inequality again, we find that

$$x^2 + y + x + y^2 = (x^2 + y^2) + (x + y) \geq \frac{1}{2}(x + y)^2 + (x + y)$$

so that

$$2^{x^2+y} + 2^{x+y^2} \geq 2^{1+\frac{1}{4}(x+y)^2+\frac{1}{2}(x+y)} = 2^{\frac{1}{4}[(x+y+1)^2+3]} \ .$$

Suppose the (x, y) satisfies the system. Then

$$\sqrt{2(x+y)} \geq 2 \Rightarrow (x + y) \geq 2$$

and

$$\frac{1}{4}[(x + y + 1)^2 + 3] \leq 3 \Rightarrow (x + y + 1)^2 \leq 9 \Rightarrow x + y + 1 \leq 3 \Rightarrow x + y \leq 2 \ .$$

Hence $x + y = 2$ and all inequalities are equalities. Therefore $x = y = 1$.

Solution 2. [A. Rodriguez] Wolog, we may assume that $x \geq 1$. Let $\sqrt{x} + \sqrt{y} = 2$; then $y = (2 - \sqrt{x})^2$. Define

$$\begin{aligned} g(x) &= x + y^2 + y + x^2 = (2 - \sqrt{x})^4 + x^2 + x + (2 - \sqrt{x})^2 \\ &= 2x^2 - 8x^{\frac{3}{2}} + 26x - 36x^{\frac{1}{2}} + 20 \ . \end{aligned}$$

Then

$$\begin{aligned} g'(x) &= 4x - 12x^{\frac{1}{2}} + 26 - 18x^{-\frac{1}{2}} = 2x^{-\frac{1}{2}}(2x^{\frac{3}{2}} - 6x + 13x^{\frac{1}{2}} - 9) \\ &= 2x^{-\frac{1}{2}}(x^{\frac{1}{2}} - 1)(2x - 4x^{\frac{1}{2}} + 9) = 2x^{-\frac{1}{2}}(x^{\frac{1}{2}} - 1)[2(x^{\frac{1}{2}} - 1)^2 + 7] > 0 \end{aligned}$$

for $x > 1$. Hence $g(x)$ is strictly increasing for $x > 1$, so that $g(x) \geq g(1) = 4$ for $x \geq 1$ with equality if and only if $x = 1$. Thus, if the first equation holds, then

$$8 = 2^{x^2+y} + 2^{x+y^2} \geq 2\sqrt{2^{g(x)}} \Rightarrow 16 \geq 2^{g(x)} \Rightarrow g(x) \leq 4 \ .$$

Hence $g(x) = 4$, so that $x = 1$ and $y = 1$. Thus, $(x, y) = (1, 1)$ is the only solution.

Solution 3. [S. Yazdani] Set $\sqrt{x} = 1 + u$ and $\sqrt{y} = 1 - u$. Then $x^2 + y = (1 + u)^4 + (1 - u)^2$ and $x + y^2 = (1 - u)^4 + (1 + u)^2$, so

$$8 = 2^{x^2+y} + 2^{x+y^2} = 2^{u^4+7u^2+2}\left(2^{4u^3+2u} + \frac{1}{2^{4u^3+2u}}\right) \geq 2^2(2) = 8$$

with equality if and only if $u = 0$. Since the extremes of this inequality are equal, we must have $u = 0$, so $x = y = 1$.

Solution 4. [C. Hsia] With $\sqrt{x} = 1 + u$ and $\sqrt{y} = 1 - u$, we can write the first equation as

$$2^{4u^3+2u} + \frac{1}{2^{4u^3+2u}} = 2^{1-7u^2-u^4} \ .$$

Let $z = 2^{4u^3+2u}$. We note that the quadratic $z^2 - 2^{1-7u^2-u^4}z + 1 = 0$ is solvable, and so has nonnegative discriminant. Hence

$$2^{2-14u^2-2u^4} \geq 4 = 2^2 \Rightarrow -14u^2 - 2u^4 \geq 0 \Rightarrow u = 0 \ .$$

Hence $x = y = 1$.

Solution 5. [M. Boase] $2(x+y) \geq (x+y)+2\sqrt{xy} = (\sqrt{x}+\sqrt{y})^2 = 4$ so that $x+y \geq 2$. Let $f(t) = t(t+1)$. For positive values of t, $f(t)$ is an increasing strictly convex function of t. Hence

$$f(x) + f(y) \geq 2f(\frac{1}{2}(x + y)) \geq 2f(1) = 4$$

so that $x^2 + x + y^2 + y \geq 4$. Equality occurs if and only if $x = y = 1$. Applying the Arithmetic-Geometric Means Inequality, we find that

$$4 = \frac{1}{2}(2^{x^2+y} + 2^{x+y^2}) \geq 2^{\frac{1}{2}(x^2+y^2+x+y)}$$

so that $x^2 + x + y^2 + y \leq 4$. Hence $x^2 + x + y^2 + y = 4$ and so $x = y = 1$.

Comment. Note that $2(x^2 + y^2) \leq (x + y)^2$ with equality if and only if $x = y$. Hence

$$x^2 + y^2 + x + y \geq \frac{1}{2}(x + y)^2 + (x + y) \geq 4$$

with equality if and only if $x = y = 1$. This avoids the use of the convexity of the function f.

Solution 6. [J. Chui] Wolog, let $x \geq y$ so that $\sqrt{x} \geq 1 \geq \sqrt{y}$. Suppose that $\sqrt{x} = 1 + u$ and $\sqrt{y} = 1 - u$. Then $x + y = 2 + 2u^2 \geq 2$ and $xy = (1 - u^2)^2 \leq 1$. Thus

$$\begin{aligned} 8 &= 2^{x^2+y} + 2^{x+y^2} \geq 2\sqrt{2^{x^2+y+x+y^2}} \\ &= 2\sqrt{2^{(x+y)(x+y+1)-2xy}} \geq 2\sqrt{2^{2\cdot 3-2\cdot 1}} = 2^3 = 8 \end{aligned}$$

with equality if and only if $x = y$.

Solution 7. [C. Deng] By the Root-Mean-Square, Arithmetic Mean Inequality, we have that

$$\frac{x^2 + y^2}{2} \geq \left(\frac{x + y}{2}\right)^2 \geq \left(\frac{\sqrt{x} + \sqrt{y}}{2}\right)^4 = 1 \ ,$$

with equality if and only if $x = y = 1$. By the Arithmetic-Geometric Means Inequality, we have

$$\begin{aligned} 4 &= \frac{2^{x^2+y} + 2^{x+y^2}}{2} \geq \sqrt{2^{x^2+y^2+x+y}} \\ &\geq \sqrt{2^{2+2}} = 4 \ . \end{aligned}$$

Since equality must hold throughtout, $x = y$, and thus the only solution to the system is $(x, y) = (1, 1)$.

635. Two unequal spheres in contact have a common tangent cone. The three surfaces divide space into various parts, only one of which is bounded by all three surfaces; it is "ring-shaped". Being given the radii r and R of the spheres with $r < R$, find the volume of the "ring-shaped" region in terms of r and R.

Solution. Let P and Q be the centres of the spheres of respective radii r and R, and let O be the apex of the cone. Consider a vertical slice of the configuration through its axis of rotation. Let A and B be points in the slice that are the tangent points of the smaller and larger spheres, respectively, with the tangent cone. Let u and V be the centres of the circles through A and B, respectively, that are perpendicular ot the axis of rotation.

From a consideration of similar triangles and pythagoras theorem, we find that

$$\begin{aligned} |OP| &= r\left(\tfrac{R+r}{R-r}\right) & |OU| &= \tfrac{4Rr^2}{R^2-r^2} \\ |UP| &= r\left(\tfrac{R-r}{R+r}\right) & |AU| &= \tfrac{2r}{R+r}\sqrt{Rr} \\ |OQ| &= R\left(\tfrac{R+r}{R-r}\right) & |OV| &= \tfrac{4R^2r}{R^2-r^2} \\ |VQ| &= R\left(\tfrac{R-r}{R+r}\right) & |BV| &= \tfrac{2R}{R+r}\sqrt{Rr} \end{aligned}$$

The volume of the cone obtained by rotating OBV is

$$\frac{1}{3}\pi|BV|^2|OV| = \frac{16\pi R^5 r^2}{3(R+r)^3(R-r)}$$

and the volume of the cone obtained by rotating OAU is

$$\frac{16\pi R^2 r^5}{3(R+r)^3(R-r)}$$

so that the volume of the frustum obtained by rotating $AUVB$ is

$$\frac{16\pi R^2 r^2(R^3-r^3)}{3(R+r)^3(R-r)} = \frac{16\pi R^2 r^2}{3(R+r)^3}(R^2+Rr+r^2) \quad .$$

The volume of a slice of a sphere of radius a and height h from the equatorial plane is

$$\pi\int_0^h (a^2-t^2)dt = \pi[a^2h - h^3/3] \ .$$

The portion of the larger sphere included within the frustum has volume

$$\begin{aligned}
&\frac{2\pi R^3}{3} - \pi\left[R^3\left(\frac{R-r}{R+r}\right) - \frac{R^3}{3}\left(\frac{R-r}{R+r}\right)^3\right] \\
&\quad = \frac{\pi R^3}{3}\left[2 - 3\left(\frac{R-r}{R+r}\right) + \left(\frac{R-r}{R+r}\right)^3\right] \\
&\quad = \frac{\pi R^3}{3(R+r)^3}[4r^3+12Rr^2] = \frac{4\pi R^2 r^2}{3(R+r)^3}[Rr+3R^2]
\end{aligned}$$

and the portion of the smaller sphere included within the frustum has volume

$$\frac{2\pi r^3}{3} + \pi\left[r^3\left(\frac{R-r}{R+r}\right) - \frac{r^3}{3}\left(\frac{R-r}{R+r}\right)^3\right] = \frac{4\pi R^2 r^2}{3(R+r)^3}[Rr+3r^2] \quad .$$

