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Problems 479-527

479. Let x, y, z be positive integer for which
$$\frac{1}{x}+\frac{1}{y}=\frac{1}{z}$$
and the greatest common divisor of x and z is 1. Prove that $x+y$, $x-z$ and $y-z$ are all perfect squares. Give two examples of triples (x,y,z) that satisfy these conditions.

480. Let a and b be positive real numbers for which $60^a=3$ and $60^b=5$. Without the use of a calculator or of logarithms, determine the value of
$$12^{\frac{1-a-b}{2(1-b)}} .$$

481. In a certain town of population $2n+1$, one knows those to whom one is known. For any set A of n citizens, there is some person among the other $n+1$ who knows everyone on A. Show that some citizen of the town knows all the others.

[This problem was published as #11262 in the *American Mathematical Monthly* (113:10 (December, 2006), 940. Solvers of this problem should send their solutions to Prof. Barbeau and are invited to submit their solutions to the problems editor for the *Monthly*: Prof. Doug Hensley, Monthly Problems, Department of Mathematics, Texas A & M University, 3368 TAMU, College Station, TX 77843-3368, USA. A pdf file of the solution may be sent to *monthlyproblems@math.tamu.edu*.]

482. A trapezoid whose parallel sides have the lengths a and b is partitioned into two trapezoids of equal area by a line segment of length c parallel to these sides. Determine c as a function of a and b.

483. Let A and B be two points on the circumference of a circle, and E be the midpoint of arc AB (either arc will do). Let P be any point on the minor arc EB and N the foot of the perpendicular from E to AP. Prove that $AN=NP+PB$.

484. ABC is a triangle with $\angle A=40°$ and $\angle B=60°$. Let D and E be respective points of AB and AC for which $\angle DCB=70°$ and $\angle EBC=40°$. Furthermore, let F be the point of intersection of DC and EB. Prove that $AF\perp BC$.

485. From the foot of each altitude of the triangle, perpendiculars are dropped to the other two sides. Prove that the six feet of these perpendiculars lie on a circle.

486. Determine all quintuplets (a,b,c,d,u) of nonzero integers for which
$$\frac{a}{b}=\frac{c}{d}=\frac{ab+u}{cd+u} .$$

487. ABC is an isosceles triangle with $\angle A=100°$ and $AB=AC$. The bisector of angle B meets AC in D. Show that $BD+AD=BC$.

488. A host is expecting a number of children, which is either 7 or 11. She has 77 marbles as gifts, and distributes them into n bags in such a way that whether 7 or 11 children come, each will receive a number of bags so that all 77 marbles will be shared equally among the children. What is the minimum value of n?

489. Suppose n is a positive integer not less than 2 and that $x_1 \geq x_2 \geq x_3 \geq \cdots \geq x_n \geq 0$,

$$\sum_{i=1}^{n} x_i \leq 400 \qquad \text{and} \qquad \sum_{i=1}^{n} x_i^2 \geq 10^4 \ .$$

Prove that $\sqrt{x_1} + \sqrt{x_2} \geq 10$. is it possible to have equality throughout? [*Bonus:* Formulate and prove a generalization.]

490. (a) Let a, b, c be real numbers. Prove that

$$\min \ [(a-b)^2, (b-c)^2, (c-a)^2] \leq \frac{1}{2}[a^2 + b^2 + c^2] \ .$$

(b) Does there exist a number k for which

$$\min \ [(a-b)^2, (a-c)^2, (a-d)^2, (b-c)^2, (b-d)^2, (c-d)^2] \leq k[a^2 + b^2 + c^2 + d^2]$$

for any real numbers a, b, c, d? If so, determine the smallest such k.
[*Bonus:* Determine if there is a generalization.]

491. Given that x and y are positive real numbers for which $x + y = 1$ and that m and n are positive integers exceeding 1, prove that

$$(1 - x^m)^n + (1 - y^n)^m > 1 \ .$$

492. The faces of a tetrahedron are formed by four congruent triangles. if α is the angle between a pair of opposite edges of the tetrahedron, show that

$$\cos \alpha = \frac{\sin(B - C)}{\sin(B + C)}$$

where B and C are the angles adjacent to one of these edges in a face of the tetrahedron.

493. Prove that there is a natural number with the following characteristics: (a) it is a multiple of 2007; (b) the first four digits in its decimal representation are 2009; (c) the last four digits in its decimal representation are 2009.

494. (a) Find all real numbers x that satisfy the equation

$$(8x - 56)\sqrt{3 - x} = 30x - x^2 - 97 \ .$$

(b) Find all real numbers x that satisfy the equation

$$\sqrt{x} + \sqrt[3]{x + 7} = \sqrt[4]{x + 80} \ .$$

495. Let $n \geq 3$. A regular $n-$gon has area S. Squares are constructed externally on its sides, and the vertices of adjacent squares that are not vertices of the polygon are connected to form a $2n-$sided polygon, whose area is T. Prove that $T \leq 4(\sqrt{3} + 1)S$. For what values of n does equality hold?

496. Is the hundreds digit of $N = 2^{2006} + 2^{2007} + 2^{2008}$ even or odd? Justify your answer.

497. Given $n \geq 4$ points in the plane with no three collinear, construct all segments connecting two of these points. It is known that the length of each of these segments is a positive integer. Prove that the lengths of at least 1/6 of the segments are multiples of 3.

498. Let a be a real parameter. Consider the simultaneous sytem of two equations:

$$\frac{1}{x+y} + x = a - 1 \; ; \tag{1}$$

$$\frac{x}{x+y} = a - 2 \; . \tag{2}$$

(a) For what value of the parameter a does the system have exactly one solution?

(b) Let $2 < a < 3$. Suppose that (x, y) satisfies the sytem. For which value of a in the stated range does $(x/y) + (y/x)$ reach its maximum value?

499. The triangle ABC has all acute angles. The bisector of angle ACB intersects AB at L. Segments LM and LN with $M \in AC$ and $N \in BC$ are constructed, perpendicular to the sides AC and BC respectively. Suppose that AN and BM intersect at P. Prove that CP is perpendicular to AB.

500. Find all sets of distinct integers $1 < a < b < c < d$ for which $abcd - 1$ is divisible by $(a-1)(b-1)(c-1)(d-1)$.

501. Given a list of $3n$ not necessarily distinct elements of a set S, determine necessary and sufficient conditions under which these $3n$ elements can be divided into n triples, none of which consist of three distinct elements.

502. A set consisting of n men and n women is partitioned at random into n disjoint pairs of people. What are the expected value and variance of the number of male-female couples that result? (The *expected value* E is the average of the number N of male-female couples over all possibilities, *i.e.* the sum of the numbers of male-female couples for the possibilities divided by the number of possibilities. The *variance* is the average of the difference $(E-N)^2$ over all possibilities, *i.e.* the sum of the values of $(E-N)^2$ for the possibilities divided by the number of possibilities.)

503. A natural number is *perfect* if it is the sum of its proper positive divisors. Prove that no two consecutive numbers can both be perfect.

504. Find all functions f taking the real numbers into the real numbers for which the following conditions hold simultaneously:

(a) $f(x + f(y) + yf(x)) = y + f(x) + xf(y)$ for every real pair (x, y);

(b) $\{f(x)/x : x \neq 0\}$ is a finite set.

505. What is the largest cubical present that can be completely wrapped (without cutting) by a unit square of wrapping paper?

506. A two-person game is played as follows. A position consists of a pair (a, b) of positive integers. Playes move alternately. A move consists of decreasing the larger number in the current position by any positive multiple of the smaller number, as long as the result remains positive. The first player unable to make a move loses. (This happens, for example, when $a = b$.) Determine those positions (a, b) from which the first player can guarantee a win with optimal play.

507. Prove that, if a, b, c are positive reals, then

$$\log^2 \frac{ab}{c} + \log^2 \frac{bc}{a} + \log^2 \frac{ca}{b} + \frac{3}{4} \geq \log(abc) \; .$$

508. Let a, b, c be integers exceeding 1 for which both $\log_a b + \log_b a$ and $\log_a^2 b + \log_b^2 a$ are rational. Prove that, for every positive integer n, $\log_a^n b + \log_b^n a$ is rational.

509. Let $ABCDA'B'C'D'$ be a cube where the point O is the centre of the face $ABCD$ and $|AB| = 2a$. Calculate the distance from the point B to the line of intersection of the planes $A'B'O$ and $ADD'A'$ and the distance between AB' and BD. (AA', BB', CC', DD' are edges of the cube.)

510. Solve the equation

$$\sqrt[3]{x^2+2} + \sqrt[3]{4x^2+3x-2} = \sqrt[3]{3x^2+x+5} + \sqrt[3]{2x^2+2x-5} \ .$$

511. Find the sum of the last 100 digits of the number

$$A = 1 \cdot 2 \cdot 3 \cdot \ldots \cdot 2005 \cdot 2006 + 2007 \ .$$

512. Prove that

$$\binom{3n}{n} = \sum_{k=0}^{n} \binom{2n}{k}\binom{n}{k}$$

when $n \geq 1$.

513. Solve the equation

$$2^{1-2\sin^2 x} = 2 + \log_2(1 - \sin^2 x) \ .$$

514. Prove that there do not exist polynomials $f(x)$ and $g(x)$ with complex coefficients for which

$$\log_b x = \frac{f(x)}{g(x)}$$

where b is any base exceeding 1.

515. Let n be a fixed positive integer exceeding 1. To any choice of n real numbers x_i satisfying $0 \leq x_i \leq 1$, we can associate the sum

$$\sum \{|x_i - x_j| : 1 \leq i < j \leq n\} \ .$$

What is the maximum possible value of this sum and for which values of the x_i is it assumed?

516. Let $n \geq 1$. Is it true that, for any $2n+1$ positive real numbers $x_1, x_2, \cdots, x_{2n+1}$, we have that

$$\frac{x_1 x_2}{x_3} + \frac{x_2 x_3}{x_4} + \cdots + \frac{x_{2n+1} x_1}{x_2} \geq x_1 + x_2 + \cdots + x_{2n+1} \ ,$$

with equality if and only if all the x_i are equal?

517. A man bought four items in a *Seven-Eleven* store. The clerk entered the four prices into a pocket calculator and *multiplied* to get a result of 7.11 dollars. When the customer objected to this procedure, the clerk realized that he should have added and redid the calculation. To his surprise, he again got the answer 7.11. What did the four items cost?

518. Let I be the incentre of triangle ABC, and let AI, BI, CI, produced, intersect the circumcircle of triangle ABC at the respective points D, E, F. Prove that $EF \perp AD$.

519. Let AB be a diameter of a circle and X any point other than A and B on the circumference of the circle. Let t_A, t_B and t_X be the tangents to the circle at the respective points A, B and X. Suppose that AX meets t_B at Z and BX meets t_A at Y. Show that the three lines YZ, t_X and AB are either concurrent (i.e. passing through a common point) or parallel.

520. The *diameter* of a plane figure is the largest distance between any pair of points in the figure. Given an equilateral triangle of side 1, show how, by a stright cut, one can get two pieces that can be rearranged to form a figure with maximum diameter

(a) if the resulting figure is convex (*i.e.* the line segment joining any two of its points must lie inside the figure);

(b) if the resulting figure is not necessaarily convex, but it is connected (*i.e.* any two points in the figure can be connected by a curve lying inside the figure).

521. On a 8×8 chessboard, either $+1$ or -1 is written in each square cell. Let A_k be the product of all the numbers in the kth row, and B_k the product of all the numbers in the kth column of the board $(k = 1, 2, \cdots, 8)$. Prove that the number

$$A_1 + A_2 + \cdots + A_8 + B_1 + B_2 + \cdots + B_8$$

is a multiple of 4.

522. (a) Prove that, in each scalene triangle, the angle bisector from one of its vertices is always "between" the median and the altitude from the same vertex.

(b) Find the measures of the angles of a triangle if the lengths of the median, the angle bisector and the altitude from one of its vertices are in the ratio $\sqrt{5} : \sqrt{2} : 1$.

523. Let ABC be an isosceles triangle with $AB = AC$. The segments BC and AC are used as hypotenuses to construct three right triangles BCM, BCN and ACP. Prove that, if $\angle ACP + \angle BCM + \angle BCN = 90°$, then the triangle MPN is isosceles.

524. Solve the irrational equation

$$\frac{7}{\sqrt{x^2 - 10x + 26} + \sqrt{x^2 - 10x + 29} + \sqrt{x^2 - 10x + 41}} = x^4 - 9x^3 + 16x^2 + 15x + 26 \ .$$

525. The circle inscribed in the triangle ABC divides the median from A into three segments of the same length. If the area of ABC is $6\sqrt{14}$, calculate the lengths of its sides.

526. For the non-negative numbers a, b, c, prove the inequality

$$4(a + b + c) \geq 3(a + \sqrt{ab} + \sqrt[3]{abc}) \ .$$

When does equality hold?

527. Consider the set A of the $2n-$digit natural numbers, with 1 and 2 each occurring n times as a digit, and the set B of the $n-$digit numbers all of whose digits are 1, 2, 3, 4 with the digits 1 and 2 occurring with equal frequency. Show that A and B contain the same number of elements (*i.e.*, have the same cardinality).

Solutions

479. Let x, y, z be positive integer for which

$$\frac{1}{x} + \frac{1}{y} = \frac{1}{z}$$

and the greatest common divisor of x and z is 1. Prove that $x + y$, $x - z$ and $y - z$ are all perfect squares. Give two examples of triples (x, y, z) that satisfy these conditions.

Solution 1. [G. Ghosn] Since $(1/y) = (x - z)/(xz)$ and gcd $(x, x - z) =$ gcd $(z, x - z) = 1$, the fractions on both sides of the equation are in lowest terms, and so $x - z = 1$ and $xz = y$. Hence $x + y = x(1 + z) = x^2$ and $y - z = z(x - 1) = z^2$.

Solution 2. Since $z(x+y) = xy$ and the greatest common divisor of x and z is 1, x, being a divisor of $z(x+y)$ must be a divisor of $x+y$ and so of y. Let $y = ux$ for some positive integer u. Then $z(1+u) = ux$. Since u and $1+u$ have greatest common divisor 1, u must divide z and $1+u$ must divide x, Hence $z = uv$ and $x = (1+u)w$, for some positive integers v and w. Therefore $uv(1+u) = u(1+u)w$, whence $v = w$.