Hence, the portions of the sphere lying within the frustum have total volume

$$\frac{4\pi R^2 r^2}{3(R+r)^3}[3R^2+2Rr+3r^2] \ .$$

Subtracting this from the volume of the frustum yields the volume of the ring-shaped region

$$\frac{4\pi R^2 r^2}{3(R+r)^3}[(4R^2+4Rr+4r^2)-(3R^2+2Rr+3r^2)] = \frac{4\pi R^2 r^2}{3(R+r)^3}[R^2+2Rr+r^2] = \frac{4\pi R^2 r^2}{3(R+r)} \quad .$$

Comment. The volume of a slice of a sphere of radius a and height h from the equatorial plane can be obtained from the volume of a right circular cone and a cylinder using the method of Cavalieri. The area of a cross-section of the slice at height t from the equator is $\pi(a^2-t^2) = \pi a^2 - \pi t^2$. The term πa^2 represents the cross-section of a cylinder of radius a and height h while πt^2 represents the area of the cross section of a cone of base radius h at distance t from the vertex. Thus the area of the each cross-section of the cylinder is the sum of the areas of the corresponding cross-sections of the spherical slice and cone. Cavalieri's principle says that the volumes of the solids bear the same relation. Thus the volume of the spherical slice is

$$\pi a^2 h - \frac{1}{3}\pi h^3 \quad .$$

636. Let ABC be a triangle. Select points D, E, F outside of ΔABC such that ΔDBC, ΔEAC, ΔFAB are all isosceles with the equal sides meeting at these outside points and with $\angle D = \angle E = \angle F$. Prove that the lines AD, BE and CF all intersect in a common point.

Solution. Let AD and BC intersect at P, $a_1 = |CP|$, $a_2 = |BP|$, $\alpha_1 = \angle CDP$, $\alpha_2 = \angle BDP$. Let BE and AC intersect at Q, $b_1 = |AQ|$, $b_2 = |CQ|$, $\beta_1 = \angle AEQ$, $\beta_2 = \angle CEQ$. Let CF and AB intersect at R, $c_1 = |BR|$, $c_2 = |AR|$, $\gamma_1 = \angle BFR$, $\gamma_2 = \angle AFR$.

Applying the Law of Sines to ΔBPD and ΔCPD, we find that

$$\frac{a_1}{\sin\alpha_1} = \frac{a_2}{\sin\alpha_2}$$

and similarly that

$$\frac{b_1}{\sin\beta_1} = \frac{b_2}{\sin\beta_2} \quad \text{and} \quad \frac{c_1}{\sin\gamma_1} = \frac{c_2}{\sin\gamma_2} \quad .$$

Let $\alpha = \angle BAE$. Then $\alpha = \angle FAC$ since $\angle FAB = \angle EAC$. Similarly, let $\beta = \angle FBC = \angle ABD$ and $\gamma = \angle BCE = \angle ACD$.

Let $|AB| = c$, $|BC| = a$, $|AC| = b$, $|AD| = u$, $|BE| = v$, $|CF| = w$. By the Law of Sines, we find that

$$\frac{v}{\sin\alpha} = \frac{c}{\sin\beta_1} \quad \text{and} \quad \frac{v}{\sin\gamma} = \frac{a}{\sin\beta_2}$$

so that

$$\frac{c\sin\alpha}{\sin\beta_1} = \frac{a\sin\gamma}{\sin\beta_2} \Longrightarrow \frac{\sin\beta_1}{\sin\beta_2} = \frac{c}{a}\cdot\frac{\sin\alpha}{\sin\gamma} \quad .$$

Similarly

$$\frac{\sin\alpha_1}{\sin\alpha_2} = \frac{b}{c}\cdot\frac{\sin\gamma}{\sin\beta} \quad \text{and} \quad \frac{\sin\gamma_1}{\sin\gamma_2} = \frac{a}{b}\cdot\frac{\sin\beta}{\sin\alpha} \quad .$$

Putting this altogether yields

$$\frac{a_1}{a_2}\cdot\frac{b_1}{b_2}\cdot\frac{c_1}{c_2} = \frac{\sin\alpha_1}{\sin\alpha_2}\cdot\frac{\sin\beta_1}{\sin\beta_2}\cdot\frac{\sin\gamma_1}{\sin\gamma_2} = \frac{b}{c}\cdot\frac{c}{a}\cdot\frac{a}{b}\cdot\frac{\sin\gamma}{\sin\beta}\cdot\frac{\sin\alpha}{\sin\gamma}\cdot\frac{\sin\beta}{\sin\alpha} = 1 \ .$$

By the converse of Ceva's Theorem, the cevians AP, BQ and CR are concurrent and the result follows.

637. Let n be a positive integer. Determine how many real numbers x with $1 \le x < n$ satisfy

$$x^3 - \lfloor x^3 \rfloor = (x - \lfloor x \rfloor)^3 \ .$$

Solution 1. Let $n - 1 \le x < n$. Then $\lfloor x^3 \rfloor = (n-1)^3 + r$ for $0 \le r < 3n(n-1)$. The equation is equivalent to

$$\lfloor x^3 \rfloor = \lfloor x \rfloor^3 + 3x\lfloor x \rfloor(x - \lfloor x \rfloor) = (n-1)^3 + 3x(n-1)(x - n + 1) \ .$$

The increasing function $(n-1)^3 + 3x(n-1)(x-n+1)$ takes the value 0 when $x = n-1$ and $3n(n-1)$ when $x = n$. Therefore, on the interval $[n-1, n)$, it assumes each of the values $0, 1, \cdots, 3n(n-1) - 1$ exactly once.

For $0 \le r < 3n(n-1)$, consider the equation

$$r = 3x(n-1)(x - n + 1) \ .$$

This is equivalent to

$$\begin{aligned}(n-1)^3 + r &= (n-1)^3 - 3x(n-1)^2 + 3x^2(n-1) \\ &= [(n-1) - x]^3 + x^3 \ ,\end{aligned}$$

When x is a solution of this equation for which $n-1 \leq x < n$, we have that $x^3 \leq (n-1)^3 + r$ and

$$x^3 = (n-1)^3 + r + [x-(n-1)]^3 < (n-1)^3 + r + 1 \ ,$$

so that $\lfloor x^3 \rfloor = (n-1)^3 + r$¿ It follows that for each value of these values of r, the given equation is satisfied and so there are $3n(n-1)$ solutions x for which $n-1 \leq x < n$.

Therefore, the total number of solutions not exceeding n is

$$\sum_{k=2}^{n} 3k(k-1) = \sum_{k=2}^{n} k^3 - (k-1)^3 - 1 = n^3 - 1 - (n-1) = n^3 - n \ .$$

Solution 2. Consider the behaviour of the two sides of the equation on the half-open interval defined by $k \leq x < k+1$ for k a nonnegative integer. The function on the right increases continuously from 0 with right limit equal to 1. The function on the left increases continuously in the same way on each half-open interval defined by $\sqrt[3]{i} \leq x < \sqrt[3]{i+1}$ for $k^3 \leq i \leq (k+1)^3 - 1 = k^3 + 3k(k+1)$. By examining the graphs, we see that they take equal values exactly once in each of the smaller intervals except the rightmost. Thus, they are equal $(k+1)^3 - k^3 - 1$ times. Therefore, over the whole of the interval defined by $1 \leq x < n^3$, they are equal exactly

$$\sum_{k=1}^{n-1} [(k+1)^3 - k^3 - 1] = n^3 - 1^3 - (n-1) = n^3 - n$$

times, so that the given equation has this many solutions.

Solution 3. Let $x = k + r$, where k is a nonnegative integer and $0 \leq r < 1$. Then

$$x^3 - \lfloor x^3 \rfloor = (k+r)^3 - (k^3 + \lfloor 3kr(k+r) + r^3 \rfloor)$$

so that the equation becomes

$$3kr(k+r) = \lfloor 3kr(k+r) + r^3 \rfloor \ .$$

This is equivalent to the assertion that $3kr(k+r)$ is an integer, so there is a solution to the equation for every x for which $3kr(k+r)$ is an integer, where $0 \leq k \leq n-1$ and $0 \leq r < 1$.

Fix k. As r increases from 0 towards but not equal to 1, $3kr(k+r)$ increases from 0 up to but not including $3k(k+1)$, so it assumes exactly $3k(k+1)$ integer values. Hence the total number of solutions is

$$\sum_{k=0}^{n-1} 3k(k+1) = n^3 - n \ .$$

638. Let x and y be real numbers. Prove that

$$\max(0, -x) + \max(1, x, y) = \max(0, x - \max(1, y)) + \max(1, y, 1-x, y-x)$$

where $\max(a, b)$ is the larger of the two numbers a and b.

Solution 1. [C. Deng] First, note that for real a, b, c, d,

$$\max(a, b) - c = \max(a-c, b-c) \ ;$$

$$\max(\max(a, b), c) = \max(a, b, c) \ ;$$

$$\max(a, b) + \max(c, d) = \max(a+c, a+d, b+c, b+d) \ .$$

[Establish these equations.] Then

$$\begin{aligned}\max(0,-x) &= \max(0,-x) + \max(1,y) - \max(1,y)\\ &= \max(1,y,1-x,y-x) - \max(1,y)\ ;\end{aligned}$$

and

$$\begin{aligned}\max(1,x,y) &= \max(1,x,y) - \max(1,y) + \max(1,y)\\ &= \max(\max(1,y),x) - \max(1,y) + \max(1,y)\\ &= \max(\max(1,y) - \max(1,y), x - \max(1,y)) + \max(1,y)\\ &= \max(0, x - \max(1,y)) + \max(1,y)\ .\end{aligned}$$

Adding these equations yields the desired result.

Solution 2. If $0 \leq x \leq 1$, then $-x \leq 0$, $x - \max(1,y) \leq x - 1 \leq 0$, $1 - x \leq 1$, $y - x \leq y$, so that both sides are equal to $\max(1,y)$. If $x \leq 0$, then $\max(0,-x) = -x$, $\max(1,x,y) = \max(1,y)$, $\max(0, x - \max(1,y)) = 0$ and $1 - x \geq 1$, $y - x \geq y$, so that

$$\max(1,y,1-x,y-x) = \max(1-x,y-x) = \max(1,y) - x$$

which is the same as the left side.

Suppose that $x \geq 1$. Then the left side is equal to $0 + \max(x,y) = \max(x,y)$. When $y \leq 1$, the right side becomes $(x-1)+1 = x = \max(x,y)$. When $1 \leq y \leq x$, the right side becomes $x - y + y = x = \max(x,y)$. When $x \leq y$, the right side is $0 + y = \max(x,y)$. Thus, the result holds in all cases.

639. (a) Let $ABCDE$ be a convex pentagon such that $AB = BC$ and $\angle BCD = \angle EAB = 90°$. Let X be a point inside the pentagon such that AX is perpendicular to BE and CX is perpendicular to BD. Show that BX is perpendicular to DE.

(b) Let N be a regular nonagon, *i.e.*, a regular polygon with nine edges, having O as the centre of its circumcircle, and let PQ and QR be adjacent edges of N. The midpoint of PQ is A and the midpoint of the radius perpendicular to QR is B. Determine the angle between AO and AB.

(a) *Solution 1.* Let AX intersect BE in Y, CE intersect BD in Z and BX intersect DE in P. Assume X lies inside the triangle BDE; a similar proof holds when X lies outside the triangle BDE. From similar right triangles and since $AB = AC$, we have that

$$BY \cdot BE = AB^2 = AC^2 = BZ \cdot BD\ .$$

Hence triangles BYZ and BDE are similar and $\angle BYZ = \angle BDE$ and $\angle BZY = \angle BED$. Thus the quadrilateral $DEYZ$ is concyclic.

The quadrilateral $BYXZ$ is also concyclic, so that $\angle BZY = \angle BXY$. Therefore $\angle BED = \angle BXY$, with the result that triangles BXY and BEP are similar. Hence $\angle EPB = \angle XYB = 90°$.

Solution 2. [K. Zhou, J. Lei] Let T be selected on DE so that $BT \perp ED$. Let AY meet BT at S and CZ meet BT at R. Because triangles BSY and BET are similar, $BY : BR = BT : BE$, so that $BR \cdot BT = BY \cdot BE = AB^2$. Similarly, $BS \cdot BT = BZ \cdot BD = AC^2 = AB^2$. Hence $BR = BS$ so that $R = S$. So R and S must be the point X where AY and CZ meet and so T is none other than P. The result follows.

(b) *Answer*: $\angle OAB = 30°$.

Solution 1. [S. Sun] Let C be the point on OR for $BC \perp OR$. Since $\angle BOC = \angle QOA = 20°$, the right triangles BOC and QOA are similar, Since $QO = 2OB$, it follows that $AO = 2OC$.

Consider the triangle AOC. We have $AO = 2OC$ and $\angle AOC = 60°$. By splitting an equilateral triangle along a median, it is possible to construct a triangle UVW for which $AO = UV = 2VW$ and $\angle UVW = 60°$. Since also $VW = OC$, triangles AOC and UVW are congruent (SAS), so that $\angle OCA = \angle VWU = 90°$. Therefore, A, B, C are collinear, and $\angle OAB = \angle OAC = \angle UWV = 30°$.

Solution 2. Let C be the intersection of the radius perpendicular to QR and the circumcircle of N. We have that $\angle POQ = \angle QOR = 40°$. Thus, triangle OPC is equilateral, so that PB and OC are perpendicular. Since also $\angle OAP = 90°$, A and B lie on the circle with diameter OP, Hence $\angle OAB = \angle OPB = 30°$.

Solution 3. [D. Brox] $OA = r \sin 70°$ and $OD = \frac{r}{2} \cos 40°$, where r is the circumradius of the nonagon and D is the foot of the perpendicular from B to OA. Hence

$$AD = r(\sin 70° - \sin 30° \cos 40°) = r \sin 40° \cos 30° \ .$$

Therefore

$$\tan \angle OAB = \frac{BD}{AD} = \frac{OD \tan 40°}{AD} = \frac{\cos 40° \tan 40°}{2 \sin 40° \cos 30°} = \frac{1}{2 \cos 30°} = \frac{1}{\sqrt{3}} \ ,$$

whence $\angle OAB = 30°$.

Solution 4. [H. Dong] Let E be the midpoint of OP so that triangle OEB is equilaterial.

$$EB = EP \Longrightarrow \angle EPB = \angle EBP = 30° \Longrightarrow \angle OBP = 30° \ .$$

Hence $OBAP$ is concyclic, so that $\angle OAB = \angle OPB = 30°$.

Solution 5. [D. Arthur] $OB = \frac{1}{2}OP = OP \cos 60° = OP \cos \angle PQB$ so that $PB \perp OC$. Thus $OPAB$ is concyclic. Since $\angle OBA = 180° - \angle OPA = 180° - 70° = 110°$, then

$$\angle OAB = 180° - (\angle AOB + \angle OBA) = 180° - (40° + 110°) = 30° \ .$$

Solution 6. [F. Espinosa] $|\overrightarrow{OB}| = \frac{r}{2}$ and $|\overrightarrow{OA}| = r \cos 20°$. Then $\overrightarrow{OR} \cdot \overrightarrow{OB} = \frac{1}{2} r^2 \cos 20°$ and $\overrightarrow{OR} \cdot \overrightarrow{OA} = r(r \cos 20°) \cos 60° = \frac{1}{2} r^2 \cos 20°$. Hence $\overrightarrow{OR} \cdot \overrightarrow{AB} = overrightarrowOR \cdot \overrightarrow{OB} - overrightarrowOR \cdot \overrightarrow{OA} = 0$ with the result that $\angle ABO = 90°$. As before, it follows that $\angle OAB = 30°$.

Solution 7. [T. Costin] Let F be the midpoint of the side ST of the nonagon $PQRST \cdots$. Then $\angle AOF = 120°$, so $\angle OAG = 30°$ and $\angle OGA = 90°$, where G is the intersection point of AF and OR. Hence $OG = \frac{1}{2}OA$.

Let H be the intersection of AP and OC, with C the midpoint of RS. Then $OG = OH \cos 20°$. Also $OA = OQ \cos 20° = OR \cos 20°$. Hence

$$OH = \frac{OG}{\cos 20°} = \frac{OA}{2 \cos 20°} = \frac{OR}{2}$$

so that $H = B$. Hence $\angle OAB = \angle OAH = 30°$.

640. Suppose that $n \geq 2$ and that, for $1 \leq i \leq n$, we have that $x_i \geq -2$ and all the x_i are nonzero with the same sign. Prove that

$$(1 + x_1)(1 + x_2) \cdots (1 + x_n) > 1 + x_1 + x_2 + \cdots + x_n \ \ ,$$

Solution 1. When $n = 2$, we have that

$$(1 + x_1)(1 + x_2) = 1 + x_1 + x_2 + x_1 x_2 > 1 + x_1 + x_2$$

since x_1 and x_2 are nonzero with the same sign. Suppose, as an induction hypothesis, that the result holds for $n = k \geq 2$. Then

$$\begin{aligned}(1+x_1)(x+x_2)\cdots(1+x_k)(1+x_{k+1}) - (1+x_1+x_2+\cdots+x_k+x_{k+1})\\ = [(1+x_1)(1+x_2)\cdots(1+x_k) - (1+x_1+x_2+\cdots+x_k)]\\ + x_{k+1}[(1+x_1)(1+x_2)\cdots(1+x_k) - 1]\\ > x_{k+1}[(1+x_1)(1+x_2)\cdots(1+x_k) - 1] \equiv A\ .\end{aligned}$$

If $x_i > 0$ $(1 \leq i \leq k+1)$, then $1 + x_i > 1$ $(1 \leq i \leq k)$ and $A > 0$.

Let $-2 \leq x_i < 0$. Then, for $1 \leq i \leq k$,

$$-1 \leq 1 + x_i < 1 \Longrightarrow -1 \leq (1+x_1)(1+x_2)\cdots(1+x_k) \leq 1$$

$$\Longrightarrow (1+x_1)(1+x_2)\cdots(1+x_k) - 1 \leq 0\ .$$

Since also $x_{k+1} < 0$, $A \geq 0$. Hence

$$(1+x_1)(1+x_2)\cdots(1+x_{k+1}) > 1 + x_1 + x_2 + \cdots + x_{k+1}\ .$$

The result follows by induction.

Solution 2. The case $n = 2$ is proved as in the first solution. Suppose that all x_i are negative and at least two, say x_1 and x_2 lie in $[-2, -1]$. Then

$$(1+x_1)(1+x_2)\cdots(1+x_n) \geq -1 \geq 1 - 1 - 1 \geq 1 + x_1 + x_2 + \cdots + x_n$$

since $-2 \leq x_i < 0$ and $|1 + x_i| \leq 1$ for $1 \leq i \leq n$.

Henceforth assume that either (i) all x_i are positive $(1 \leq i \leq n)$ or (ii) all x_i are negative with $-2 \leq x_1 < 0$ and $-1 < x_i < 0$ for $2 \leq i \leq n$. As an induction hypothesis, assume that the result holds for $n = k \geq 2$. Then $1 + x_{k+1} > 0$, so that

$$\begin{aligned}&(1+x_1)(1+x_2)\cdots(1+x_k)(1+x_{k+1})\\ &\quad > (1+x_1+x_2+\cdots+x_k)(1+x_{k+1}) \quad \text{by the induction hypothesis}\\ &\quad > 1 + x_1 + x_2 + \cdots + x_k + x_{k+1} \quad \text{by the } n = 2 \text{ case.}\end{aligned}$$

The result follows by induction.

641. Observe that $x^2 + 5x + 6 = (x+2)(x+3)$ while $x^2 + 5x - 6 = (x+6)(x-1)$. Determine infinitely many coprime pairs (m, n) of positive integers for which both $x^2 + mx + n$ and $x^2 + mx - n$ can be factored as a product of linear polynomials with integer coefficients.