Therefore $(x,y,z) = ((1+u)v, u(1+u)v, uv)$. Since x and z have greatest common divisor 1, $v = 1$ and $(x,y,z) = (1+u, u(1+u), u)$. This satisfies the given equation as well as $x + y = (1+u)^2 = x^2$, $x - z = 1$ and $y - z = u^2 = z^2$. Particular examples are $(x,y,z) = (2,2,1), (3,6,2), (4,12,3), (5,20,4)$.

Solution 3. We have that $z(x+y) = xy$ and $x(y-z) = yz$. Since gcd $(x,z) = 1$, z and x both must divide y, so that $y = vz = wx$ for some positive integers v and w. Since $z(1+w)x = xvz$, $1 + w = v$ and gcd $(v,w) = 1$. Since $wx = vz$,,we must have that $x = v$ and $z = w$ and $y = vw$. This satisfies the equation as well as $x + y = v^2$, $x - z = 1$ and $y - z = w^2$.

Solution 4. [K. Huynh] Observe that $x > y$ and $z > y$. From the equation, we obtain that $xz + yz = xy$ whence $(x-z)(y-z) = z^2$. Since gcd $(x,z) = 1$, there is no prime that divides $x - z$ and z^2, so that gcd $(x - z, z^2) = 1$. Therefore $x - z = 1$, $y - z = z^2$, $y = z^2 + z$ and $x + y = (z+1)^2$.

480. Let a and b be positive real numbers for which $60^a = 3$ and $60^b = 5$. Without the use of a calculator or of logarithms, determine the value of

$$12^{\frac{1-a-b}{2(1-b)}} .$$

Solution 1. [V. Zhou]

$$\begin{aligned}
12^{\frac{1-a-b}{2(1-b)}} &= \left(\frac{60}{5}\right)^{\frac{1-a-b}{2(1-b)}} = 60^{(1-b)\cdot(\frac{1-a-b}{2(1-b)})} \\
&= \left(\frac{60}{60^{a+b}}\right)^{\frac{1}{2}} = \left(\frac{60}{60^a \cdot 60^b}\right)^{\frac{1}{2}} \\
&= \left(\frac{60}{3 \times 5}\right)^{\frac{1}{2}} = 2 .
\end{aligned}$$

Solution 2. Since $60^b = 5$, $12^b = 5^{1-b}$ and $5 = 12^{b/(1-b)}$. Since $60^a = 3$, $2^2 5^a 12^a = 12$. Therefore

$$2^2 = 12^{1-a}5^{-a} = 12^{1-a}12^{-ab/(1-b)} = 12^{(1-a-b+ab-ab)/(1-b)} = 12^{(1-a-b)/(1-b)} .$$

Therefore $2 = 12^{(1-a-b)/2(1-b)}$.

Solution 3. [A. Guo; D. Shi] Since $a = \log_{60} 3$ and $b = \log_{60} 5$,

$$1 - (a+b) = 1 - \log_{60}(15) = \log_{60}(60/15) = \log_{60} 4 .$$

Also, $1 - b = 1 - \log_{60} 5 = \log_{60} 12$, so that

$$\frac{1-a-b}{1-b} = \frac{\log_{60} 4}{\log_{60} 12} = \log_{12} 4 = 2\log_{12} 2 .$$

Therefore

$$12^{\frac{1-a-b}{2(1-b)}} = 12^{\log_{12} 2} = 2 .$$

481. In a certain town of population $2n+1$, one knows those to whom one is known. For any set A of n citizens, there is some person among the other $n+1$ who knows everyone in A. Show that some citizen of the town knows all the others.

Solution 1. [K. Huynh] We prove that there is a set of $n+1$ people in the town, each of whom knows (and is known by) each of the rest. First, observe that for any set of k people, with $k \leq n$, there is a person not among them who knows them all. This follows by augmenting the set to n people and applying the condition of the problem.

Let p_1 be any person. There is a person, say p_2 who knows p_1. A person p_3 can be found who knows both p_1 and p_2, so that $\{p_1, p_2, p_3\}$ is a triplet each of whom knows the other two. Suppose, as an induction hypothesis, that $3 \leq k \leq n$, and $\{p_1, p_2, \cdots, p_k\}$ is a set of k people any pair of whom know each other. By the foregoing observation, there is another person p_{k+1} who knows them all. By induction, we can find a set $\{p_1, p_2, \cdots, p_{n+1}\}$, each pair of whom know each other.

Consider the remaining n people. There must be one among the p_i who knows all of these remaining people. This person p_i therefore knows everyone.

Solution 2. Let us suppose that the persons are numbered from 0 to $2n$ inclusive. The notation $(a : a_1, a_2, \cdots, a_k)$ will mean that a is knows and is known by each of $a_1, a_2, \cdots, a_k$. Begin with the set $\{1, 2, \cdots, n\}$; some person, say 0, knows everyone in this set, so that

$$(0 : 1, 2, 3, \cdots, n) \ .$$

If person 0, knows everyone else, then we are done. Otherwise, there is a person, say, $n+1$, not known to 0, so that everyone in the set $\{n+1, n+2, \cdots, 2n\}$, is known by a person in the first set, say 1, so that

$$(1 : 0, n+1, n+2, \cdots, 2n) \ .$$

Consider the set $\{0, 2, 3, \cdots, n\}$. If 1 knows everyone in this set, then 1 knows everyone and we are done. If 1 does not know everyone in this set, then there is someone else, say $n+1$, who does, so that

$$(n+1 : 0, 1, \cdots, n) \quad \text{and} \quad (0 : 1, 2, \cdots, n+1) \ .$$

If 0 knows everyone in the set $\{1, n+2, \cdots, 2n\}$, then 0 knows everyone; if $n+1$ knows everyone in this set, then $n+1$ knows everyone, and we are done. If not, then there is a person 2, say, who knows everyone in the set:

$$(2 : 0, 1, n+1, n+2, \cdots, 2n) \ .$$

Consider the set $\{0, 3, \cdots, n, n+1\}$. If 1 or 2 knows everyone in this set, then 1 or 2 knows everybody and we are done. Otherwise, there is a person, say $n+2$ who knows everyone in the set, so that

$$(n+2 : 0, 1, 2, \cdots, n+1) \quad \text{and} \quad (0 : 1, 2, \cdots, n+1, n+2) \ .$$

We can continue on in this way either until we find someone that knows everyone, or until we reach the ith stage for which

$$(i : 0, 1, 2, \cdots, i-1, n+1, \cdots, 2n) \quad \text{and} \quad (n+i : 0, 1, 2, \cdots, n, n+1, \cdots, n+i-1) \ .$$

If we get to the nth stage, then n and $2n$ each know everyone.

482. A trapezoid whose parallel sides have the lengths a and b is partitioned into two trapezoids of equal area by a line segment of length c parallel to these sides. Determine c as a function of a and b.

Solution. Let u be the distance between the segment of length a and that of length c, and v the distance between the segment of length c and that of length b. Then

$$\frac{u+v}{u} = \frac{b-a}{c-a} \ .$$

From the area condition, we have that

$$2\left(\frac{c+a}{2}\right)u = \left(\frac{b+a}{2}\right)(u+v) = \left(\frac{b^2-a^2}{2(c-a)}\right)u \ ,$$

whence $2(c^2-a^2) = b^2-a^2$ and $c^2 = \frac{1}{2}(a^2+b^2)$. Therefore

$$c = \sqrt{\frac{a^2+b^2}{2}} \ .$$

483. Let A and B be two points on the circumference of a circle, and E be the midpoint of arc AB (either arc will do). Let P be any point on the minor arc EB and N the foot of the perpendicular from E to AP. Prove that $AN = NP + PB$.

Solution 1. Produce ANP to M so that $AN = NM$. Then $EM = AE = EB$. Hence $\angle EBM = \angle EMB$, so that

$$\angle PBM = \angle EBM - \angle EBP = \angle EMB - \angle EAP = \angle EMB - \angle EMA = \angle PMB \ .$$

Therefore $PB = PM$, so that

$$AN = NM = NP + PM = NP + PB \ .$$

Solution 2. [V. Zhou] Determine Q on AN so that $AQ = BP$. Then, also, $\angle EAQ = \angle EAP = \angle EPB$ and $AE = EB$, so that triangles AEQ and BEP are congruent. Hence $EQ = EP$ and so $QN = NP$. Therefore $AN = QN + AQ = NP + PB$.

Solution 3. [Y. Wang] Let O be the centre and r the radius of the circle. Let F and G be the respective midpoints of AP and AB. Then $FG \| BP$ and, since $\angle AFO = \angle AGO = 90°$, the quadrilateral $AFGO$ is concyclic.

Let $\alpha = \angle AOF = \angle AGF$ and $\beta = \angle AOE = \angle BOE$. Then

$$\angle PAB = \angle FAG = \angle FOG = \angle FOE = \angle NEO = \beta - \alpha \ .$$

Also, $|FN| = |OE|\sin(\beta-\alpha) = r\sin(\beta-\alpha)$ and $|AF| = r\sin\alpha$. By the Law of Sines applied to triangle AFG,

$$\frac{|FG|}{\sin(\beta-\alpha)} = \frac{|AF|}{\sin\alpha} = r,$$

whence $|FG| = r\sin(\beta-\alpha) = |FN|$. Hence $AN = PF + FN = PN + 2FN = PN + 2FG = NP + PB$.

484. ABC is a triangle with $\angle A = 40°$ and $\angle B = 60°$. Let D and E be respective points of AB and AC for which $\angle DCB = 70°$ and $\angle EBC = 40°$. Furthermore, let F be the point of intersection of DC and EB. Prove that $AF \perp BC$.

Solution 1. [J. Schneider] Let AH be the altitude from A to BC. We apply the converse of Ceva's Theorem in the trigonometric form to show that the cevians AH, BE and CD concur.

$$\frac{\sin 30° \sin 40° \sin 10°}{\sin 10° \sin 20° \sin 70°} = \frac{\sin 30° (2 \sin 20° \cos 20°)}{\sin 20° \cos 20°} = 2\sin 30° = 1 \ .$$

Hence AH, BE and CD concur, so that AH passes through F and the result follows.

Solution 2. [A. Siddhour] In triangle BCF, since $\angle CBF = 40°$ and $\angle CBF = 40°$, it follows that $\angle BFC = 70° = \angle CBF$ and $BF = BC$. Hence $|BF| = a$ (using the standard convention for lengths of the sides of the triangle ABC). Assign coordinates:

$$B \sim (0,0), \quad C \sim (a,0), \quad A \sim (c\cos 60°, c\sin 60°), \quad F \sim (a\cos 40°, a\sin 40° \ .$$

By the Law of sines, we have that $c\sin 40° = a\sin 80°$, whence $c = 2a\cos 40°$.

We have that

$$\begin{aligned}\overrightarrow{FA}\cdot\overrightarrow{BC} &= (c\cos 60° - a\cos 40°, c\sin 60° - a\sin 60°)\cdot(a,0)\\ &= a(2a\cos 40°\cos 60° - a\cos 40° = a\cos 40° - a\cos 40° = 0\ ,\end{aligned}$$

from which it follows that $AF \perp BC$.

Solution 3. [Y. Wang] The result will follow if one can show that $\angle FAC = 10°$. Since $\angle FCA = \angle BCA - \angle DCB = 80° - 70° = 10°$, it is enough to show that the perpendicular from F to AC bisects AC, *i.e.*, $2|CF|\cos\angle FCA = |AC|$.

Since $\angle FBC = 40°$ and $\angle BCF = 70°$, it follows that $\angle BFC = 70°$ so that $|CF| = 2|BC|\cos 70°$. Since $BC : AC = \sin\angle BAC : \sin\angle ABC = \sin 40° : \sin 60°$,

$$2|CF|\cos\angle FCA = 4|BC|\cos 70°\cos 10° = 4|AC|\sin 40°\sin 20°\sin 80°/\sin 60°\ .$$

For each angle θ,

$$\begin{aligned}4\sin\theta\sin(60°+\theta)\sin(60°-\theta) &= 2\sin\theta[\cos 2\theta - \cos 120°]\\ &= 2\sin\theta\cos 2\theta + 2\sin\theta\sin 30°\\ &= \sin 3\theta - \sin\theta + \sin\theta = \sin 3\theta\ .\end{aligned}$$

When $\theta = 20°$, this becomes $4\sin 20°\sin 40°\sin 80° = \sin 60°$. so that $2|CF|\cos\angle FCA = |AC|$, as desired.

Solution 4. Since $\angle BFC = 70° = \angle BCD$, $BF = BC$. Let $|BF| = |BC| = 1$, $|AF| = u$ and $|CF| = v$. Let $\angle BAF = \theta$, so that $\angle CAF = 40° - \theta$. By the Sine law applied to triangles BFC and AFC,

$$\frac{\sin 40°}{\sin 70°} = v = \frac{u\sin(40°-\theta)}{\sin 10°}\ .$$

By the Sine Law applied to triangle ABF, $u = \sin 20°/\sin\theta$. Hence

$$\frac{\sin 40°}{\sin 70°} = \frac{\sin 20°\sin(40°-\theta)}{\sin 10°\sin\theta}\ ,$$

so that

$$\sin 10°\sin 40°\sin\theta = \sin 20°\cos 20°\sin(40°-\theta)\ ,$$

whence

$$2\sin 10°\sin\theta = \sin(40°-\theta) = \sin 40°\cos\theta - \cos 40°\sin\theta$$

and

$$\sin\theta(2\sin 10° + \cos 40°) = \cos\theta\sin 40°\ .$$

Now

$$\begin{aligned}2\sin 10° + \cos 40° &= \sin 10° + (\sin 10° + \sin 50°)\\ &= \sin 10° + 2\sin 30°\cos 20° = \sin 10° + \sin 70°\\ &= 2\sin 40°\cos 30° = \sqrt{3}\sin 40°\ .\end{aligned}$$

Hence $\sqrt{3}\sin\theta = \cos\theta$, so that $\cot\theta = \sqrt{3}$. Hence $\theta = 30°$ and the result follows.