Solution 1. For the factorizations to occur, both discriminants must be squares: $m^2 - 4n = u^2$, $m^2 + 4n = v^2$ for some integers u and v. Suppose m^2 can be expressed as the sum of two squares: $m^2 = p^2 + q^2$. Then $2m^2 = (p+q)^2 + (p-q)^2$. Write $u = p - q$, $v = p + q$. Then $8n = v^2 - u^2 = 4pq$ so that $n = \frac{1}{2}pq$.

Now we construct our examples. Let r and s be a coprime pair of integers with opposite parity. Define $p = r^2 - s^2$, $q = 2rs$, $m = r^2 + s^2$ and $n = rs(r^2 - s^2)$. Then any prime power divisor of m and n must divide both $r^2 + s^2$ and one of r, s and $r^2 - s^2$, and hence both $2r^2$ and $2s^2$. Hence it must divide 2. But we have arranged for m to be odd. Hence m and n are coprime. Observe that

$$x^2 + (r^2 + s^2)x + rs(r^2 - s^2) = (x + s(r+s))(x + r(r-s))$$

and

$$x^2 + (r^2 + s^2)x - rs(r^2 - s^2) = (x + r(r+s))(x - s(r-s))\ .$$

Solution 2. With the notation of the first solution, we have that $m^2 - u^2 = v^2 - m^2$, whence $v^2 - 2m^2 = -u^2$. Let us take $u = 1$. We show that $v^2 - 2m^2 = -1$ has infinitely many solutions with m odd. Let $(v, m) = (v_k, m_k)$ where

$$(v_1, m_1) = (1, 1) \quad \text{and} \quad v_{k+1} + m_{k+1}\sqrt{2} = (3 + 2\sqrt{2})(v_k + m_k\sqrt{2})$$

so

$$v_{k+1} = 3v_k + 4m_k \qquad m_{k+1} = 2v_k + 3m_k$$

for $k \geq 1$. By induction, it is proved that, for each k, $v_k^2 - 2m_k^2 = -1$. Let $(m, n) = (m_k, \frac{1}{4}(m_k^2 - 1))$. Then it is readily shown that (m, n) are both integers and satisfy the condition of the problem. For example, we have

$$(u, v; m, n) = (1, 1; 1, 0), (1, 7; 5, 6), (1, 41; 29, 210), \cdots$$

so that, for example, $x^2 + 29x + 210 = (x + 14)(x + 15)$ and $x^2 + 29x - 210 = (x + 35)(x - 6)$.

Solution 3. [J. Rickards, M. Boase] Let a be even and let $(m, n) = (a^2 + 1, a^3 - a)$. Then the greatest common divisor of m and a is 1, as is the greatest common divisor of m and $a^2 - 1$. Then

$$x^2 + (a^2 + 1)x + (a^3 - a) = (x + \overline{a^2 - a})(x + \overline{a + 1})$$

and

$$x^2 + (a^2 + 1)x - (a^3 - a) = (x + \overline{a^2 + a})(x - \overline{a - 1}) \ .$$

Comment. This is a special case of Solution 1.

Solution 4. [S. Hemmati] Let k be an integer and let

$$m = 4k^2 + 1 = (2k + 1)^2 - 4k = (2k - 1)^2 + 4k$$

and $n = 2k(2k + 1)(2k - 1)$. The three factors of n are pairwise coprime, and it follows that the greatest common divisor of m and n is 1. We have that

$$x^2 + mx - n = [x + 2k(2k + 1)][x - (2k - 1)]$$

and

$$x^2 + mx + n = [x + 2k(2k - 1)][x + (2k + 1)] \ .$$

Solution 5. [C. Deng] Suppose that $x^2 + mx - n$ has roots a and $-b$ and that $x^2 + mx + n$ has roots r and s. Then $n = ab = rs$ and $m = b - a = -r - s$. In terms of a and b, the values of r and s are given by

$$\frac{a - b \pm \sqrt{a^2 - 6ab + b^2}}{2} \ .$$

Since $a - b$ and $a^2 - 6ab + b^2$ have the same parity, these are integers if and only if $a^2 - 6ab + b^2$ is a square. Any values of a and b which make this quantity square will yield acceptable values of m and n.

Let $(a_1, b_1) = (1, 6)$, and for $k \geq 1$,

$$a_{k+1} = 6a_k - b_k \ , \qquad b_{k+1} = a_k \ .$$

Then

$$a_{k+1}^2 - 6a_{k+1}b_{k+1} + b_{k+1}^2 = (6a_k - b_k)^2 - 6(6a_k - b_k)a_k + a_k^2 = a_k^2 - 6a_kb_k + b_k^2 \ ,$$

so that, by induction, we see that this quantity is equal to 1 for all $k \geq 1$. Thus

$$(r_k, s_k) = \left(\frac{a_k - b_k - 1)}{2}, \frac{a_k - b_k + 1}{2}\right) \ .$$

Observe that the greatest common divisor of a_{k+1} and b_{k+1} is equal to that of a_k and b_k, and so, by induction, equal to 1, the greatest common divisor of 1 and 6. It follows, for all k, that a_kb_k and $a_k - b_k$ are relativeoly prime. Thus, the pair

$$(x^2 + (b_k - a_k)x - a_kb_k, x^2 + (b_k - a_k)x + a_kb_k)$$

satisfy the desired conditions for $k \geq 1$.

In particular, we find that

$$x^2 + 5x - 6 = (x + 6)(x - 1) \ , \quad x^2 + 5x + 6 = (x + 2)(x + 3) \ ;$$

$$x^2 + 29x - 210 = (x + 35)(x - 6) \ , \quad x^2 + 29x + 210 = (x + 14)(x + 15) \ ;$$

$$x^2 + 169x - 7140 = (x + 204)(x - 35) \ , \quad x^2 + 169x + 7140 = (x + 84)(x + 85) \ .$$

Solution 5. [K. Zhou] Suppose that $x^2 + mx - n$ has integer roots $-a$ and $-b$ and that $x^2 + mx + n$ has an integer root c. Then

$$x^2 + mx - n = (x + a)(x + b)$$

so that $m = a + b$ and $n = -ab¿$ The two roots of the polynomial $x^2 + mx + n$ are $-c$ and $c - m$, where $-ab = n = c(a + b - c)$. Therefore $a(b + c0 = c^2 - bc$ so that

$$a = c - 2b + \frac{2b^2}{b + c} \ .$$

Thus, $b + c$ must divide $2b^2$ as all the other terms in the equation are integers.

To construct our examples, let a, b, c be chosen so that a and b are integers, $b+c = 1$ and $a = c-2b+2b^2$. Then c is an integer and

$$c = 1 - b \ , \qquad a = 1 - 3b + 2b^2 = (1 - b)(1 - 2b) \ .$$

Therefore, let

$$\begin{aligned} m &= a + b = 1 - 2b + 2b^2 = (1 - b)^2 + b^2 \\ &= (1 - b) + b(2b - 1) \end{aligned}$$

and

$$n = -ab = -b(1 - b)(1 - 2b) \ .$$

Suppose, if possible, that some prime p divides both m and n. From the factorization of n, it follows that p divides one of b, $1 - b$ and $1 - 2b$, and from the expressions for m, we see that it must divide all three of these numbers. But this is impossible, as b and $1 - b$ are coprime. Therefore, for all integers b, m and n are coprime.

We verify that these values of m and

$$x^2 + (1 - 2b + b^2)x + b(b - 1)(2b - 1) = (x + b)(x + (b - 1)(2b - 1)) \ ;$$

$$x^2 + (1 - 2b + b^2)x - b(b - 1)(2b - 1) = (x - b + 1)(x + b(2b - 1) \ .$$

642. In a convex polyhedron, each vertex is the endpoint of exactly three edges and each face is a concyclic polygon. Prove that the polyhedron can be inscribed in a sphere.

Solution. Let us begin with a couple of preliminary observations. Since three edges are incident with each vertex, exactly three faces of the polyhedron meet at each vertex. The centre of the circumscribing circle of any face is the point common to the right bisectors of the edges. The planes that right bisect the edges of a face intersect in a line perpendicular to the face, and this line is the locus of the centres of spheres which contain all the vertices of the face. Finally, any two vertices of the polyhedron can be joined by a path of edges of the polyhedron.

Any two adjacent faces of the polyhedron are inscribed in a unique sphere. Let the edge AB be common to two faces α and β which have respective circumcentres P and Q. The respective lines m and n to these faces through their circumcentres are non-parallel lines on the plane right-bisecting AB and so intersect in a unique point. This point is the centre of the only sphere that contains all of the vertices of α and β.

The three faces meeting at any vertex are contained in the same sphere. Suppose that vertex A belongs to the edges AB, AC and AD. The right bisecting planes of the edges AB and AC meet in a line through the centre of and perpendicular to the circumcircle of ABC. The right bisecting plane of AD is not parallel to this line and does not contain it, and so meets it in a single point. This point, lying on the perpendicular to each of the three faces adjacent to A and passing through their circumcentres is equidistance from all the vertices of these faces and so is the centre of a sphere containing these faces.

There is a circumscribing sphere for the polyhedron. Suppose this is false. Then there must be two vertices for which the spheres circumscribing the faces about the vertices differ. Join the two vertices by a path of edges. For one of the edges, say RS, the sphere circumscribing the faces meeting at R must be different from the sphere circumscribing the faces meeting at S. But then this means that the two faces adjacent to RS must be circumscribed by two separate spheres, contrary to what was shown above. Hence the desired result follows.

643. Let n^2 distinct integers be arranged in an $n \times n$ square array ($n \geq 2$). Show that it is possible to select n numbers, one from each row and column, such that if the number selected from any row is greater than another number in this row, then this latter number is less than the number selected from its column.

Solution. We proceed in a number of rounds. In Round 1, select the least element in each row. If each column has one such number, we stop; otherwise, deselect in any column all but the largest of the selected numbers. Any row that does not contain a selected number, we call *free*. In each subsequent round, pick the least element not yet tried from each free row, and then deselect all but the biggest number in each column. Since any row can be freed at most $n - 1$ times, there are at most $n(n-1)+1$ rounds. In the final round, each column must have exactly one element.

Example:

$$\begin{array}{ccc} 6 & \mathbf{5} & 11 \\ 4 & \mathbf{2} & 3 \\ 7 & 10 & \mathbf{1} \end{array} \quad \longrightarrow \quad \begin{array}{ccc} 6 & \mathbf{5} & 11 \\ 4 & 2 & \mathbf{3} \\ 7 & 10 & \mathbf{1} \end{array} \quad \longrightarrow \quad \begin{array}{ccc} 6 & \mathbf{5} & 11 \\ 4 & 2 & \mathbf{3} \\ \mathbf{7} & 10 & 1 \end{array}$$

Suppose, wolog (by shuffling the columns if necessary), that the entries $a_{11}, a_{22}, \cdots, a_{nn}$ are selected from the array (a_{ij}). If $a_{ik} < a_{ii}$, then this number must have been an earlier possible selection and was rejected in favour of a larger number in its column. Hence $a_{ik} < a_{kk}$.

Comment: There is a dual procedure taking the largest element of each column and rejecting all but the smallest selected number in each row.

644. Given a point P, a line $\mathfrak{L}$ and a circle $\mathfrak{C}$, construct with straightedge and compasses an equilateral triangle PQR with one vertex at P, another vertex Q on $\mathfrak{L}$ and the third vertex R on $\mathfrak{C}$.

Solution 1. Analysis. Suppose that we have the required triangle PQR with $Q \in \mathfrak{L}$ and $R \in \mathfrak{C}$. Then a 60° rotation with centre P takes $\mathfrak{C}$ to a circle $\mathfrak{C}'$ and R to the point lying on the intersection of $\mathfrak{C}'$ and $\mathfrak{L}$. Accordingly, we need to construct a rotated image $\mathfrak{C}'$ of $\mathfrak{C}$ and, if this intersects $\mathfrak{L}$, then we can construct the triangle.

Construction. Let O be the centre of $\mathfrak{C}$. Construct an equilateral triangle POO' and with centre O' construct a circle $\mathfrak{C}'$ with radius equal to that of $\mathfrak{C}$. If this circle $\mathfrak{C}'$ intersects $\mathfrak{L}$ at R, then there are two constructible points which with P and R are the vertices of an equilateral triangle; one of them Q will lie on $\mathfrak{C}$.

Proof. The circle $\mathfrak{C}'$ is the image of $\mathfrak{C}$ under a $60°$ rotation with centre P that carries O to O'. The point R lies on $\mathfrak{C}'$ so its inverse image Q under the rotation lies on $\mathfrak{C}$. Since $PQ = PR$ and $\angle P = 60°$, PQR is an equilateral triangle.

Comment. There are two possible images of $\mathfrak{C}$ yielding up to four possibilities for R. However, it is also possible that neither images intersects $\mathfrak{L}$ and the construction is not possible.

645. Let $n \geq 3$ be a positive integer. Are there n positive integers $a_1, a_2, \cdots, a_n$ not all the same such that for each i with $3 \leq i \leq n$ we have

$$a_i + S_i = (a_i, S_i) + [a_i, S_i] \quad .$$

where $S_i = a_1 + a_2 + \cdots + a_i$, and where $(\cdot,\cdot)$ and $[\cdot,\cdot]$ represent the greatest common divisor and least common multiple respectively?

Solution 1. Letting $b_i = (a_i, S_i)$, we find that

$$[a_i, S_i] = \frac{a_i S_i}{(a_i, S_i)} = \frac{a_i S_i}{b_i} \ .$$

The given condition is equivalent to $a_i + S_i = b_i + (a_i S_i / b_i)$, which is equivalent to

$$0 = b_i^2 - (a_i + S_i) b_i + a_i S_i = (b_i - a_i)(b_i - S_i) \ .$$

We can achieve the condition by making $a_i = (a_i, S_i)$ and $S_i = [a_i, S_i]$. Let $a_1 = a_2 = 1$, $a_i = 2^{i-2}$ for $i \geq 3$. Then

$$S_i = 1 + \sum_{j=2}^{i} 2^{j-2} = 1 + (2^{i-1} - 1) = 2^{i-1}$$

$$\Longrightarrow (a_i, S_i) = 2^{i-2} \ , \quad [a_i, S_i] = 2^{i-1}$$

for $i \geq 3$.

Solution 2. Let $a_i = 1$, $a_2 = 2$ and $a_i = 3 \cdot 2^{i-3}$ for $i \geq 3$. Then

$$S_i = 1 + 2 + 3 \sum_{j=0}^{i-3} 2^j = 1 + 2 + 3(2^{i-2} - 1) = 3 \cdot 2^{i-2}$$

$$\Longrightarrow (a_i, S_i) = 3 \cdot 2^{i-3} \ , \quad [a_i, S_i] = 3 \cdot 2^{i-2}$$

for $i \geq 3$.

Solution 3. [K. Purbhoo] Choose a_1 at will, and let $a_i = S_{i-1}$ for $i \geq 2$. Then $S_i = S_{i-1} + a_i = 2a_i$, $a_i + S_i = 3a_i$, $(a_i, S_i) = a_i$ and $[a_i, S_i] = 2a_i$ for $i \geq 2$.

Solution 4. [K. Yeats] Let $a_1 = 1$, $a_2 = 3$ and $a_n = 2^{n-1}$ for $n \geq 3$. Then $S_n = 2^n$ for $n \geq 2$ and, for each i with $3 \leq i \leq n$, $(a_i, S_i) = 2^{i-1}$ and $[a_i, S_i] = 2^i$.

646. Let ABC be a triangle with incentre I. Let AI meet BC at L, and let X be the contact point of the incircle with the line BC. If D is the reflection of L on X, we construct B' and C' as the reflections of D with respect to the lines BI and CI, respectively. Show that the quadrailateral $BCC'B'$ is cyclic.

Solution 1. Without loss of generality, we may assume that $AC \geq AB$. Observe that B' lies on the side AB and C' lies on the side AC. We use the conventional notation for the sides a, b, c of the triangle and $2s = a + b + c$ for the permimeter. Let Z and Y be the tangent points of the incircle with sides AB and AC respectively. Observe that $IX \perp BC$, $IY \perp AC$ and $IZ \perp AB$.

We have that $XL = XD = ZB' = YC'$,

$$|XC| = s - c \ , \quad |AY| = |AZ| = s - a \ ;$$

$$|LC| = \frac{ab}{b+c} \ .$$

Therefore

$$|XL| = s - c - \frac{ab}{b+c} \ ,$$

$$\begin{aligned} |AB'| &= |AZ| + |ZB| \\ &= s - a + s - c - \frac{ab}{b+c} = b - \frac{ab}{b+c} \\ &= b\left(1 - \frac{a}{b+c}\right) \ ; \end{aligned}$$

and

$$\begin{aligned} |AC'| &= |AY| - |YC'| \\ &= s - a - s + c + \frac{ab}{b+c} \\ &= \frac{-ab + bc - ac + c^2 + ab}{b+c} \\ &= c\left(\frac{b+c-a}{b+c}\right) = c\left(1 - \frac{a}{b+c}\right) \ . \end{aligned}$$

Therefore

$$AC' : AB' = c : b = AB : AC$$

so that triangles ABC and $AC'B'$ are similar and $\angle ABC = \angle AC'B'$, $\angle ACB = \angle AB'C'$. Therefore the quadrilateral $BCC'B'$ is concyclic.

Solution 2. [S. Sun] Suppose that $AC \geq AB$. Use the same notation as in Solution 1, and let $t = |BL|$. Then $|ZB'| = |YC'| = |DX| = |XL| = t - (s - b)$. We have that

$$|AB'| = (s - a) + t - (s - b) = t + (b - a)$$

and

$$|AC'| = (s - a) - t + (s - b) = c - t \ ,$$

whereupon

$$AB' : AC' = [t + (b - a)] : [c - t] = [b - (a - t)] : [c - t] \ .$$

However, as AL is an angle bisector, we have that $(a - t) : t = b : c$, so that

$$[b - (a - t) : [c - t] = b : c = AC : AB \ .$$

Therefore, triangles $AC'B'$ and ABC are similar, and we can conclude, as in Solution 1, that $BCC'B'$ is concyclic.

Comment. Notice that Solutions 1 and 2 follow the same strategy, but the second solution is cleaner as it avoid the actual computation of t and merely exploited a relationship involving this variable.

Solution 3. [A. Murali] We again assume that $AC \geq AB$ and use the notation of Solution 1. We first show that $AB'IC'$ is concyclic. Observe that $\angle ZB'I = \angle LD'I = \angle YC'I$, so that triangles $IB'Z$ and $IC'Y$ are similar and $\angle ZIB' = \angle YIC'$. Thus $\angle B'AC' = 180° - \angle ZIY = 180° - \angle B'IC'$ and $AB'IC'$ is concycl c. It follows that $\angle B'C'I = \angle BAI = \frac{1}{2}\angle BAC = \angle LAC$.

We have that
$$\angle CC'I = \angle IDC = \angle ILB = \angle LAC + \angle ACL \ ,$$
whence
$$\angle CC'B' = \angle CC'I + \angle B'C'I = (\angle LAC + \angle ACL) + \angle LAC = \angle ACB + \angle BAC = 180° - \angle ABC \ .$$
Therefore, $BCC'B'$ is concyclic.

647. Find all continuous functions $f : \mathbf{R} \to \mathbf{R}$ such that
$$f(x + f(y)) = f(x) + y$$
for every $x, y \in \mathbf{R}$.