Solution 5. [K. Huynh] Let a, b, c be the sides of triangle ABC according to convention. Since $\angle BFC = \angle FCB = 70°$, $|BF| = |BC| = a$. Let the respective feet of the perpendiculars from A and F to BC be P and Q. Then $|BP| = c\cos 60° = c/2$ and $|BQ| = a\cos 40°$. From the Law of Sines, $a\sin 80° = c\sin 40°$, so that $c = 2a\cos 40°$. Hence $BP = BQ$, and the result follows.

Solution 6. [G. Ghosn] Applying the Law of Sines to triangles BCE and BEA using their common side BE, we obtain that

$$\frac{|EC|}{|EA|} = \left(\frac{\sin 40^\circ}{\sin 80^\circ}\right)\left(\frac{\sin 40^\circ}{\sin 20^\circ}\right) = \frac{\sin^2 40^\circ}{\sin 20^\circ \sin 80^\circ} = \frac{2\cos 20^\circ \sin 40^\circ}{\sin 80^\circ} \ .$$

Similarly,

$$\frac{|DA|}{|DB|} = \frac{\sin 10^\circ \sin 60^\circ}{\sin 40^\circ \sin 70^\circ} \ .$$

By Ceva's therem

$$\begin{aligned} 1 &= \frac{|EC|}{|EA|}\frac{|DA|}{|DB|}\frac{|MB|}{|MC|} \\ &= \frac{2\cos 20^\circ \sin 40^\circ \sin 10^\circ \sin 60^\circ}{\sin 80^\circ \sin 40^\circ \sin 70^\circ}\frac{|MB|}{|MC|} \\ &= \frac{2\cos 80^\circ \sin 60^\circ}{\sin 80^\circ}\frac{|MB|}{|MC|} \ , \end{aligned}$$

whence we find that $|MB| : |MC| = \tan 80^\circ : \tan 60^\circ$.

Let AN be an altitude of triangle ABC, so that $|AN| = |NB|\tan 60^\circ = |CN|\tan 80^\circ$. Then $MB : MC = NB : NC$, so that $M = N$ and the desired result follows.

485. From the foot of each altitude of the triangle, perpendiculars are dropped to the other two sides. Prove that the six feet of these perpendiculars lie on a circle.

Solution 1. Let ABC be the triangle with altitudes AP, BQ and CR; let H be the orthocentre. Let $PU \perp AB$, $QV \perp BC$, $RW \perp CA$, $PX \perp CA$, $QY \perp AB$ and $RZ \perp BC$, where $U, Y \in AB$; $V, Z \in BC$; and $W, X \in CA$.

Consider triangles AQR and ABC. Since $ARHQ$ is concyclic (right angles at Q and R),

$$\angle ARQ = \angle AHQ = \angle BHP = 90^\circ - \angle HBP = 90^\circ - \angle QBC = \angle ACB \ .$$

Similarly, $\angle AQR = \angle ABC$. Thus, triangles AQR and ABC are similar, the similarity being implemented by a dilatation of centre A followed by a reflection about the bisector of angle BAC. Since QY and RW are altitudes of triangle AQR, triangle AYW is formed from triangle AQR as triangle AQR is formed from triangle ABC. Hence triangles AYW and AQR are similar by the combination of a dilatation with centre A and a reflection about the bisector of angle BAC.

Therefore, triangle AYW and ABC are directly similar and $YW \| BC$. Similarly triangles BZU and BCA as well as triangles CXV and CAB are similar and $ZU \| CA$ and $XV \| AB$. (We note that this means that $XWYUZV$ is a hexagon with opposite sides parallel, although this is not needed here.)

Since $PX \| HQ$ and $PU \| HR$, $AU : AR = AP : AH = AX : AQ$, so that there is a dilatation taking $U \rightarrow R$, $P \rightarrow H$ and $X \rightarrow Q$. Therefore $UX \| RQ$ and triangle AXU is similar to triangle AQR and to triangle ABC.

Consider quadrilateral $UZVX$.

$$\begin{aligned} \angle UZV + \angle UXV &= (180^\circ - \angle BZU) + (180^\circ - \angle AXU - \angle CXV) \\ &= (180^\circ - \angle ACB) + (180^\circ - \angle ABC - \angle BAC) = 180^\circ \ . \end{aligned}$$

Hence $UZVX$ is concyclic. Similarly, $VXWY$ and $WYUZ$ are concyclic.

Since triangles AYW and AXU are similar with $\angle AWY = \angle AUX$ and $\angle AYW = \angle AXU$, $XWYU$ is concyclic. Similarly, $YUZV$ and $ZVXW$ are concylclic. Hence $XWYUZV$ is a hexagon, any consecutive four vertices of which are concylcic, and so is itself concyclic.

Solution 2. [K. Huynh] Let a, b, c be the lengths of the sides and A, B, C the angles of the triangle ABC according to convention. Use the notation of Solution 1. We have that $|BU| = |BP|\cos B = (c\cos B)\cos B = c\cos^2 B$. Similarly, $|BZ| = a\cos^2 B$, $|AY| = c\cos^2 A$ and $|CV| = a\cos^2 C$. Therefore, $|BY| = c(1-\cos^2 A) = c\sin^2 A$ and $|CV| = a(1-\cos^2 C) = a\sin^2 C$.

Since $a\sin C = c\sin A$,

$$\begin{aligned}|BU||BY| &= (c\cos^2 B)(a\sin^2 A) = \cos^2 B(c\sin A)^2 \\ &= \cos^2 B(a\sin C)^2 = (a\cos^2 B)(a\sin^2 C) = |BZ||BV| \ .\end{aligned}$$

from which, by a power-of-the-point argument [give details!], we deduce that $YUZV$ is concyclic. Similarly, $ZVXW$ and $XWYU$ are concyclic.

Suppose that the circumcircle of $YUZV$ intersects AZ at L and the circumcircle of $ZVXW$ intersects AZ at M. Since $XWYU$ is concyclic, $|AY||AU| = |AW||AX|$. Therefore,

$$|AL||AZ| = |AY||AU| = |AW||AX| = |AM||AZ| \ .$$

Hence $L = M$. Thus, the circumcircles of $YUZV$ and $ZVXW$ share three noncollinear points, Z, V and $L = M$, and so must coincide. Similarly, each coincides with the circumcircle of $XWYU$ and the result follows.

486. Determine all quintuplets (a,b,c,d,u) of nonzero integers for which

$$\frac{a}{b} = \frac{c}{d} = \frac{ab+u}{cd+u} \ .$$

Solution 1. If $a = b$ and $c = d$, then an integer u exists as in the problem if and only if $|a| = |c|$. Suppose that none of the fractions is equal to 1.

Let $a/b = c/d = r/s$ where the greatest common divisor of r and s is 1. Then there are integers v and w for which $(a,b) = (vr, vs)$ and $(c,d) = (wr, ws)$. If there exists a number u satisfying the conditions of the problem, then

$$\frac{r}{s} = \frac{v^2rs+u}{w^2rs+u} \Leftrightarrow rs(w^2r - v^2s) = (s-r)u \ .$$

Since the greatest common divisor of r and s is 1, it follows that gcd $(rs, r-s) = 1$ so that $s-r$ must divide

$$w^2r - v^2s = (w^2-v^2)r - v^2(s-r) \ ,$$

and so must divide $w^2 - v^2$.

For the converse, suppose that r and s are chosen arbitrarily, and that v and w are chosen to satisfy $w^2 - v^2 = (s-r)n$ for some integer n. Let $(a,b,c,d) = (vr, vs, wr, ws)$. Then $w^2r - v^2s = (s-r)(rn-v^2)$. Let $u = rs(rn - v^2)$. Thence,

$$\begin{aligned}\frac{ab+u}{cd+v} &= \frac{v^2rs + rs(rn-v^2)}{w^2rs + rs(rn-v^2)} = \frac{v^2+rn-v^2}{w^2+rn-v^2} \\ &= \frac{rn}{(s-r)n+rn} = \frac{rn}{sn} = \frac{r}{s} \ ,\end{aligned}$$

as desired.

Hence (a,b,c,d,u) satisfies the given condition if and only if $a:c = b:d = v:w$, $a:b = c:d = r:s$ where $w^2 - v^2 \equiv 0 \pmod{s-r}$ and $u = rs(rn - v^2)$ for some integer n.

Solution 2. Let $a/b = c/d = r/s$ where gcd $(r,s) = 1$. Let $(a,b) = (vr, vs)$ and $(c,d) = (wr, ws)$. Then

$$\frac{r}{s} = \frac{v^2rs + u}{w^2rs + u} \ ,$$

so that $v^2rs + u = pr$ and $w^2rs + u = ps$ for some integer p. Since gcd $(r,s) = 1$, $u = qrs$ for some integer q, and

$$\frac{r}{s} = \frac{v^2 + q}{w^2 + q} \ .$$

Thus, $v^2 + q = hr$ and $w^2 + q = hs$ for some integer h and $w^2 - v^2 = h(s - r)$. We can now conclude as in the first solution.

Solution 3. The equation $a/c = (ab + u)/(cd + u)$ leads to $u(a - c) = ac(b - d)$. Similarly, $u(b - d) = bd(a - c)$. Multiplying the two equations leads to

$$u^2(a - c)(b - d) = (abcd)(a - c)(b - d) = (ad)^2(a - c)(b - d) \ .$$

Therefore, either $(a,b) = (c,d)$ or $ad = bc = u$.

Conversely, pick any positive number u and let a and c be positive divisors of u. Suppose that $d = u/a$ and $b = u/c$. Then

$$\begin{aligned}\frac{ab + u}{cd + u} &= \frac{(au/c) + u}{(cu/a) + u} \\ &= \frac{au(a + c)}{cu(a + c)} = \frac{a}{c} = \frac{b}{d} \ .\end{aligned}$$

Therefore the system of equations is satisfied when $(a,b) = (c,d)$ or a,b,c,d are integers for which $ad = bc = u$.

Comment. A. Siddour discovered the particular family

$$(a, b, c, d, u) = (tr^2, trs, trs, ts^2, t^2r^2s^2) \ ,$$

for parameters r, s, t.

487. ABC is an isosceles triangle with $\angle A = 100°$ and $AB = AC$. The bisector of angle B meets AC in D. Show that $BD + AD = BC$.

Solution 1. We have that $\angle ACB = \angle ABC = 40°$ and $\angle ABD = \angle DBC = 20°$. Locate E on BC so that $\angle BED = 80°$. Then $\angle BDE = 80°$ so that $BD = BE$.

Since $\angle DEC = 100°$ and $\angle DCE = 40°$, $\angle EDC = 40°$, so that $DE = EC$.

Since $\angle BAD + \angle BED = 180°$, the quadrilateral $ADEB$ is concyclic. Because BD bisects angle ABE, it bisects arc ADE and so $AD = DE$. It follows that

$$BD + AD = BE + EC = BC \ .$$

Comment. A variant on this argument involves a point F on the segment BE so that $BF = BA$ and $\angle DFE = \angle DEF = 80°$. From the congruence of triangles ABD and FBD, we get $AD = DF = DE = EC$, from which the result follows.

Solution 2. [A. Siddour] By the Law of Sines, $AD : BD = \sin 20° : \sin 100°$ and $BC : BD = \sin 60° : \sin 40°$. Hence

$$\begin{aligned}1 + \frac{AD}{BD} &= 1 + \frac{\sin 20°}{\sin 100°} = \frac{\sin 100° + \sin 20°}{\sin 80°} \\ &= \frac{2\sin 60° \cos 40°}{2\sin 40° \cos 40°} = \frac{\sin 60°}{\sin 40°} = \frac{BC}{CD} \ ,\end{aligned}$$

from which the result follows.

488. A host is expecting a number of children, which is either 7 or 11. She has 77 marbles as gifts, and distributes them into n bags in such a way that whether 7 or 11 children come, each will receive a number of bags so that all 77 marbles will be shared equally among the children. What is the minimum value of n?

Solution: 17 bags will suffice. A systematic way to approach the problem is to number the marbles and line them up in order. For a distribution to 11 children, place a marker after every seventh one (numbers 7, 14, *etc.*); for a distribution to 7 children, place a marker after every eleventh one (number 11, 22, *etc.*). This will require $6 + 10 = 16$ markers. This partitions the marbles into seventeen groups, that can be bagged accordingly. With $a \times b$ representing a bags with b marbles, we have 5×7, 2×6, 2×5, 2×4, 2×3, 2×2 and 2×1 for a distribution of

$$7, 4 + 3, 7, 1 + 6, 5 + 2, 7, 2 + 5, 6 + 1, 7, 3 + 4, 7$$

to eleven children and

$$7 + 4, 3 + 7 + 1, 6 + 5, 2 + 7 + 2, 5 + 6, 1 + 7 + 3, 4 + 7$$

to seven children.

Alternative distributions are 6×7, 5×4, 4×3, 1×2, 1×1 for distributions of

$$7, 7, 7, 7, 7, 7, 4 + 3, 4 + 3, 4 + 3, 4 + 3, 4 + 2, +1$$

and

$$7 + 4, 7 + 4, 7 + 4, 7 + 4, 7 + 4, 7 + 3 + 1, 3 + 3 + 3 + 2\ ,$$

and

$$7 \times 7, 4 \times 4, 3 \times 3, 3 \times 1$$

for distributions of

$$7, 7, 7, 7, 7, 7, 7, 4 + 3, 4 + 3, 4 + 3, 4 + 1 + 1 + 1$$

and

$$7 + 4, 7 + 4, 7 + 4, 7 + 4, 7 + 3 + 1, 7 + 3 + 1, 7 + 3 + 1\ .$$

Solution 1: Fewer than 17 bags will not suffice. [B. Wu] Construct a graph whose vertices consist of a set $X \cup Y$ and edges consist of a set E, where $\#X = 7$, $\#Y = 11$, representing the children if, respectively, seven and eleven show up, and the only edges join pairs (x, y) with $x \in X$ and $y \in Y$ when a bag assigned to x in one distribution goes to y in the other distribution.

This graph is connected (*i.e.*, one can construct a path of edges that is incident with every vertex). If a connected component contains m vertices of X and n vertices of Y, then the number of marbles represented by these vertices is $11m = 7n$. Since 11 and 7 are coprime, $(m, n) = (7, 11)$ and each connected component must contain all of the vertices. Since each connected graph with 18 vertices must contain at least 17 edges, at least 17 bags are necessary.