Solution 1. Setting $(x, y) = (t, 0)$ yields $f(t + f(0)) = f(t)$ for all real t. Setting $(x, y) = (0, t)$ yields $f(f(t)) = f(0) + t$ for all real t. Hence $f(f(f(t))) = f(t)$ for all real t, *i.e.*, $f(f(z)) = z$ for each z in the image of f. Let $(x, y) = (f(t), -f(0))$. Then
$$f(f(t) + f(-f(0))) = f(f(t)) - f(0) = f(0) + t - f(0) = t$$
so that the image of f contains every real and so $f(f(t)) \equiv t$ for all real t.

Taking $(x, y) = (u, f(v))$ yields
$$f(u + v) = f(u) + f(v)$$
since $v = f(f(v))$ for all real u and v. In particular, $f(0) = 2f(0)$, so $f(0) = 0$ and $0 = f(-t+t) = f(-t) + f(t)$. By induction, it can be shown that for each integer n and each real t, $f(nt) = nf(t)$. In particular, for each rational r/s, $f(r/s) = rf(1/s) = (r/s)f(1)$. Since f is continuous, $f(t) = f(t \cdot 1) = tf(1)$ for all real t. Let $c = f(1)$. Then $1 = f(f(1)) = f(c) = cf(1) = c^2$ so that $c = \pm 1$. Hence $f(t) \equiv t$ or $f(t) \equiv -t$. Checking reveals that both these solutions work. (For $f(t) \equiv -t$, $f(x + f(y)) = -x - f(y) = f(x) + y$, as required.)

Solution 2. Taking $(x, y) = (0, 0)$ yields $f(f(0)) = f(0)$, whence $f(f(f(0))) = f(f(0)) = f(0)$. Taking $(x, y) = (0, f(0))$ yields $f(f(f(0))) = 2f(0)$. Hence $2f(0) = f(0)$ so that $f(0) = 0$. Taking $x = 0$ yields $f(f(y)) = y$ for each y. We can complete the solution as in the Second Solution.

Solution 3. [J. Rickards] Let $(x, y) = (x, -f(x))$ to get
$$f(x + f(-f(x)) = f(x) - f(x) = 0$$
for all x. Thus, there is at least one element u for which $f(u) = 0$. But then, taking $(x, y) = (0, u)$, we find that $f(0) = f(0 + f(u)) = f(0) + u$, so that $u = 0$.

Therefore $f(f(y)) = y$ for each y, so that f is a one-one onto function. Also, $x + f(-f(x)) = 0$, so that $-f(x) = f(f(-f(x)) = f(-x)$ for each value of x.

Since $f(x)$ is continuous and vanishes only for $x = 0$, we have either (1) $f(x)$ is positive for $x > 0$ and negative for $x < 0$, or (2) $f(x)$ is negative for $x > 0$ and positive for $x < 0$. Suppose that situation (1) obtains. Then, for every real number x, $f(x - f(x)) = f(x + f(-x)) = f(x) - x = -(x - f(x))$. Since $f(x - f(x))$ and $x - f(x)$ have the same sign, we must have $f(x) = x$. Suppose that situation (2) obtains. Then, for every real x, $f(x + f(x)) = f(x) + x$, from which we deduce that $f(x) = -x$. Therefore, there are two functions $f(x) = x$ and $f(x) = -x$ that satisfy the equation and both work.

648. Prove that for every positive integer n, the integer $1 + 5^n + 5^{2n} + 5^{3n} + 5^{4n}$ is composite.

Solution. Observe the following representations:

$$x^8 + x^6 + x^4 + x^2 + 1 = (x^4 + x^3 + x^2 + x + 1)(x^4 - x^3 + x^2 - x + 1) \ . \tag{1}$$

and

$$x^4 + x^3 + x^2 + x + 1 = (x^2 + 3x + 1)^2 - 5x(x + 1)^2 \ . \tag{2}$$

When $n = 2k$ is even, we can substitute $x = 5^k$ into equation (1) to get a factorization. When $n = 2k - 1$ is odd, we can substitute $x = 5^{2k-1}$ into equation (2) to get a difference of squares, which can then be factored.

649. In the triangle ABC, $\angle BAC = 20°$ and $\angle ACB = 30°$. The point M is located in the interior of triangle ABC so that $\angle MAC = \angle MCA = 10°$. Determine $\angle BMC$.

Solution 1. [S. Sun] Construct equilateral triangle MDC with M and D on opposite sides of AC and equilateral triangle AME with M and Z on opposite sides of AB. Since $AM = MC$, these equilateral triangles are congruent. Since $AM = MD$ and

$$\angle AMD = \angle AMC - \angle DMC = 160° - 60° = 100° \ ,$$

$\angle MAD = \angle MDA = 40°$. Since $ME = AM = MC$, triangle EMC is isosceles. Since

$$\angle EMC = 360° - \angle EMA - \angle AMC = 360° - 60° - 160° = 140° \ ,$$

$\angle EMC = \angle MCE = 20°$. As $\angle MCB = 20° = \angle MCE$, E, B, C are collinear. Now

$$\begin{aligned} \angle EBA &= \angle BAC + \angle BCA = 20° + 30° = 50° \\ &= 60° - 10° = \angle EAM - \angle BAM = \angle EAB \ , \end{aligned}$$

so that $BE = AE = ME$ and triangle BEM is isosceles. Since $\angle BEM = \angle BEA - \angle MEA = 80° - 60° = 20°$, it follows that

$$\angle BMC = 360° - \angle EMB - \angle EMA - \angle AMC = 360° - 80° - 60° - 160° = 60° \ .$$

Solution 2. Let O be the circumcentre of the triangle BAC; this lies on the opposite side of AC to B. Since the angle subtended at the centre by a chord is double that subtended at the circumference, we have that

$$\angle AOC = 2(180° - \angle ABC) = 2(180° - 130°) = 100° \ .$$

The right bisector of the segment AC passes through the apex of the isosceles triangle MAC and the centre O of the circumcircle of triangle BAC. We have that $\angle AOM = 50°$, $\angle AMO = \frac{1}{2}\angle AMC = 80°$, and

$$\angle MAO = 180° - 50° - 80° = 50° \ .$$

Therefore, triangle MAO is isosceles with $MA = MO$.

Observe that $\angle BAO = \angle BAC + \angle MAO - \angle MAC = 60°$ and that $AO = BO$, so that triangle BAO is equilateral and so $BA = BO$. Since B and M are both equidistant from A and O, the line BM must right bisect the segment AO at N, say. Therefore, $\angle MNO = 90°$, so that $\angle NMO = 40°$. It follows that

$$\angle BMC = 180° - \angle CMO - \angle NMO = 180° - 80° - 40° = 60° \ .$$

Solution 3. [M. Essafty] Let $\alpha = \angle MBA$, so that $\angle MBC = 130° - \alpha$. From the trigonometric version of Ceva's Theorem, we have that

$$\sin\alpha \sin 20° \sin 10° = \sin(130° - \alpha)\sin 10° \sin 10°$$

$$\Rightarrow 2\sin alpha \sin 10^\circ \cos 10^\circ = \sin(130^\circ - \alpha)\sin 10^\circ$$

$$\Rightarrow 2\sin\alpha\cos 10^\circ = \cos(40^\circ - \alpha) = \cos 40^\circ \cos\alpha + \sin 40^\circ \sin\alpha \ .$$

Dividing both sides by $\cos 40^\circ \cos\alpha$ yields that

$$2\cos\alpha\left(\frac{2\cos 10^\circ}{\cos 40^\circ} - \frac{\sin 40^\circ}{\cos 40^\circ}\right) = 1 \ .$$

Therefore

$$\begin{aligned}\cot\alpha &= \frac{\cos 10^\circ + \cos 10^\circ - \cos 50^\circ}{\cos 40^\circ} \\ &= \frac{\cos 10^\circ + 2\sin 30^\circ \sin 20^\circ}{\cos 40^\circ} \\ &= \frac{\cos 10^\circ + \sin 20^\circ}{\cos 40^\circ} = \frac{\cos 10^\circ + \cos 70^\circ}{\cos 40^\circ} \\ &= \frac{2\cos 40^\circ \cos 30^\circ}{\cos 40^\circ} = 2\cos 30^\circ = \sqrt{3} \ .\end{aligned}$$

Therefore $\alpha = 30^\circ$.

650. Suppose that the nonzero real numbers satisfy

$$\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = \frac{1}{xyz} \ .$$

Determine the minimum value of

$$\frac{x^4}{x^2+y^2} + \frac{y^4}{y^2+z^2} + \frac{z^4}{z^2+x^2} \ .$$

Solution 1. [W. Fu] Let $f(x,y,z)$ denote the expression

$$\frac{x^4}{x^2+y^2} + \frac{y^4}{y^2+z^2} + \frac{z^4}{z^2+x^2} \ .$$

Then

$$\begin{aligned}f(x,y,z) - f(x,z,y) &= \left(\frac{x^4}{x^2+y^2} + \frac{y^4}{y^2+z^2} + \frac{z^4}{z^2+x^2}\right) - \left(\frac{x^4}{x^2+z^2} + \frac{z^4}{z^2+y^2} + \frac{y^4}{y^2+x^2}\right) \\ &= \frac{x^4-y^4}{x^2+y^2} + \frac{y^4-z^4}{y^2+z^2} + \frac{z^4-x^4}{z^2+x^2} \\ &= (x^2-y^2) + (y^2-z^2) + (z^2-x^2) = 0 \ .\end{aligned}$$

Thus, $f(x,y,z) = f(x,z,y)$ and

$$\begin{aligned}f(x,y,z) &= \frac{1}{2}(f(x,y,z) + f(x,z,y)) \\ &= \frac{1}{2}\left[\frac{x^4+y^4}{x^2+y^2} + \frac{y^4+z^4}{y^2+z^2} + \frac{z^4+x^4}{z^2+x^2}\right] \\ &= \frac{1}{2}\left[\left(x^2+y^2 - \frac{2x^2y^2}{x^2+y^2}\right) + \left(y^2+z^2 - \frac{2y^2z^2}{y^2+z^2}\right) + \left(z^2+x^2 - \frac{2z^2x^2}{z^2+x^2}\right)\right] \\ &= (x^2+y^2+z^2) - \frac{1}{2}\left(\frac{2x^2y^2}{x^2+y^2} + \frac{2y^2z^2}{y^2+z^2} + \frac{2z^2x^2}{z^2+x^2}\right)\end{aligned}$$

Observe that

$$x^2+y^2+z^2=\frac{1}{2}[(x^2+y^2)+(y^2+z^2)+(z^2+x^2)]\geq xy+yz+zx=1$$

and that $2x^2y^2\leq x^4+y^4$. Hence

$$\begin{aligned} f(x,y,z) &\geq 1-\frac{1}{2}\left(\frac{x^4+y^4}{x^2+y^2}+\frac{y^4+z^4}{y^2+z^2}+\frac{x^4+x^4}{z^2+x^2}\right) \\ &= 1-\frac{1}{2}[f(x,y,z)+f(x,z,y)]=1-f(x,y,z)\ , \end{aligned}$$

from which $f(x,y,z)\geq\frac{1}{2}$. Equality occurs if and only if $x=y=z=1/\sqrt{3}$.

Solution 2. [S. Sun] From the Arithmetic-Geometric Means Inequality, we have that

$$\frac{x^4}{x^2+y^2}+\frac{1}{4}(x^2+y^2)\geq x^2$$

with a similar inequality for the other pairs of variables. Adding the three inequalities obtained, we find that

$$\frac{x^4}{x^2+y^2}+\frac{y^4}{y^2+z^2}+\frac{z^4}{z^2+x^2}+\frac{1}{2}(x^2+y^2+z^2)\geq x^2+y^2+z^2$$

from which

$$\frac{x^4}{x^2+y^2}+\frac{y^4}{y^2+z^2}+\frac{z^4}{z^2+x^2}\geq\frac{1}{2}(x^2+y^2+z^2)\ ,$$

with equality if and only if $x=y=z$. Since $(x-y)^2+(y-z)^2+(z-x)^2\geq 0$, it follows that $x^2+y^2+z^2\geq xy+yz+zx=1$. Therefore

$$\frac{x^4}{x^2+y^2}+\frac{y^4}{y^2+z^2}+\frac{z^4}{z^2+x^2}\geq\frac{1}{2}$$

with equality if and only if $x=y=z=1/\sqrt{3}$.

Solution 3. [K. Zhou; G. Ajjanagadde; M. Essafty] Since $(x-y)^2\geq 0$, *etc.*, we have that $x^2+y^2+z^2\geq xy+yz+zx$. By the Cauchy-Schwarz Inequality, we have that

$$\begin{aligned} &\left[\left(\frac{x^2}{\sqrt{x^2+y^2}}\right)^2+\left(\frac{y^2}{\sqrt{y^2+z^2}}\right)^2+\left(\frac{z^2}{\sqrt{z^2+x^2}}\right)^2\right][(\sqrt{x^2+y^2})^2+(\sqrt{y^2+z^2})^2+(\sqrt{z^2+x^2})^2] \\ &\qquad\geq(x^2+y^2+z^2)^2\ , \end{aligned}$$

whence

$$\left(\frac{x^4}{x^2+y^2}+\frac{y^4}{y^2+z^2}+\frac{z^4}{z^2+x^2}\right)[(x^2+y^2)+(y^2+z^2)+(z^2+x^2)]\geq(x^2+y^2+z^2)^2\ ,$$

so that

$$\frac{x^4}{x^2+y^2}+\frac{y^4}{y^2+z^2}+\frac{z^4}{z^2+x^2}\geq\frac{x^2+y^2+z^2}{2}\geq\frac{xy+yz+zx}{2}=\frac{1}{2}\ .$$

Equality occurs when $x=y=z=1/\sqrt{3}$.

Solution 4. Observe that the given condition is equivalent to $xy+yz+zx=1$. Since the expression to be minimized is the same when (x,y,z) is replaced by $(-x,-y,-z)$ and since two of the variables must have the same sign, we may assume that x and y are both positive.

Suppose, first, that $z>0$. Since $x^2+y^2\geq 2xy$, we have that

$$\frac{x^4}{x^2+y^2}=x^2-\frac{x^2y^2}{x^2+y^2}\geq x^2-\frac{xy}{2}\ ,$$

with similar inequalities for the other pairs of variables. Therefore, the expression to be minimized is not less that

$$(x^2 + y^2 + z^2) - \frac{1}{2}(xy + yz + zx) \geq (xy + yz + zx) - \frac{1}{2}(xy + yz + zx) = \frac{1}{2} \ .$$

Equality occurs if and only if $x = y = z = 1/\sqrt{3}$.

Regardless of the signs of the variables, if the largest of x^2, y^2, z^2 is at least 2, we show that the expression is not less that 1. For example, if $x^2 \geq 2$, $x^2 \geq y^2$, we find that

$$\frac{x^4}{x^2 + y^2} \geq \frac{x^4}{2x^2} = \frac{x^2}{2} \geq 1 \ .$$

Henceforth, assume that x^2, y^2, z^2 are less than 2 and that $z < 0$. Then $xy < 2$. Since $0 > z = (1 - xy)/(x + y)$, then $xy > 1$, so that $x + y \geq 2\sqrt{xy} > 2$. Hence

$$|z| = \frac{xy - 1}{x + y} \leq \frac{1}{2} \ .$$

If $x > y$, then (because $xy > 1$), $x > 1$, so that

$$\frac{x^4}{x^2 + y^2} > \frac{x^4}{2x^2} > \frac{1}{2} \ .$$

If $y > z$, then $y > 1 > |z|$ and

$$\frac{y^4}{y^2 + z^2} > \frac{y^4}{2y^2} > \frac{1}{2} \ .$$

In any case, when $z < 0$, the quantity to be minimized exceeds $1/2$. Therefore, the minimum value is $1/2$, achieved when $(x, y, z) = (3^{-1/2}, 3^{-1/2}, 3^{-1/2})$.

Solution 5. [B. Wu] We first establish a lemms: *if a, b, u, v are positive, then*

$$\frac{a^2}{u} + \frac{b^2}{v} \geq \frac{(a + b)^2}{u + v}$$

with equality if and only if $a : u = b : v$. To see this, subtract the right side from the left to get a fraction whose numerator is $(av - bu)^2$.

Applying this to the given expression yields that

$$\begin{aligned} \frac{(x^2)^2}{y^2 + z^2} + \frac{(y^2)^2}{z^2 + x^2} + \frac{(z^2)^2}{x^2 + y^2} \\ &\geq \frac{(x^2 + y^2 + z^2)^2}{2(x^2 + y^2 + z^2)} = \frac{x^2 + y^2 + z^2}{2} \\ &\geq \frac{xy + yz + zx}{2} = \frac{1}{2} \ . \end{aligned}$$

Equality occurs if and only if $x = y = z = 1/\sqrt{3}$.

Solution 6. [M. Essafty] Squaring both sides of the equation $2x^2 = (x^2 + y^2) + (x^2 - y^2)$ yields that

$$\begin{aligned} 4x^4 &= (x^2 + y^2)^2 + (x^2 - y^2)^2 + 2(x^2 + y^2)(x^2 - y^2) \\ &\geq (x^2 + y^2)^2 + 2(x^2 + y^2)(x^2 - y^2) \end{aligned}$$

whence

$$\frac{4x^4}{x^2 + y^2} \geq 3x^2 - y^2 \ .$$

Taking account of similar inequalities for other pairs of variables, we obtain that

$$\frac{4x^4}{x^2+y^2}+\frac{4y^4}{y^2+z^2}+\frac{4z^4}{z^2+x^2}\geq 2(x^2+y^2+z^2)\geq 2(xy+yz+zx)=2\ ,$$

from which we conclude that the minimum value is $\frac{1}{2}$. This is attained when $x=y=z=1/\sqrt{3}$.

Solution 7. [O. Xia] Recall that, for $r>0$, $r+(1/r)\geq 2$, so that $r\geq 2-(1/r)$. It follows that

$$\begin{aligned}\frac{x^4}{x^2+y^2}&=\left(\frac{x^2}{2}\right)\left(\frac{2x^2}{x^2+y^2}\right)\\&\geq\left(\frac{x^2}{2}\right)\left(2-\frac{x^2+y^2}{2x^2}\right)\\&=x^2-\frac{x^2+y^2}{4}\end{aligned}$$

with similar equalities for the other two terms in the problem statement. Equality occurs if and only if $x^2=y^2=z^2$.

Adding the three equalities yields that Determine the minimum value of

$$\frac{x^4}{x^2+y^2}+\frac{y^4}{y^2+z^2}+\frac{z^4}{z^2+x^2}\geq\frac{x^2+y^2+z^2}{2}\ .$$

As before, we see that the right member assumes its minimum value of $\frac{1}{2}$ when $x=y=z=1/\sqrt{3}$.

651. Determine polynomials $a(t)$, $b(t)$, $c(t)$ with integer coefficients such that the equation $y^2+2y=x^3-x^2-x$ is satisfied by $(x,y)=(a(t)/c(t),b(t)/c(t))$.

Solution. The equation can be rewritten $(y+1)^2=(x-1)^2(x+1)$. Let $x+1=t^2$ so that $y+1=(t^2-2)t$. Thus, we obtain the solution

$$(x,y)=(t^2-1,t^3-2t-1)\ .$$

With these polynomials, both sides of the equation are equal to $t^6-4t^4+4t^2-1$.

652. (a) Let m be any positive integer greater than 2, such that $x^2\equiv 1\pmod m$ whenever the greatest common divisor of x and m is equal to 1. An example is $m=12$. Suppose that n is a positive integer for which $n+1$ is a multiple of m. Prove that the sum of all of the divisors of n is divisible by m.

(b) Does the result in (a) hold when $m=2$?

(c) Find all possible values of m that satisfy the condition in (a).