Solution 2: Fewer than 17 bags will not suffice. Since we must accommodate 11 children, no bag can contain more than 7 marbles. There cannot be more than seven bags with 7 marbles; otherwise, one of the seven children must get two bags of 7 and so more than 11 marbles. Let a *large bag* contain 4 to 6 marbles and a *small bag* contain 1 to 3 marbles.

If there are five or fewer bags with 7 marbles, then at least six of eleven children each get two bags, and we will need at least $5 + 6 \times 2 = 17$ bags. Suppose that there are six bags with 7 marbles. Then six of eleven children get 1 bag with 7 marbles and the other five get at least two bags. If any of the five get more than two bags, then we need at least 17 bags.

Suppose, if possible, that exactly six of eleven children get 1 bag with seven marbles and the other five get exactly two bags each. Then each of these five must get a large bag and a small bag. Thus, there are six bags with 7 marbles, five large and five small bags. Let us consider how they might be distributed among seven children. Six of the seven get a bag with 7 marbles and the seventh can get no more than two large bags. Thus, at least three large bags go to children who have a bag of seven marbles. If only three large bags go to such children, then the other three children with bags of seven marbles must have at least two small bags each, requiring six small bags. This is not possible. If four large bags go to children with bags of seven marbles, then the other children must each get three bags requiring at least $4 \times 2 + 3 \times 3 = 17$ bags. Finally, if five large bags go to children with a bag of seven marbles, then these five have two bags each, the sixth has at least three bags and the seventh, with only small bags, needs four bags, for a total of at least seventeen bags.

Finally, suppose that there are seven bags with 7 marbles. Then each of seven children would get one, and no remaining bag could hold more than 4 marbles. For a distribution to eleven children, seven children get a bag with 7 marbles and the remaining four get at most one bag with 4 marbles. Hence, there are at most four bags with 4 marbles. Thus, at most four of seven children get 2 bags and the remaining three get at least three bags, for a total of at least seventeen bags.

Solution 3: Fewer than 17 bags will not suffice. If, in the distribution to eleven children, five or fewer get a 7-marble bag, then we need at least $5 + 6 \times 2 = 17$ bags. If, in the distribution to seven children, four or fewer get two bags, then we need at least $4 \times 2 + 3 \times 3 = 17$ bags. So we can, henceforth, restrict attention to the situations where there are at least six 7-marble bags and at least five of seven children get two bags. Observe that, for a distribution to eleven children, all the 7-marble and 4-marble bags must go to distinct children, so that the number of 7-marble plus the number of 4-marble bags cannot exceed 10. Since there are at least six 7-marble bags and at least five of seven children get two bags, there must be at least four 4-marble bags. There cannot be seven 7-marble bags, since there would not be enough 4-marble bags available to give five of seven children two bags. Thus, we have left to consider the following situations:

Case (i): 6 7-marble bags and 5 4-marble bags. In the distribution to eleven children, we need at least $6 \times 1 + 5 \times 2 = 16$ bags, with equality if and only if we have 5 3-marble bags as well. But then there would be no combination of bags to go with one of the 7-marble bags to make a presentation of eleven marbles to one of seven children. Thus, we need at least seventeen bags.

Case(ii): 6 7-marble bags and 4 4-marble bags. Once again we need at least 16 bags to distribute to eleven children, with equality if and only if six children get one bag and five children get two. The sixteen bags must include 6 7-marble, 4 4-marble and 4 3-marble bags and, either, a 5-marble and 2-marble bag, or, a 6-marble and 1-marble bag. In either case, we cannot make a distribution to seven children. Thus, we need at least seventeen bags.

489. Suppose n is a positive integer not less than 2 and that $x_1 \geq x_2 \geq x_3 \geq \cdots \geq x_n \geq 0$,

$$\sum_{i=1}^{n} x_i \leq 400 \qquad \text{and} \qquad \sum_{i=1}^{n} x_i^2 \geq 10^4 \ .$$

Prove that $\sqrt{x_1} + \sqrt{x_2} \geq 10$. is it possible to have equality throughout? [*Bonus:* Formulate and prove a generalization.]

Solution. If $x_1 \geq 100$, the result holds trivially. Suppose that $x_1 < 100$. Then

$$\begin{aligned} 10^4 &\leq x_1^2 + \sum_{j=2}^{n} x_j^2 \leq x_1^2 + x_2 \sum_{j=2}^{n} x_j \\ &\leq x_1^2 + x^2(400 - x_1) = x_1(x_1 - x_2) + 400x_2 \\ &< 100(x_1 - x_2) + 400x_2 = 100x_1 + 300x_2 \ . \end{aligned}$$

Hence $x_1 + 3x_2 \geq 100$.

Therefore
$$\begin{aligned}(\sqrt{x_1}+\sqrt{x_2})^2 &= x_1+x_2+2\sqrt{x_1x_2} \geq x_1+x_2+2x_2 \\ &= x_1+3x_2 \geq 100\ ,\end{aligned}$$
so that $\sqrt{x_1}+\sqrt{x_2} \geq 10$. Equality holds, for example, when $n=16$ and $x_1=x_2=\cdots=x_{16}=25$.

490. (a) Let a,b,c be real numbers. Prove that
$$\min\ [(a-b)^2,(b-c)^2,(c-a)^2] \leq \frac{1}{2}[a^2+b^2+c^2]\ .$$

(b) Does there exist a number k for which
$$\min\ [(a-b)^2,(a-c)^2,(a-d)^2,(b-c)^2,(b-d)^2,(c-d)^2] \leq k[a^2+b^2+c^2+d^2]$$
for any real numbers a,b,c,d? If so, determine the smallest such k.
[*Bonus:* Determine if there is a generalization.]

Solution 1. (a) Wolog, $a \leq b \leq c$. Let $b-a=x\geq 0$, $c-b=y\geq 0$ and m be the minimum of x and y. Then
$$\begin{aligned}\frac{1}{2}[a^2+b^2+c^2] &= \frac{1}{2}[a^2+(a+x)^2+(a+x+y)^2] \\ &= \frac{1}{2}[a^2+2(a+x)^2+2(a+x)y+y^2] \\ &= \frac{1}{2}[(a+y)^2+2(a+x)^2+2xy] \geq xy \geq m^2\ ,\end{aligned}$$
with equality if and only if $x=y=-a$. Since m^2 is equal to the left member of the inequality, the result follows.

(b) Suppose that $a\leq b\leq c\leq d$ and let m be the minimum of the nonnegative quantities $b-a$, $c-b$, $d-c$. The m^2 is the value of the left member of the inequality.

Now,
$$\begin{aligned}20m^2 &= 3m^2+2(2m)^2+(3m)^2 \\ &\leq [(b-a)^2+(c-b)^2+(d-c)^2]+[(c-a)^2+(d-b)^2]+[(d-a)^2] \\ &= 3(a^2+b^2+c^2+d^2)-2(ab+ac+ad+bc+bd+cd) \\ &= 4(a^2+b^2+c^2+d^2)-(a+b+c+d)^2 \leq 4(a^2+b^2+c^2+d^2)\ ,\end{aligned}$$
so that $m^2 \leq \frac{1}{5}(a^2+b^2+c^2+d^2)$. Equality occurs if and only if $b-a=c-b=d-c$ and $a+b+c+d=0$. This occurs, for example, when $(a,b,c,d)=(-3,-1,1,3)$.

Solution 2. (a) Observe that
$$\begin{aligned}a^2+b^2+c^2 &= \frac{1}{3}[(a-b)^2+(b-c)^2+(c-a)^2+(a+b+c)^2 \\ &\geq \frac{1}{3}[(a-b)^2+(b-c)^2+(c-a)^2]\ .\end{aligned}$$

Wolog, assume that $a\leq b\leq c$ and note that
$$2\ \min\ \{(b-a),(c-b)\} \leq (b-a)+(c-b)=(c-a)\ ,$$

so that
$$\begin{aligned}(a-b)^2+(b-c)^2+(c-a)^2 &\geq 2\ \min\ \{(b-a)^2,(c-b)^2\}+4\ \min\ \{(b-a)^2,(c-b)^2 \\ &= 6\ \min\ \{(b-a)^2,(c-b)^2\ .\end{aligned}$$

The desired result follows.

(b) Wolog, let $a \leq b \leq c \leq d$ and supppose that $m = \min\,(d-c, c-b, b-a)$. Then

$$\begin{aligned} a^2+b^2+c^2+d^2 &= \frac{1}{4}[(d-c)^2+(c-b)^2+(b-a)^2+(d-b)^2+(c-a)^2+(d-a)^2+(a+b+c+d)^2] \\ &\geq \frac{1}{4}[3m^2+2(2m)^2+(3m)^2+0] = 5m^2 \ , \end{aligned}$$

from which the result follows. Equality occurs if and only if $d-c = c-b = b-a$ and $a+b+c+d = 0$.

Generalization. [B. Wu] Let $n \geq 3$ and suppose that $a_1 \leq a_2 \leq \cdots \leq a_n$. Let

$$x = a_1^2 + a_2^2 + \cdots + a_n^2$$

and

$$y = \min\,\{(a_i - a_j)^2 : 1 \leq i < j \leq n\} \ .$$

If $i < j$, then

$$(a_j - a_i)^2 = [(a_j - a_{j-1}) + \cdots + (a_{i+1} - a_i)]^2 \geq (j-i)^2 y \ ,$$

whence

$$\sum_{1\leq i<j\leq n} (j-i)^2 y \leq \sum_{1\leq i<j\leq n} (a_j - a_i)^2 = n\sum_{i=1}^n a_i^2 - (\sum_{i=1}^n a_i)^2 \leq nx \ .$$

Therefore,

$$\begin{aligned} \frac{x}{y} &\geq \frac{1}{n} \sum_{1\leq i<j\leq n} (j-i)^2 = \frac{1}{n}\left[n\sum_{i=1}^n i^2 - (\sum_{i=1}^n i)^2\right] \\ &= \frac{1}{n}\left[\frac{n^2(n+1)(2n+1)}{6} - \frac{n^2(n+1)^2}{4}\right] \\ &= \frac{n(n+1)}{12}[2(2n+1) - 3(n+1)] = \frac{n(n+1)}{12}(n-1) = \frac{n(n^2-1)}{12} \ . \end{aligned}$$

Hence

$$\min\,\{(a_i - a_j)^2 : 1 \leq i, j \leq n\} \leq \frac{12}{n(n^2-1)} \sum_{i=1}^n a_i^2 \ .$$

This yields the coefficients $\frac{1}{2}$ and $\frac{1}{5}$ for $n = 3$ and $n = 4$, respectively. For equality, we require that $a_{i+1} - a_i$ are all equal and that $a_1 + a_2 + \cdots + a_n = 0$. This occurs for example when n is even and

$$(a_1, a_2, \cdots, a_n) = (-(n-1), -(n-3), \cdots, (n-3), (n-1))$$

and when n is odd and

$$(a_1, a_2, \cdots, a_n) = (-\frac{1}{2}(n-1), \frac{1}{2}(n-3), \cdots, -1, 0, 1, \cdots \frac{1}{2}(n-3), \frac{1}{2}(n-1)) \ .$$

491. Given that x and y are positive real numbers for which $x + y = 1$ and that m and n are positive integers exceeding 1, prove that

$$(1 - x^m)^n + (1 - y^n)^m > 1 \ .$$

Solution 1. (Probability) Suppose that 0's and 1's are to be placed at random in the slots of an $m \times n$ rectangular array. Let x be the probability that 1 goes in any given slot and y the probability that 0 goes into the slot. Then x^m is the probability that all the m slots in a given column receive 1 and $1 - x^m$ the probability that at least one slot in the column receives 0. Thus, $(1 - x^m)^n$ is the probability that at least one slot in every column receives 0. Similarly, $(1-y^n)^m$ is the probability that at least one slot in every row receives 1.

It is possible for both of these events to occur. [Explain how to do this.] However, also, at least one of these events must occur. For, suppose there is a column that contains no 0; then all of its entries must be 1 and so each row contains at least one 1.

Since these two events exhaust all possibilities and are not mutually exclusive, the sum of their probabilities exceeds one:

$$(1-x^m)^n + (1-y^n)^m > 1 \ .$$

Solution 2. [A. Logan] Observe first that, if f is a convex real function and if $p \leq q \leq r \leq s$ and $p + s = q + r$, then

$$f(q) + f(r) \leq f(p) + f(s) \ .$$

To see this, add the inequalities

$$f(q) \leq \left(\frac{s-q}{s-p}\right) f(p) + \left(\frac{q-p}{s-p}\right) f(s)$$

and

$$f(r) \leq \left(\frac{s-r}{s-p}\right) f(p) + \left(\frac{r-p}{s-p}\right) f(s) \ .$$

If f is strictly convex, then the inequalities are strict.

Consider the substitution $f(t) = t^n$ (for $t \geq 0$), which is strictly convex, $p = y(1-x^m)$, $q = 1-x = y$, $r = 1-x^m$, $s = 1-x^{m+1}$. Then, for $m, n \geq 1$,

$$y^n + (1-x^m)^n \leq y^n(1-x^m)^n + (1-x^{m+1})^n \ ,$$

with strict inequality when $n \geq 2$.

Fix $n \geq 2$. When $m = 1$, we have that

$$(1-x^m)^n + (1-y^n)^m = y^n + 1 - y^n = 1 \ .$$

We prove that the inequality of the problem holds for each $m > 1$ by induction on m. Assuming it for m, we have that

$$\begin{aligned}(1-y^n)^{m+1} &= (1-y^n)^m(1-y^n) \geq [1-(1-x^m)^n](1-y^n) \\ &= 1 - y^n - (1-x^m)^n + y^n(1-x^m)^n > 1 - (1-x^{m+1})^n \ ,\end{aligned}$$

using the induction hypothesis and the result of the previous paragraph.

Solution 3. (Calculus) Let

$$\phi(x) = (1-x^m)^n + [1-(1-x)^n]^m - 1 \ .$$

Then

$$\begin{aligned}\phi'(x) &= n(1-x^m)^{n-1}(-mx^{m-1}) + m[1-(1-x^n)]^{m-1} n(1-x)^{n-1} \\ &= -n(1-x)^{n-1}(1+x+\cdots+x^{m-1})^{n-1} mx^{m-1} \\ &\qquad + mx^{m-1}[1+(1-x)+\cdots+(1-x)^{n-1}]^{m-1} n(1-x)^{n-1} \\ &= mn(1-x)^{n-1}x^{m-1}[f(x)-g(x)] \ ,\end{aligned}$$

where

$$f(x) = [1+(1-x)+\cdots+(1-x)^{n-1}]^{m-1}$$

and

$$g(x) = [1+x+\cdots+x^{m-1}]^{n-1} \ .$$

The function $f(x)$ decreases from $f(0) = n^{m-1}$ to $f(1) = 1$ and $g(x)$ increases from $g(0) = 1$ to $g(1) = m^{n-1}$ for $0 \leq x \leq 1$. Hence $f(x) - g(x)$ is a decreasing function from $f(0) - g(0) = n^{m-1} - 1 > 0$ to $f(1) - g(1) = 1 - m^{n-1} < 0$ Therefore, there exists a number c with $0 < c < 1$ for which $\phi'(x) > 0$ for $0 < x < c$, $\phi'(c) = 0$ and $\phi'(x) < 0$ for $c < x < 1$. Hence $\phi(x)$ increases for $0 < x < c$ and decreases for $c < x < 1$. Since $\phi(0) = \phi(1) = 0$, it follows that $\phi(x) > 0$ as desired.

Comment. If $m = n > 1$ and $x = y = \frac{1}{2}$, the inequality becomes $2(1 - 2^{-n})^n > 1$, which reduces to the intriguing

$$\frac{1}{2^n} + \frac{1}{2^{1/n}} < 1 .$$

This can be shown directly by noting that $n \leq 2^{n-1}$ for $n \geq 2$ and

$$\left(1 - \frac{1}{2^n}\right)^n > 1 - \frac{n}{2^n} \geq \frac{1}{2} .$$

492. The faces of a tetrahedron are formed by four congruent triangles. if α is the angle between a pair of opposite edges of the tetrahedron, show that

$$\cos \alpha = \frac{\sin(B - C)}{\sin(B + C)}$$

where B and C are the angles adjacent to one of these edges in a face of the tetrahedron.

Solution 1. Let the vertices of the tetrahedron by $ABCD$. We must have $|BC| = |AD| = a$, , $|AC = |BD = b$, $|AB| = |CD| = c$. Let the opposite edges in question be BC and AD, both of length a, and let the angle between $\overrightarrow{BD}$ and $\overrightarrow{BC}$ be B and the angle between $\overrightarrow{CD}$ and $\overrightarrow{CB}$ be C. These are two angles of each triangular face, the remaining angle being $180° - (B + C)$ whose sine is the same as that of $B + C$. By the Law of Sines, we have that $a \sin C = b \sin(B + C)$ and $a \sin B = c \sin(B + C)$.

Then

$$\begin{aligned} a^2 \cos \alpha &= \overrightarrow{BC} \cdot \overrightarrow{DA} = (\overrightarrow{DC} - \overrightarrow{DB}) \cdot \overrightarrow{DA} \\ &= \overrightarrow{DC} \cdot \overrightarrow{DA} - \overrightarrow{DB} \cdot \overrightarrow{DA} = ca \cos C - cb \cos B \\ &= [\sin(B + C)]^{-1}[a^2 \sin B \cos C - a^2 \sin C \cos B] = a^2[\sin(B + C)]^{-1} \sin(B - C) , \end{aligned}$$

which yields the desired result.

Comment. Note that there is a sign ambiguity in the answer, depending on which ends of the edge are assigned the angles B and C; in the solution, the choice was made to make the answer come out right.

Solution 2. Complete the parallelogram $ABCE$ in the plane of triangle ABC so that $AE \| BC$ and $EC \| AB$. Suppose that AC and BE intersect at F; then F is the midpoint of both AC and BE. Let $|AC| = |BD| = b$, $|BC| = |AD| = a$, $|AB| = |CD| = c$, $|BF| = |DF| = |EF| = x$ and $|DE| = u$.

Then $2(a^2 + c^2) = b^2 + 4x^2$ and $2(u^2 + b^2) = 4x^2 + 4x^2 = 8x^2$, whence $u^2 = 2(a^2 + c^2 - b^2)$. Since $u^2 = 2a^2(1 - \cos \alpha)$,

$$\cos \alpha = 1 - \frac{u^2}{2a^2} = \frac{b^2 - c^2}{a^2} .$$

Now

$$\begin{aligned} \sin(B - C) &= \sin B \cos C - \sin C \cos B \\ &= \frac{b}{2R} \cdot \frac{a^2 + b^2 - c^2}{2ab} - \frac{c}{2R} \cdot \frac{a^2 + c^2 - b^2}{2ac} \\ &= \frac{2(b^2 - c^2)}{4aR} , \end{aligned}$$

and, similarly, $\sin(B+C) = 2a^2/4aR$. Hence

$$\frac{\sin(B-C)}{\sin(B+C)} = \frac{b^2-c^2}{a^2} = \cos\alpha \ .$$

493. Prove that there is a natural number with the following characteristics: (a) it is a multiple of 2007; (b) the first four digits in its decimal representation are 2009; (c) the last four digits in its decimal representation are 2009.

Solution. First, we show that there is a positive integer n for which $n \cdot 10^4 + 2$ is a multiple of 2007. For otherwise, if none of the numbers of the form $n \cdot 10^4 + 2$ with $n = 0, 1, 2, \cdots, 2006$ were a multiple of 2007, then, by the Pigeonhole Principle, there must be two of them congruent modulo 2007. Hence, their difference, a number of the form $(k-m) \cdot 10^4$ with $0 \le k, m \le 2006$ would be a multiple of 2007, an impossibility.

Now use this number n to compose the number $M = \overline{2009007n2009}$. Since

$$\begin{aligned} M &= 2009007 \cdot 10^k + n \cdot 10^4 + 2009 \\ &= 2007000 \cdot 10^k + 2007 \cdot 10^k + 2007 + (n \cdot 10^4 + 2) \end{aligned}$$

for $5 \le k \le 8$, M is a multiple of 2007. Since it has all three of the desired characteristics, the problem is solved.

494. (a) Find all real numbers x that satisfy the equation

$$(8x - 56)\sqrt{3-x} = 30x - x^2 - 97 \ .$$

(b) Find all real numbers x that satisfy the equation

$$\sqrt{x} + \sqrt[3]{x+7} = \sqrt[4]{x+80} \ .$$

Solution. (a) We must have $x \le 3$. The equation can be rewritten

$$0 = 8(x-7)\sqrt{3-x} + x^2 - 30x + 97 \ .$$

Let y be positive with $y^2 = 3 - x$. Then

$$\begin{aligned} 0 &= -8(y^2+4)y + (y^2-3)^2 - 30(3-y^2) + 97 \\ &= y^4 - 8y^3 + 24y^2 - 32y + 16 = (y-2)^4 \ . \end{aligned}$$

Hence $y = 2$, so that $x = -1$. This solution is valid.

(b) The domain of the equation is given by $x \ge 0$. One solution is $x = 1$; we prove that it is the only solution. The equation is equivalent to

$$x^2 + 4x^{3/2}(x+7)^{1/3} + 6x(x+7)^{2/3} + 4x^{1/2}(x+7) + (x+7)^{4/3} = x + 80 \ .$$

Let

$$f(x) = 4x^{3/2}(x+7)^{1/3} + 6x(x+7)^{2/3} + 4x^{1/2}(x+7) + (x+7)^{4/3}.$$

This function $f(x)$ is increasing and $f(1) = 80$. If $x > 1$, then also $x^2 > x$ and $f(x) > 80$, so that $x^2 + f(x) > x + 80$. Similarly, when $0 \le x < 1$, then $x^2 < x$ and $f(x) < 80$, so that $x^2 + f(x) < x + 80$. Hence, there is no solution to the equation save 1.

495. Let $n \geq 3$. A regular $n-$gon has area S. Squares are constructed externally on its sides, and the vertices of adjacent squares that are not vertices of the polygon are connected to form a $2n-$sided polygon, whose area is T. Prove that $T \leq 4(\sqrt{3}+1)S$. For what values of n does equality hold?

Solution. Wolog, let the sidelength of the given polygon be 1. The $2n-$sided polygon is composed of the regular $n-$gon, n squares with sidelength 1 and n isosceles triangles with equal sides of length 1 and angle between these sides equal to

$$2\pi - \left(\frac{\pi}{2} + \frac{\pi}{2} + \frac{(n-2)\pi}{n} \right) = \frac{2\pi}{n} \ .$$

Therefore

$$T = S + n + n \cdot \frac{1}{2} \sin \frac{2\pi}{n} \ .$$

On the other hand, S is the sum of the areas of n isosceles triangles, each with base 1, apex angle $2\pi/n$ and height $(1/2)\cot(\pi/n)$. Hence $S = (n/4)\cot(\pi/n)$, so that $n = 4S\tan(\pi/n)$. Therefore

$$T = S + n + n \cdot \frac{1}{2} \sin \frac{2\pi}{n} = S \left(1 + 4 \tan \frac{\pi}{n} + 2 \tan \frac{\pi}{n} \sin \frac{2\pi}{n} \right) \ .$$

Apply $\sin 2\theta = 2\tan\theta/(1+\tan^2\theta)$ to $\sin(2\pi/n)$ to obtain that

$$\begin{aligned} T &= S\left(1 + 4\tan\frac{\pi}{n} + \frac{4\tan^2\frac{\pi}{n}}{1+\tan^2\frac{\pi}{n}} \right) \\ & S\left(1 + 4\tan\frac{\pi}{n} + \frac{4\tan^2\frac{\pi}{n} + 4 - 4}{1+\tan^2\frac{\pi}{n}} \right) \\ &= S\left[5 + 4\left(\tan\frac{\pi}{n} - \frac{1}{1+\tan^2\frac{\pi}{n}} \right) \right] \ . \end{aligned}$$

Since $n \geq 3$ and the tangent function is increasing, $0 < \tan(\pi/n) \leq \tan(\pi/3) = \sqrt{3}$. so that

$$\tan\frac{\pi}{n} - \frac{1}{1+\tan^2(\pi/n)} \leq \sqrt{3} - \frac{1}{1+(\sqrt{3})^2} = \sqrt{3} - \frac{1}{4} \ .$$

Therefore. $T \leq 4S(\sqrt{3}+1)$, as desired. Equality holds when $n = 3$ and the polygon is an equilateral triangle.

496. Is the hundreds digit of $N = 2^{2006} + 2^{2007} + 2^{2008}$ even or odd? Justify your answer.

Solution. Observe that

$$N = 2^{2006}(1+2+4) = 7 \cdot 2^6 \cdot 2^{2000} = 7 \cdot 2^6 \cdot (2^{20})^{100} \ .$$

However, modulo 100,

$$2^{20} = 1024^2 \equiv 24^2 = 576 \equiv 76$$

and $76^n \equiv 76$ for each positive integer n. Hence

$$N = 7 \cdot 2^6 \cdot (2^{20})^{100} \equiv 7 \cdot 64 \cdot 76 \equiv 48$$

(mod 100). Denote the hundreds digit of N by h. Since N is a multiple of 8, the three digit number $h48$ must be a multiple of 8 as well. This is possible only if h is even.

Thus, the hundreds digit of the number N is even.

497. Given $n \geq 4$ points in the plane with no three collinear, construct all segments connecting two of these points. It is known that the length of each of these segments is a positive integer. Prove that the lengths of at least 1/6 of the segments are multiples of 3.

Solution. First, we prove a lemma: *Four points are given in the plane with no three collinear. The length of each of the segments joining two of these points is an integer. Therefore, at least one of the segments has a length divisible by 3.* Denote the four points by A, B, C, D; wolog, assume that $\angle BAD = \angle BAC + \angle CAD$. Let $\angle BAC = \alpha$, $\angle CAD = \beta$ and $\angle BAD = \gamma$, so that $\gamma = \alpha + \beta$. Applying the Law of Cosines to triangles ABC, ACD and ABD, we find that

$$BC^2 = AB^2 + AC^2 - 2AB \cdot AC \cdot \cos\alpha$$

$$CD^2 = AD^2 + AC^2 - 2AD \cdot AC \cdot \cos\beta$$

and

$$BD^2 = AB^2 + AD^2 - 2AB \cdot AD \cdot \cos\gamma \ .$$

Assume, if possible, that the lengths of all six segments AB, AC, AD, BC, BD, CD are not multiples of 3. Then

$$AB^2 \equiv AC^2 \equiv AD^2 \equiv BC^2 \equiv BD^2 \equiv CD^2 \equiv 1$$

modulo 3, from which it follows that

$$2AB \cdot AC \cdot \cos\alpha \equiv 2AD \cdot AC \cdot \cos\beta \equiv 2AB \cdot AD \cdot \cos\gamma \equiv 1$$

modulo 3. Therefore

$$AC^2 \cdot AB \cdot AD \cdot \cos\alpha \cos\beta \equiv (2AB \cdot AC \cdot \cos\alpha)(2AD \cdot AC \cdot \cos\beta) \equiv 1$$

modulo 3.

From the foregoing equations, each of $\cos\alpha$, $\cos\beta$ and $\cos\gamma$ are rational. Let $\cos\alpha = p/q$ and $\cos\beta = r/s$, in lowest terms, where p, q, r, s are integers. None of these four integers can be multiples of 3. The denominators q and s cannot be multiples of 3 for they must cancel into side lengths, and the numerators p and r cannot be multiples of 3 since the terms containing the cosines are not divisible by 3. Hence $p^2 \equiv q^2 \equiv r^2 \equiv s^2 \equiv 1 \pmod 3$.