(a) *Solution 1.* Let $n+1$ be a multiple of m. Then $\gcd(m,n)=1$. We observe that n cannot be a square. Suppose, if possible, that $n=r^2$. Then $\gcd(r,m)=1$. Hence $r^2\equiv 1\pmod m$. But $r^2+1\equiv 0\pmod m$ by hypothesis, so that 2 is a multiple of m, a contradiction.

As a result, if d is a divisor of n, then n/d is a distinct divisor of n. Suppose $d|n$ (read "d divides n"). Since m divides $n+1$, therefore $\gcd(m,n)=\gcd(d,m)=1$, so that $d^2=1+bm$ for some integer b. Also $n+1=cm$ for some integer c. Hence

$$d+\frac{n}{d}=\frac{d^2+n}{d}=\frac{1+bm+cm-1}{d}=\frac{(b+c)m}{d}\quad .$$

Since $\gcd(d,m)=1$ and $d+n/d$ is an integer, d divides $b+c$ and so $d+n/d\equiv 0\pmod m$.

Hence

$$\sum_{d|n}d=\sum\{(d+n/d):d|n,d<\sqrt{n}\}\equiv 0\pmod m$$

as desired.

Solution 2. Suppose that $m > 1$ and m divides $n+1$. Then gcd $(m, n) = 1$. Suppose, if possible, that $n = r^2$ for some r. Then, since gcd $(m, r) = 1$, $r^2 \equiv 1 \pmod{r}$. Therefore m divides both $r^2 + 1$ and $r^2 - 1$, so that $m = 2$. But this gives a contradiction. Hence n is not a perfect square.

Suppose that d is a divisor of n. Then the greatest common divisor of m and d is 1, so that $d^2 \equiv 1 \pmod{n}$. Suppose that $de = n$. Then $e \neq 1d$ and the greatest common divisor of m and e is 1. Therefore, there are numbers u and v for which both du and ev are congruent to 1 modulo m. Since $n \equiv -1$ and $d^2 \equiv 1 \pmod{m}$, it follows that

$$d + e \equiv d + un \equiv u(d^2 + n) \equiv u(1 - 1) = 0$$

mod m), from which it can be deduced that m divides the sum of all the divisors of n.

Solution 3. Suppose that $n + 1 \equiv 0 \pmod{m}$. As in the first solution, it can be established that n is not a perfect square. Let x be any positive divisor of n and suppose that $xy = n$; x and y are distinct. Since gcd $(x, m) = 1$, $x^2 \equiv 1 \pmod{m}$, so that

$$y = x^2 y \equiv xn \equiv -x \pmod{m}$$

whence $x + y$ is a multiple of m. Thus, the divisors of n comes in pairs, each of which has sum divisible by m, and the result follows.

Solution 4. [M. Boase] As in the second solution, if $xy = n$, then $x^2 \equiv y^2 \equiv 1 \pmod{m}$ so that

$$0 \equiv x^2 - y^2 \equiv (x - y)(x + y) \pmod{m}.$$

For any divisor r of m, we have that

$$x(x - y) \equiv x^2 - xy \equiv 2 \pmod{r}$$

from which it follows that the greatest common divisor of m and $x - y$ is 1. Therefore, m must divide $x + y$ and the solution can be completed as before.

(b) *Solution.* When $m = 2$, the result does not hold. The hypothesis is true. However, the conclusion fails when $n = 9$ since $9 + 1$ is a multiple of 2, but $1 + 3 + 9 = 13$ is odd.

(c) *Solution 1.* By inspection, we find that $m = 1, 2, 3, 4, 6, 8, 12, 24$ all satisfy the condition in (a).

Suppose that m is odd. Then $\gcd(2, m) = 1 \Rightarrow 2^2 = 4 \equiv 1 \pmod{m} \Rightarrow m = 1, 3$.

Suppose that m is not divisible by 3. Then $\gcd(3, m) = 1 \Rightarrow 9 = 3^2 \equiv 1 \pmod{m} \Rightarrow m = 1, 2, 4, 8$. Hence any further values of m not listed in the above must be even multiples of 3, that is, multiples of 6.

Suppose that $m \geq 30$. Then, since $25 = 5^2 \not\equiv 1 \pmod{m}$, m must be a multiple of 5.

It remains to show that in fact m cannot be a multiple of 5. We observe that there are infinitely many primes congruent to 2 or 3 modulo 5. [To see this, let $q_1, \cdots, q_s$ be the s smallest odd primes of this form and let $Q = 5q_1 \cdots q_s + 2$. Then Q is odd. Also, Q cannot be a product only of primes congruent to ± 1 modulo 5, for then Q itself would be congruent to ± 1. Hence Q has an odd prime factor congruent to ± 2 modulo 5, which must be distinct from $q_1, \cdots, q_s$. Hence, no matter how many primes we have of the desired form, we can always find one more.] If possible, let m be a multiple of 5 with the stated property and let q be a prime exceeding m congruent to ± 2 modulo 5. Then $\gcd(q, m) = 1 \Rightarrow q^2 \equiv 1 \pmod{m} \Rightarrow q^2 \equiv 1 \pmod{5}$ $\Rightarrow q \not\equiv \pm 2 \pmod{5}$, yielding a contradiction. Thus, we have given a complete collection of suitable numbers m.

Solution 2. [J. Rickards] Suppose that a suitable value of m is equal to a power of 2, Then $3^2 \equiv 1$ (mod m) implies that m must be equal to 4 or 8. It can be checked that both these values work.

Suppose that $m = p^a q$, where p is an odd prime and p and q are coprime. By the Chinese Remainder Theorem, there is a value of x for which $x \equiv 1 \pmod{q}$ and $x \equiv 2 \pmod{p^a}$. Then $x^2 \equiv 1 \pmod{m}$, so that $4 \equiv x^2 \equiv 1 \pmod{p^a}$ and thus p must equal 3. Therefore, m must be divisible by only the primes 2 and 3. Therefore $25 = 5^2 \equiv 1 \pmod{m}$, with the result that m must divide 24. Checking reveals that the only possibilities are $m = 3, 4, 6, 8, 12, 24$.

Solution 3. [D. Arthur] Suppose that $m = ab$ satisfies the condition of part (a), where the greatest common divisor of a and b is 1. Let gcd $(x, a) = 1$. Since a and b are coprime, there exists a number t such that $at \equiv 1 - x \pmod{b}$, so that $z = x + at$ and b are coprime. Hence, the greatest common divisor of z and ab equals 1, so that $z^2 \equiv 1 \pmod{ab}$, whence $x^2 \equiv z^2 \equiv 1 \pmod{a}$. Thus a (and also b) satisfies the condition of part (a).

When m is odd and exceeds 3, then gcd $(2, m) = 1$, but $2^2 = 4 \not\equiv 1 \pmod{m}$, so m does not satisfy the condition. When $m = 2^k$ for $k \geq 4$, then gcd $(3, m) = 1$, but $3^2 = 9 \not\equiv 1 \pmod{m}$. It follows from the first paragraph that if m satisfies the condition, it cannot be divisible by a power of 2 exceeding 8 nor by an odd number exceeding 3. This leaves the possibilities $1, 2, 3, 4, 6, 8, 12, 24$, all of which satisfy the condition.

653. Let $f(1) = 1$ and $f(2) = 3$. Suppose that, for $n \geq 3$, $f(n) = \max\{f(r) + f(n-r) : 1 \leq r \leq n-1\}$. Determine necessary and sufficient conditions on the pair (a, b) that $f(a+b) = f(a) + f(b)$.

Solution 1. From the first few values of $f(n)$, we conjecture that $f(2k) = 3k$ and $f(2k+1) = 3k+1$ for each positive integer k. We establish this by induction. It is easily checked for $k = 1$. Suppose that it holds up to $k = m$.

Suppose that $2m+2$ is the sum of two positive even numbers $2x$ and $2y$. Then $f(2x)+f(2y) = 3(x+y) = 3(m+1)$. If $2m+2$ is the sum of two positive odd numbers $2u+1$ and $2v+1$, then

$$f(2u+1) + f(2v+1) = (3u+1) + (3v+1) = 3(u+v) + 2 < 3(u+v+1) = 3(m+1) .$$

Hence $f(2(m+1)) = 3(m+1)$.

Suppose $2m+3$ is the sum of $2z$ and $2w+1$. Then $z + w = m + 1$ and

$$f(2z) + f(2w+1) = 3z + 3w + 1 = 3(z+w) + 1 = 3(m+1) + 1 .$$

Hence $f(2(m+1)+1) = 3(m+1)+1$. The conjecture is established by induction.

By checking cases on the parity of a and b, one verifies that $f(a+b) = f(a) + f(b)$ if and only if at least one of a and b is even. (If a and b are both odd, the left side is divisible by 3 while the right side is not.)

Solution 2. [K. Yeats] By inspection, we conjecture that $f(n+1) = f(n) + 2$ when n is odd, and $f(n+1) = f(n)+1$ when n is even. This is true for $n = 1, 2$. Suppose it holds up to $n = 2k$. If $2k+1 = i+j$ with i even and j odd, then $f(i-1) + f(j+1) = f(i) - 2 + f(j) + 2 = f(i) + f(j)$ and $f(i+1) + f(j-1) = f(i)+1+f(j)-1 = f(i)+f(j)$ (where defined), so in particular $f(2k+1) = f(2k)+f(1) = f(2k)+1$. Note that this also tells us that $f(2k+1) = f(i) + f(j)$ whenever $i + j = 2k+1$. Now consider $2k+2 = i+j$. If i and j are both even, then

$$f(i+1) + f(j-1) = f(i) + 1 - f(j) - 2 = f(i) + f(j) - 1$$

while if i and j are both odd, then

$$f(i+1) + f(j-1) = f(i) + 2 - f(j) - 1 = f(i) + f(j) + 1 .$$

Thus, $f(2k+2) = f(i) + f(j)$ if and only if i and j are both even. In particular, $f(2k+2) = f(2k) + f(2) = f(2k+1) - 1 + 3 = f(2k) + 2$. We thus find that $f(a+b) = f(a) + f(b)$ if and only if at least one of a and b is even.