Since $\cos\gamma = \cos\alpha\cos\beta - \sin\alpha\sin\beta$, we have, from $2AB \cdot AD \cos\gamma \equiv 1$ and $AC^2 \equiv 1$, modulo 3, that

$$2AC^2 \cdot AB \cdot AD \cdot \cos\gamma \equiv 1 \Longleftrightarrow$$

$$2AC^2 \cdot AB \cdot AD \cdot \cos\alpha \cdot \cos\beta - 2AC^2 \cdot AB \cdot AD \cdot \frac{\sqrt{q^2 - p^2}}{q} \cdot \frac{\sqrt{s^2 - r^2}}{s} \equiv 1 \ .$$

The second product on the left side is a multiple of 3, so that

$$2AC^2 \cdot AB \cdot AD \cdot \cos\alpha \cos\beta \equiv 1$$

. This contradicts an earlier statement and establishes the lemma. ♣

Let $n \geq 4$. There are $\binom{n}{4}$ sets of four points, so by the lemma, there are at least this many segments whose lengths are multiples of 3, counting multiplicity (some counted more than once). Since each of the segments is counted at most $\binom{n-2}{2}$ times (for the sets of four points containing the endpoints of the segment), it follows that there are at least $\binom{n}{4}/\binom{n-2}{2}$ distinct segments whose lengths are multiples of 3.

Since

$$\frac{\binom{n}{4}}{\binom{n-2}{2}} = \frac{2n(n-1)(n-2)(n-3)}{4!(n-2)(n-3)} = \frac{1}{6} \cdot \frac{n(n-1)}{2} = \frac{1}{6} \cdot \binom{n}{2} \ ,$$

there are at least as many segments with lengths divisible by 3 as one-sixths of the number of pairs of segments, and the result follows.

498. Let a be a real parameter. Consider the simultaneous sytem of two equations:

$$\frac{1}{x+y} + x = a - 1 \; ; \tag{1}$$

$$\frac{x}{x+y} = a - 2 \; . \tag{2}$$

(a) For what value of the parameter a does the system have exactly one solution?

(b) Let $2 < a < 3$. Suppose that (x, y) satisfies the sytem. For which value of a in the stated range does $(x/y) + (y/x)$ reach its maximum value?

Solution. From the identification of the coefficients of a quadratic in terms of the sum and product of the roots, we see that $1/(x + y)$ and x are the solutions of the quadratic equation

$$0 = t^2 - (a - 1)t + (a - 2) = (t - \overline{a - 2})(t - 1) \; .$$

There are two options.

Option 1. $1/(x + y) = a - 2$, $x = 1$, so that

$$(x, y) = \left(1, \frac{3-a}{a-2}\right)$$

with $a \neq 2$.

Option 2. $1/(x + y) = 1$, $x = a - 2$, so that

$$(x, y) = (a - 2, 3 - a) \; .$$

(a) For the system to have exactly one solution, either the two options produce the same pair or only one of the options is possible. In the first instance, we have $a = 3$ and the unique solution $(x, y) = (1, 0)$ and in the second, we have $a = 2$ and the unique solution is $(x, y) = (0, 1)$.

(b) When $2 < a < 3$, both x/y and y/x are positive for either solution of the system. By the Arithmetic-Geometric Means Inequality,

$$\frac{x}{y} + \frac{y}{x} \geq 2$$

with equality if and only if $x/y = y/x$. The condition for equality is equivalent to $(a - 2)/(3 - a) = (3 - a)/(a - 2)$, or $a = 5/2$. Thus, $(x/y) + (y/x)$ attains its minimum value of 2 when $a = 5/2$.

499. The triangle ABC has all acute angles. The bisector of angle ACB intersects AB at L. Segments LM and LN with $M \in AC$ and $N \in BC$ are constructed, perpendicular to the sides AC and BC respectively. Suppose that AN and BM intersect at P. Prove that CP is perpendicular to AB.

Solution 1. Let m be a line through C parallel to AB and let AN and BM intersect m at F and E, respectively. Let CP and AB intersect at D. Triangles ADP and FCP are similar, as are triangles DBP and CEP. Hence

$$AD : CF = PD : PC = DB : CE \; .$$

Therefor $AD : BD = CF : CE$.

On the other hand, triangles ABM and CEM are similar, and triangles ABN and FCN are similar. Therefore, $AM : MC = AB : CE$ and $BN : CN = AB : CF$. However, right triangles CLM and CLN are congruent and $CM = CN$, so that $AM : BN = CF : CE$. Together with $AD : BD = CF : CE$, this yields $AD : BD = AM : BN$ and $AM : AD = BN : BD$.

Let the altitude from C to AB intersects AB at a point H. Since triangles ALM and ACH are similar, $AL : AC = AM : AH$. Similarly, from the similarity of triangles BLN and BCH, $BL : BC = BN : BH$. By the angle-bisector theorem, $AL : AC = BL : BC$. It follows that $AM : AH = BN : BH$.

Since $AM : AD = BN : BD$ and $AM : AH = BN : BH$, it follows that D and H divide AB internally in the same ratio, and so $D = H$. Thus, $CP \perp AB$ and the statement is established.

Solution 2. [J. Kileel] Use the same notation as in Solution 1. Let H be the foot of the perpendicular from C to AB. It suffices to show that AN, BM and CH are concurrent. By Ceva's Theorem, this is equivalent to showing that

$$\frac{AM}{MC} \cdot \frac{CN}{NB} \cdot \frac{BH}{HA} = 1 \ .$$

Triangles MCL and NCL are congruent (ASA), so that $CM = CN$. Triangles ALM and ACH are similar, so that $AM : HA = LM : HC$. Likewise, triangles BLN and BCH are similar; therefore,

$$BH : NB = HC : LN = HC : LM \ .$$

It follows that

$$\frac{AM}{MC} \cdot \frac{CN}{NB} \cdot \frac{BH}{HA} = \frac{AM}{HA} \cdot \frac{BH}{NB} \cdot \frac{CN}{MC} = \frac{LM}{HC} \cdot \frac{HC}{LM} \cdot \frac{CN}{MC} = 1 \ .$$

500. Find all sets of distinct integers $1 < a < b < c < d$ for which $abcd - 1$ is divisible by $(a-1)(b-1)(c-1)(d-1)$.

Solution. Let $x = abcd - 1$ and $y = (a-1)(b-1)(c-1)(d-1)$. Suppose that y divides x. If any one of a, b, c, d is even, then x is odd and so y must be odd and all of a, b, c, d are even. Thus, a, b, c, d are all of the same parity. Note that

$$\frac{x}{y} < \frac{abcd}{(a-1)(b-1)(c-1)(d-1)} = \left(\frac{a}{a-1}\right)\left(\frac{b}{b-1}\right)\left(\frac{c}{c-1}\right)\left(\frac{d}{d-1}\right) \ .$$

Observe that, if $a \geq 5$, then

$$\frac{x}{y} < \left(\frac{5}{4}\right)\left(\frac{6}{5}\right)\left(\frac{7}{6}\right)\left(\frac{8}{7}\right) = 2 \ ,$$

and there are no possibilities. If $a = 4$, x is odd, but

$$\frac{x}{y} < \left(\frac{4}{3}\right)\left(\frac{6}{5}\right)\left(\frac{8}{7}\right)\left(\frac{10}{9}\right) < 3 \ ,$$

and again there are no possibilities. If $a = 3$, then

$$\frac{x}{y} < \left(\frac{3}{2}\right)\left(\frac{5}{4}\right)\left(\frac{7}{6}\right)\left(\frac{9}{8}\right) < 3 \ ,$$

and so $x/y = 2$. Since $x = 2y$ and x is not divisible by 3, then neither is y, so that none of $b-1$, $c-1$, $d-1$ is a multiple of 3. Therefore, if $b \neq 5$, then $b \geq 9$ and

$$\frac{x}{y} < \left(\frac{3}{2}\right)\left(\frac{9}{8}\right)\left(\frac{11}{10}\right)\left(\frac{15}{14}\right) < 2 \ ,$$

and there are no possibilities. Therefore, when $a = 3$, $b = 5$ and we are led to the equation $15cd - 1 = 2y = 16(c-1)(d-1)$, which is equivalent to

$$(c - 16)(d - 16) = 239 \ .$$

Since 239 is prime, the solution is unique: $(c,d) = (17, 255)$. We obtain the solution

$$(a,b,c,d) = (3,5,17,255) \ .$$

Finally, suppose that a,b,c,d are all even and x/y is odd. Since

$$\frac{x}{y} < \left(\frac{2}{1}\right)\left(\frac{4}{3}\right)\left(\frac{6}{5}\right)\left(\frac{8}{7}\right) < 4 \ ,$$

and we find that $x/y = 3$. Thus, none of b,c,d is divisible by 3. If $b \geq 4$, then

$$\frac{x}{y} < \left(\frac{2}{1}\right)\left(\frac{8}{7}\right)\left(\frac{10}{9}\right)\left(\frac{14}{13}\right) < 3 \ ,$$

and there are no possibilities. Then $b = 4$ and we obtain that $8cd - 1 = 3y = 9(c-1)(d-1)$, or

$$(c-9)(d-9) = 71 \ .$$

As 71 is prime, the solution is unique: $(c,d) = (10, 80)$. Thus, we obtain the second and final solution

$$(a,b,c,d) = (2,4,10,80) \ .$$

501. Given a list of $3n$ not necessarily distinct elements of a set S, determine necessary and sufficient conditions under which these $3n$ elements can be divided into n triples, none of which consist of three distinct elements.

Solution. A necessary and sufficient condition is that at most n of the $3n$ elements in the list occur an odd number of times.

Suppose that this condition is satisfied. Distribute one of each element appearing an odd number of times into distinct triples. Then each of the remaining elements in S occurs an even number of times, so we can sort them into at least n equal pairs (not necessarily all distinct). Sort one of these pairs into each triple, and then (if necessary) fill up the triples in any way with what is left over. Then each triple contains a pair of like elements.

On the other hand, suppose that we have distributed the $3n$ elements as specified. Then there must be a least n pairs of like elements (again, not necessarily distinct pairs). There are n elements not in these pairs, and these are the only possibilities for elements that occur an odd number of times. The result follows.

502. A set consisting of n men and n women is partitioned at random into n disjoint pairs of people. What are the expected value and variance of the number of male-female couples that result? (The *expected value* E is the average of the number N of male-female couples over all possibilities, *i.e.* the sum of the numbers of male-female couples for the possibilities divided by the number of possibilities. The *variance* is the average of the difference $(E-N)^2$ over all possibilities, *i.e.* the sum of the values of $(E-N)^2$ for the possibilities divided by the number of possibilities.)

Comment. The answer is

$$E(X) = \frac{n^2}{2n-1} \qquad \text{Var }(X) = \frac{2n^2(n-1)^2}{(2n-1)^2(2n-3)}$$

where X is the number of man-woman matches. The solution relies on some statistical theory and is given in *American Mathematical Monthly* 107 (1998), 866-867.

A direct assault seems difficult. There are

$$u_n = \frac{\binom{2n}{2}\binom{2n-2}{2}\cdots\binom{4}{2}\binom{2}{2}}{n!} = \frac{(2n)!}{2^n n} = (2n-1)(2n-3)\cdots(3)(1)$$

ways of pairing off the $2n$ people. The number of men not paired off with women must be even, as they are paired with each other, and similarly for the women. Suppose that we want $n-2k$ man-woman pairs. There are $\binom{n}{2k}$ ways of picking the men to be paired off with each other and so $\binom{n}{2k}u_k$ of selecting and pairing them off. Similarly, there are $\binom{n}{2k}u_k$ of selecting and pairing off the $2k$ women not in the couples. As for the $n-2k$ men to be paired off with women, there are $(n-2k)!$ ways of pairing them off with the women not paired with other women. Therefore the number of ways of pairing so that there are exactly $n-2k$ man-woman couples is

$$\left[\binom{n}{2k}u_k\right]^2 (n-2k)!$$

and the average number of couples over all the pairings is

$$\left[\binom{n}{2k}u_k\right]^2 (n-2k)!(n-2k) \div u_n \ .$$

503. A natural number is *perfect* if it is the sum of its proper positive divisors. Prove that no two consecutive numbers can both be perfect.

Solution. We review basic information about the sum of divisors function. For any positive integer n, the function $\sigma(n)$ is the sum of all the positive divisors of n, including both 1 and n. For $n \geq 2$, $\sigma(n) \geq n+1$ with equality if and only if n is prime. If m and n have greatest common divisor equal to 1, then $\sigma(mn) = \sigma(m)\sigma(n)$. For any prime p and positive integer exponent c, we have that $\sigma(p^c) = (p^{c+1}-1)/(p-1)$. A positive integer n is perfect if and only if $\sigma(n) = 2n$.

Lemma. (Euclid-Euler) An even positive integer r is perfect if and only if it is of the form $r = 2^{p-1}(2^p-1)$ where both p and $2^p - 1$ are prime.

Proof of Lemma. Let n be an even perfect number. Then $n = 2^k m$ where $k \geq 1$ and m is odd, so that

$$2^{k+1}m = 2n = \sigma(n) = \sigma(2^k)\sigma(m) = (2^{k+1}-1)\sigma(m) \ .$$

Since the greatest common divisor of 2^{k+1} and $2^{k+1}-1$ is 1, there exists a positive integer w for which $m = (2^{k+1}-1)w$ and $\sigma(m) = 2^{k+1}w$.

Suppose, if possible, that $w > 1$. Then the divisors of m include the distinct $1, w, m$ so that

$$\sigma(m) \geq 1 + w + m = 1 + 2^{k+1}w > \sigma(m) \ ,$$

an impossibility. Therefore $w = 1$ and $\sigma(m) = 2^{k+1} = m+1$, so that $m = 2^{k+1}-1$ is prime. This forces also $k+1$ to be prime (why?) and the necessity of the condition follows. It is straightforward to verify that each number of the given form is perfect. ♣

Since 5 and 7 are not perfect numbers, we need not consider the case where the even number of the pair is $6 = 2(2^2-1)$. We obtain the desired result by showing that for any odd prime p, neither of the numbers $u = 2^{p-1}(2^p-1) - 1$ and $v = 2^{p-1}(2^p-1)+1$ is perfect.

Observe that

$$2^{p-1}(2^p-1) = 2 \times 4^{p-1} - 4^{(p-1)/2} \equiv 2 \times 4 - 4 = 4$$

modulo 12, since every positive integer power of 4 is congruent to 4 modulo 12. Hence $u \equiv 3$ and $v \equiv 5$ (mod 12), and so neither u nor v are squares. Hence each has an even number of divisors, that can be paired off as $(d, u/d)$ and $(d, v/d)$ respectively, where d is less than the square root of u and v respectively.

We have that $u \equiv -1 \pmod 4$, so that, as $d(u/d) = u \equiv -1$, $\{d, u/d\} \equiv \{1, -1\}$ and $d + (u/d) \equiv 0 \pmod 4$. Hence

$$\sigma(u) = \sum\{d + (n/d) : d|u, d < \sqrt{u}\} \equiv 0$$

modulo 4, while $2u \equiv 6 \pmod 4$. Hence $\sigma(u) \neq 2u$ and u is not perfect.

We have that $v \equiv -1 \pmod 3$, so that $\{d, v/d\} \equiv \{1, -1\}$ and $d + (v/d) \equiv 0 \pmod 3$, for every divisor d of v. Hence

$$\sigma(v) = \sum\{d + (n/d) : d|v, d < \sqrt{v}\} \equiv 0$$

modulo 3, while $2v \equiv 10 \pmod{12}$. Hence $\sigma(v) \neq 2v$ and v is not perfect.

504. Find all functions f taking the real numbers into the real numbers for which the following conditions hold simultaneously:

(a) $f(x + f(y) + yf(x)) = y + f(x) + xf(y)$ for every real pair (x, y);

(b) $\{f(x)/x : x \neq 0\}$ is a finite set.

Solution. The function f must be the identity function $f(x) = x$ for all x.

By (b), there exists a number k such that the set $S_k = \{x : f(x) = kx\}$ is infinite. From (a), for all $x \in S_k$,

$$f(x + kx + kx^2) = f(x + f(x) + xf(x)) = x + kx + kx^2 \ .$$

Hence S_1 has infinitely many elements.

Suppose, if possible, that there exists y such that $f(y)/y = m \neq 1$. Then, for all $x \in S_1$,

$$f(x + my + yx) = f(x + f(y) + yf(x)) = y + f(x) + xf(y) = y + x + mxy \ .$$

As x ranges over the infinite set S_1, by (b), it is not possible that

$$\frac{f(x + my + yx)}{x + my + yx} = \frac{x(1 + my) + y}{x(1 + y) + my}$$

takes infinitely many values. Hence, there are x_1 and x_2 in S for which $x_1 \neq x_2$ and

$$\frac{f(x_1 + my + yx_1)}{x_1 + my + yx_1} = \frac{f(x_2 + my + yx_2)}{x_2 + my + yx_2} \ .$$

Then,

$$[x_1(1 + my) + y][x_2(1 + y) + my] = [x_2(1 + my) + y][x_1(1 + y) + my]$$

so that, since $(x_1 - x_2)y \neq 0$,

$$(x_1 - x_2)my(1 + my) = (x_1 - x_2)y(1 + y) \Leftrightarrow (m^2 - 1)y + (m - 1) = 0 \Leftrightarrow (m + 1)y + 1 = 0 \ .$$

Thus, $m = -(y + 1)/y$, so that $f(y) = my = -(y + 1)$. This means that y is uniquely determined by m and that S_m is a singleton.

For all $x \in S_1$, by condition (i), we find that

$$\begin{aligned} f((y + 1)(x - 1)) &= f(x - (y + 1) + yx) = f(x + f(y) + yf(x)) = y + f(x) + xf(y) \\ &= y + x - x(y + 1) = -y(x - 1) = \frac{-y}{y + 1}(y + 1)(x - 1) \\ &= \frac{-1}{m}(y + 1)(x - 1) \ . \end{aligned}$$

But $\{(y+1)(x-1) : x \in S_1\}$ is an infinite set, so $S_{-1/m}$ must be infinite.

However, we can go through the foregoing argument with m replaced by $-1/m$ and deduce, either that $f(x)/x$ takes infinitely many values or that $S_{-1/m}$ is a singleton, both of which yield constradictions. The result follows.

505. What is the largest cubical present that can be completely wrapped (without cutting) by a unit square of wrapping paper?

Solution. The side of the largest possible cubical present that can be wrapped is $\sqrt{2}/4$.

Let s be the side length of the cube. For any point on the cube, there is another point whose distance away is at least $2s$, so that for each point on the square, there is a point whose distance away from it is at least $2s$. This is true in particular for the centre of the square, so the diagonal of the square is at least $4s$. Hence $\sqrt{2} \geq 4s$, so that $s \leq \sqrt{2}/4$.

On the other hand, if we have a cube of this size, we can place it right in the centre of the wrapping square with its sides parallel to the diagonals of the square, and fold the corners of the square over the lateral faces of the cube with them meeting in the middle of the top face of the cube.

506. A two-person game is played as follows. A position consists of a pair (a, b) of positive integers. Playes move alternately. A move consists of decreasing the larger number in the current position by any positive multiple of the smaller number, as long as the result remains positive. The first player unable to make a move loses. (This happens, for example, when $a = b$.) Determine those positions (a, b) from which the first player can guarantee a win with optimal play.

Solution. Let $\phi = \frac{1}{2}(1+\sqrt{5})$, the larger root of the polynomial $x^2 - x - 1$. Note that ϕ is irrational, that $1/\phi = \phi - 1$ and that $0 < 1/\phi < 1 < \phi < 2$. We show that the first player can guarantee a win if and only if the ratio of the larger to the smaller number exceeds ϕ.

Let $\mathfrak{W}$ be the set of pairs $\{a, b\}$ with $a > \phi b$ and $\mathfrak{L}$ the set of pairs with $b < a < \phi b$. The result will follow if we prove that, from any pair in $\mathfrak{W}$, the next player can leave a pair in $\mathfrak{L}$ and that, from any pair in $\mathfrak{L}$, the next player must go to a pair in $\mathfrak{W}$.

Suppose that $\{a, b\} \in \mathfrak{W}$. Then $a/b > \phi$ and there is a (unique) poaitive integer k for which $\phi - 1 < (a/b) - k < \phi$. Choosing this k yields the pair $\{b, a - bk\}$. Since

$$\frac{b}{a - bk} = \frac{1}{(a/b) - k} < \frac{1}{\phi - 1} = \phi \ ,$$

the pair $\{b, a - bk\} \in \mathfrak{L}$.

On the other hand, if $\{a, b\} \in \mathfrak{L}$, then $b < a < \phi b$, so that $a < 2b$. The only legal move leads to the pair $\{b, a - b\}$. But then

$$\frac{b}{a - b} = \frac{1}{(a/b) - 1} > \frac{1}{\phi - 1} = \phi \ ,$$

so that $\{a, a - b\} \in \mathfrak{W}$.

Therefore, the first player has a winning strategy if and only if presented with a pair $\{a, b\}$ with $a > \phi b$.

Solutions for July.

507. Prove that, if a, b, c are positive reals, then

$$\log^2 \frac{ab}{c} + \log^2 \frac{bc}{a} + \log^2 \frac{ca}{b} + \frac{3}{4} \geq \log(abc) \ .$$

Solution 1. [A. Dhawan] By the Cauchy-Schwarz Inequality,

$$\left(\log^2 \frac{ab}{c} + \log^2 \frac{bc}{a} + \log^2 \frac{ca}{b}\right)(1^2 + 1^2 + 1^2) \geq \left(\log \frac{ab}{c} + \log \frac{bc}{a} + \log \frac{ca}{b}\right)^2 = \log^2(abc) \ ,$$

whence

$$\left(\log^2 \frac{ab}{c} + \log^2 \frac{bc}{a} + \log^2 \frac{ca}{b}\right) \geq \frac{1}{3}\log^2(abc) \ .$$

Also,

$$\begin{aligned} \frac{1}{3}\log^2(abc) + \frac{3}{4} - \log(abc) &= \frac{1}{12}(4\log^2(abc) - 12\log(abc) + 9) \\ &= \frac{1}{12}(2\log(abc) - 3)^2 \geq 0 \ . \end{aligned}$$

Hence

$$\log^2 \frac{ab}{c} + \log^2 \frac{bc}{a} + \log^2 \frac{ca}{b} + \frac{3}{4} \geq \frac{1}{3}\log^2(abc) + \frac{3}{4} \geq \log(abc) \ .$$

Solution 2. The inequality is equivalent to

$$(\log a + \log b - \log c)^2 + (\log b + \log c - \log a)^2 + (\log c + \log a - \log b)^2 + \frac{3}{4} \geq \log a + \log b + \log c \ .$$

Denoting $\log a + \log b - \log c = x$, $\log b + \log c - \log a = y$ and $\log c + \log a - \log b = z$, we get that $x + y + z = \log a + \log b + \log c$.

Since $(x - \frac{1}{2})^2 + (y - \frac{1}{2})^2 + (z - \frac{1}{2})^2 \geq 0$, it follows that $x^2 + y^2 + x^2 + \frac{3}{4} \geq x + y + z$, which is the desired inequality. Equality holds if and only if $x = y = z = \frac{1}{2}$, which requires that $a = b = c$.

508. Let a, b, c be integers exceeding 1 for which both $\log_a b + \log_b a$ and $\log_a^2 b + \log_b^2 a$ are rational. Prove that, for every positive integer n, $\log_a^n b + \log_b^n a$ is rational.

Solution. The result holds for $n = 0, 1, 2$. Assume, as an induction hypothesis, that $\log_a^k + \log_b^k$ is rational for $0 \leq k \leq n$.

Taking $x = \log_a b$ and $y = \log_b a$ in the identity

$$x^{k+1} + y^{k+1} = (x + y)(x^k + y^k) - xy(x^{k-1} + y^{k-1}) \ ,$$

and noting that $xy = 1$, we find that, by induction, $\log_a^{k+1} b + \log_b^{k+1} a$ is rational. The result follows.

509. Let $ABCDA'B'C'D'$ be a cube where the point O is the centre of the face $ABCD$ and $|AB| = 2a$. Calculate the distance from the point B to the line of intersection of the planes $A'B'O$ and $ADD'A'$ and the distance between AB' and BD. (AA', BB', CC', DD' are edges of the cube.)

Solution. Let M be the intersection of the plane $A'B'O$ and the line AD. Then M is the midpoint of DA. Therefore, the intersection between the planes $A'B'O$ and $ADD'A'$ is the line $A'M$. Hence, the distance from the point B to the line of intersection of the planes $A'B'O$ and $ADD'A'$ is the height of the isosceles triangle $BA'M$ from B. Since $|MB| = |A'M| = a\sqrt{5}$ and $|BA'| = 2a\sqrt{2}$, we get the height $2a\sqrt{6}/\sqrt{5}$.

For the distance between AB' and BD, we note that BD and AB' are included in the parallel planes $C'DB$ and $AB'D'$, respectively. Therefore, the distance between BD and AB' is the distance between the planes $C'DB$ and $AB'D'$, namely $(1/3)|AC'| = 2a\sqrt{3}/3$.

510. Solve the equation

$$\sqrt[3]{x^2 + 2} + \sqrt[3]{4x^2 + 3x - 2} = \sqrt[3]{3x^2 + x + 5} + \sqrt[3]{2x^2 + 2x - 5} \ .$$

Solution. Let the four members of the equation be denoted by m, n, p, q respectively, so that $m+n = p+q$. Also

$$(m+n)^3 - 3mn(m+n) = m^3 + n^3 = p^3 + q^3 = (p+q)^3 - 3pq(p+q) \ ,$$

whence $mn(m+n) = pq(p+q)$.

If $m+n = p+q = 0$, then $5x^2 + 3x = 0$, so that $x = 0$ or $x = -3/5$. Both of these work.

If $m+n = p+q \neq 0$, then $mn = pq$. Thus

$$\begin{aligned} 0 &= (3x^2 + x + 5)(2x^2 + 2x - 5) - (x^2 + 2)(4x^2 + 3x - 2) \\ &= 2x^4 + 5x^3 - 9x^2 - x - 21 = (x^2 + 2x - 7)(2x^2 + x + 3) \ , \end{aligned}$$

so that $x = -1 \pm 2\sqrt{2}$. The second quadratic factor has nonreal roots. Therefore, the solution set of the equation is

$$\{-1 - 2\sqrt{2}, -3/5, 0, -1 + 2\sqrt{2}\} \ .$$

511. Find the sum of the last 100 digits of the number

$$A = 1 \cdot 2 \cdot 3 \cdot \ldots \cdot 2005 \cdot 2006 + 2007 \ .$$

Solution. The exponent of the power of 5 that divides 2006! is equal to

$$\left\lfloor \frac{2006}{5} \right\rfloor + \left\lfloor \frac{2006}{5^2} \right\rfloor + \left\lfloor \frac{2006}{5^3} \right\rfloor + \left\lfloor \frac{2006}{5^4} \right\rfloor = 401 + 80 + 16 + 3 = 500 \ .$$

The exponent of the power of 2 that divides 2006! is at least this big, so that the decimal expansion of 2006! ends in 500 zeros. Hence the sum of the last 100 digits of the given expression is $2 + 0 + 0 + 7 = 9$.

512. Prove that

$$\binom{3n}{n} = \sum_{k=0}^{n} \binom{2n}{k}\binom{n}{k}$$

when $n \geq 1$.

Solution. The result is obtained by comparing the coefficients of x^n when both sides of the equation

$$(1+x)^{3n} = (1+x)^{2n}(1+x)^n$$

are explanded binomially.

513. Solve the equation

$$2^{1-2\sin^2 x} = 2 + \log_2(1 - \sin^2 x) \ .$$

Solution. Let $t = \cos 2x$. The equation becomes $2^t = 1 + \log_2(1+t)$. For $0 < t < 1$, we have that

$$2^t < t + 1 < 1 + \log_2(1+t) \ ,$$

and for $-1 \leq t < 0$, we have that

$$2^t > t + 1 > 1 + \log_2(1+t) \ .$$

(Sketch the graphs of 2^t and $t+1$.) Therefore $t = 0$ or $t = 1$ and so $x = 0$ or $x = \pi/4$.

Solutions for September.

514. Prove that there do not exist polynomials $f(x)$ and $g(x)$ with complex coefficients for which

$$\log_b x = \frac{f(x)}{g(x)}$$

where b is any base exceeding 1.

Solution 1. Suppose that the given equation is possible. Then we must have, for each positive integer n,

$$\frac{f(x^n)}{g(x^n)} = \log_b x^n = n \log_b x = \frac{nf(x)}{g(x)} \ ,$$

so that $f(x^n)g(x) = nf(x)g(x^n)$. However, by comparing the leading coefficients of the two sides, we see that this is impossible.

Solution 2. We presume that the relation is to be an identity for all x for which both sides are defined, in particular when x is a positive integer. Since $\log_b x = \log_2 x / \log_2 b$, $\log_b x$ is a constant multiple of $\log_2 x$. Thus, it is enough to prove the result with $b = 2$.

It is easily established by induction that, for each positive integer x, $x \leq 2^{x-1}$, so that $\log_2 x \leq x - 1$. Applying this to $x^{1/2}$, where x is a square, this yields

$$\log_2 x = 2 \log_2(x^{1/2}) \leq 2(x^{1/2} - 1) < 2x^{1/2} \ .$$

Suppose, if possible, that there exist polynomials $f(x)$ and $g(x)$ such that, for every positive integer x,

$$\log_2 x = \frac{f(x)}{g(x)} \ .$$

We may suppose that the leading coefficient of $g(x)$ is 1 and that the respective degrees of $f(x)$ and $g(x)$ are the positive integers m and n.

Suppose that

$$f(x) = a_m x^m + a_{m-1}x^{m-1} + \cdots + a_1 x + a_0 = a_m x^m (1 + a_{m-1}x^{-1} + \cdots + a_0 x^{-m})$$

and

$$f(x) = x^n + b_{n-1}x^{n-1} + \cdots + b_1 x + b_0 = x^n(1 + b_{n-1}x^{-1} + b_{n-2}x^{-2} + \cdots + b_0 x^{-n}) \ .$$

Since

$$|x^{-1}(a_{m-1} + \cdots + a_0 x^{1-m})| \leq x^{-1}(|a_{m-1}| + \cdots + |a_0|) \leq Mx^{-1}$$

and

$$|x^{-1}(b_{n-1} + \cdots + b_0 x^{1-n})| \leq x^{-1}(|b_{n-1}| + \cdots + |a_0|) \leq Mx^{-1}$$

where M is the maximum of $|a_{m-1}| + \cdots + |a_0|$ and $|b_{n-1}| + \cdots + |b_0|$, and we can select N such that $Mx^{-1} < 1/2$, for $x > N$, we have that

$$\frac{1}{2} < 1 - |a_{m-1}x^{-1} + \cdots + a_0 x^{-m}| \leq 1 + a_{m-1}x^{-1} + \cdots + a_0 x^{-m} < 1 + |a_{m-1}x^{-1} \cdots a_o x^{-m}| < \frac{3}{2}$$

and

$$\frac{1}{2} < 1 - |b_{n-1}x^{-1} + \cdots + b_0 x^{-m}| \leq 1 + b_{n-1}x^{-1} + \cdots + b_0 x^{-m} < 1 + |b_{m-1}x^{-1} \cdots b_0 x^{-m}| < \frac{3}{2} \ ,$$

so that

$$\frac{1}{3} < \frac{1 + a_{m-1}x^{-1} + \cdots + a_0 x^{-m}}{1 + b_{n-1}x^{-1} + \cdots + b_0 x^{-}} < 3 \ .$$

Then, for $x > 2N$, $f(x)/g(x)$ lies between $\frac{1}{3}a_m x^{m-n}$ and $3a_m x^{m-n}$, so that a_m must be positive.

If $m \leq n$, then $f(x)/g(x)$ is bounded whereas $\log_2 x$ is not. Hence $m > n$. But then for x a large square integer exceeding $2M$,

$$1 = (\log_2 x)^{-1}(f(x)/g(x)) > (1/2x^{1/2})[(1/3)a_m x^{m-n}] = (1/6)a_m x^{m-(1/2)-n} \ .$$

This is a contradiction for sufficiently large x and the result follows.

515. Let n be a fixed positive integer exceeding 1. To any choice of n real numbers x_i satisfying $0 \leq x_i \leq 1$, we can associate the sum

$$\sum\{|x_i - x_j| : 1 \leq i < j \leq n\} \ .$$

What is the maximum possible value of this sum and for which values of the x_i is it assumed?

Solution 1. Wolog, we may suppose that $1 \geq x_1 \geq x_2 \geq \cdots \geq x_n \geq 0$. Then the sum in question is equal to

$$\sum\{x_i - x_j : 1 \leq i < j \leq n\} = (n-1)x_1 + (n-3)x_2 + \cdots + (n+1-2i)x_i + \cdots - (n-1)x_n \ ,$$

Since $0 \leq x_i \leq 1$ for each i, this sum is dominated by $(n-1) + (n-3) + \cdots$, where the sum is taken over all the indices yielding positive coefficients and equality occurs when $x_i = 1$ for these indices.

When $n = 2m$ is even, this maximum sum is equal to $(2m-1)+(2m-3)+\cdots+3+1 = m^2 = n^2/4$; we can achieve it with $x_1 = \cdots = x_m = 1$ and $x_{m+1} = x_{m+2} = \cdots = x_{2m} = 0$. When $n = 2m+1$ is odd, this maximum sum is equal to $(2m) + (2m-2) + \cdots + 2 = 2(m + (m-1) + \cdots + 1) = m(m+1) = (n^2-1)/4$; it is achieved with $x_1 = x_2 = \cdots = x_m = 1$ and $x_{m+2} = x_{m+3} = \cdots = x_{2m+1} = 0$ (the value of x_{m+1} being immaterial).

In summary, the maximum sum can be rendered as $\lfloor n^2/4 \rfloor$.

Solution 2. We may assume that $1 \geq x_1 \geq x_2 \geq \cdots \geq x_n \geq 0$. For $1 \leq i \leq n-1$, let $d_i = x_i - x_{i+1}$, so that $|x_i - x_j| = x_i - x_j = d_i + d_{i+1} + \cdots + d_{j-1}$ when $i < j$. Suppose $1 \leq p \leq n$ is chosen so that $k(n-k) \leq p(n-p)$ for each $1 \leq k \leq n$.

Note that, in the given sum, each d_k occurs in the expansion of terms of the form $x_i - x_j$ where $1 \leq i \leq k$ and $k+1 \leq j \leq n$; there are $k(n-k)$ such terms. Therefore

$$\begin{aligned}\sum\{|x_i - x_j| : 1 \leq i < j \leq n\} &= \sum_{k=1}^{n-1} k(n-k)d_k \\ &\leq p(n-p)\sum_{k=1}^{n-1} d_k \leq p(n-p) \ .\end{aligned}$$

with equality occurring if $d_1 = d_2 = \cdots = d_{p-1} = d_{p+1} = \cdots = d_{n-1} = 0$ and $d_p = x_p - x_{p+1} = 1$, *i.e.* $x_1 = \cdots = x_p = 1$ and $x_{p+1} = \cdots = x_n = 0$.

Since $p(n-p) - k(n-k) = (p-k)(n-p-k) \geq 0$ for all k if and only if $k \leq p \leq n-k$ or $(n-k) \leq p \leq k$ for all k, we see that $p = n/2$ when n is even and $p = (n \pm 1)/2$ when n is odd. This produces the answer in Solution 1.

516. Let $n \geq 1$. Is it true that, for any $2n+1$ positive real numbers $x_1, x_2, \cdots, x_{2n+1}$, we have that

$$\frac{x_1 x_2}{x_3} + \frac{x_2 x_3}{x_4} + \cdots + \frac{x_{2n+1}x_1}{x_2} \geq x_1 + x_2 + \cdots + x_{2n+1} \ ,$$

with equality if and only if all the x_i are equal?

Solution 1. Let $n = 1$. Then we have that

$$2\left(\frac{x_1x_2}{x_3} + \frac{x_2x_3}{x_1} + \frac{x_3x_1}{x_2}\right) = \left(\frac{x_1x_2}{x_3} + \frac{x_2x_3}{x_1}\right) + \left(\frac{x_2x_3}{x_1} + \frac{x_3x_1}{x_2}\right) + \left(\frac{x_3x_1}{x_2} + \frac{x_1x_2}{x_3}\right)$$
$$\geq 2x_1 + 2x_2 + 2x_3 = 2(x_1 + x_2 + x_3) \ ,$$

with equality if and only of $x_1 = x_2 = x_3$, by the arithmetic-geometric means inequality. Thus, the inequality holds for $n = 1$.

The inequality is not generally true for $n \geq 2$. For each positive integer n, define the function

$$L_n(x_1, x_2, \cdots, x_{2n+1}) = \frac{x_1x_2}{x_3} + \frac{x_2x_3}{x_4} + \cdots + \frac{x_{2n+1}x_1}{x_2} \ .$$

Observe that

$$L_{n+1}(x_1, x_2, \cdots, x_{2n+1}, x_1, x_2) = L_n(x_1, x_2, \cdots, x_{2n+1}) + \frac{x_1x_2}{x_1} + \frac{x_2x_1}{x_2} = L_n(x_1, x_2, \cdots, x_{2n+1}) + x_1 + x_2 \ .$$

Thus, if we can determine $(x_1, x_2, \cdots, x_{2n+1})$ to contradict the inequality, then $(x_1, x_2, \cdots, x_{2n+1}, x_1, x_2)$ contradicts the inequality at the nest higher level. Accordingly, to prove our assertion, is suffices to find a counterexample when $n = 2$.

Since

$$L_2(90, 3, 9, 1, 1) = 30 + 27 + 9 + (1/90) + 30 < 97 < 90 + 3 + 9 + 1 + 1 \ ,$$

it follows that the inequality generally fails for $n \geq 2$.

Solution 2. By the Cauchy-Schwarz Inequality, we have in the case $n = 1$,

$$\left(\frac{x_1x_2}{x_3} + \frac{x_2x_3}{x_1} + \frac{x_3x_1}{x_2}\right)\left(\frac{x_1x_3}{x_2} + \frac{x_2x_1}{x_3} + \frac{x_3x_2}{x_1}\right) \geq \left(\sqrt{\frac{x_1^2x_2x_3}{x_3x_2}} + \sqrt{\frac{x_2^2x_3x_1}{x_1x_2}} + \sqrt{\frac{x_3^2x_1x_2}{x_2x_1}}\right)^2 = (x_1 + x_2 + x_3)^2 \ .$$

from which the desired inequality follows.

Suppose that $n \geq 2$. Let $x_1 = 3^4$, $x_2 = 3$, $x_3 = 3^2$, $x_4 = x_5 = \cdots = x_{2n+1} = 1$, so that $x_1 + x_2 + \cdots + x_{2n+1} = 81 + 3 + 9 + (2n - 2) = 91 + 2n$. Then

$$\frac{x_1x_2}{x_3} = \frac{x_2x_3}{x_4} = \frac{x_{2n+1}x_1}{x_2} = 3^3 \ ,$$

$$\frac{x_3x_4}{x_1} = 3^2 \ , \qquad \frac{x_{2n}x_{2n+1}}{x_1} = \frac{1}{3^4} \ ,$$

and

$$\frac{x_ix_{i+1}}{x_{i+2}} = 1 \qquad \text{for } \ i = 4, \cdots, 2n-1 \ .$$

(The last case is vacuous when $n = 2$.) The sum of the left side of the purported inequality is $3 \times 3^3 + 3^2 + (1/3^4) + (2n - 4) < 81 + 9 + 1 + (2n - 4) = 87 + 2n$. Thus, the left side is less than the right side and we have a counterexample.

Comment. The case $n = 1$ was proven by W.P. Wen and the general counterexample is adapted from one given by A. Remorov.

517. A man bought four items in a *Seven-Eleven* store. The clerk entered the four prices into a pocket calculator and *multiplied* to get a result of 7.11 dollars. When the customer objected to this procedure, the clerk realized that he should have added and redid the calculation. To his surprise, he again got the answer 7.11. What did the four items cost?

Solution. Let the cost in cents of the four items be a, b, c, d. Then a, b, c, d are whole numbers with $a + b + c + d = 711 = 3^2 \times 79$ and

$$\left(\frac{a}{100}\right)\left(\frac{b}{100}\right)\left(\frac{c}{100}\right)\left(\frac{d}{100}\right) = \frac{711}{100} \ .$$

so that $abcd = 711 \times 10^6 = 2^6 \times 3^2 \times 5^6 \times 79$. Exactly one price (in cents) is a multiple of 79, and at most three prices (in cents) are even or are a multiple of 5,

It is not possible for three prices to be a multiple of 25. Otherwise, the remaining price would be the multiple of 79, and the sum of the three remaining prices would also be a multiple of 79 as well as of 25. But $79 \times 25 > 711$, and this is not possible. Hence, at least one of the prices is a multiple of $5^3 = 125$; this price is clearly not a multiple of 79.

Case 1: One of the prices is $5 \times 79 = 395$. Suppose that $a = 5 \times 79 = 395$. Suppose that b is a multiple of $5^3 = 125$. Since not all four prices can be a multiple of 5, one price, c, say, must be a multiple of $5^2 = 25$.

If $(a, b) = (395, 125)$, then, modulo 25, $a + b + c \equiv 20$, so that $d \equiv 11 - 20 \equiv 16$. Since d can have only 2, 3, 5 as prime divisor, $d = 16$. But this leads to $c = 175 = 7 \times 5^2$, which is not possible. If $(a, b) = (395, 250)$, again $d = 16$ so that $c = 50 = 2 \times 2 \times 5^2$. But then $abcd$ is not divisible by 3. Since $a + b < 711$, this exhausts the possibilities and Case 1 cannot occur.

Case 2. One of the prices, say a is one of the multiples 79, 158, 231, 316, 474 of 79 and another, say b is one of the multiples 125, 250, 375, 500, 625 of 125. Examining the cases and conducting an analysis similar to that of Case 1, we arrive at the unique solution

$$(a, b, c, d) = (316, 125, 150, 120) = (2^2 \times 79, 5^3, 2 \times 3 \times 5^2, 2^3 \times 3 \times 5 \ .$$

Therefore the four items cost \$1.20, \$1.25, \$1.50 and \$3.16.

Comments. There are a couple of "near misses" where the product is off by a prime factor: $(45, 100, 250, 316) = (3^2 \times 5, 2^2 \times 5^2, 2 \times 5^3, 2^2 \times 79)$ and $(25, 120, 250, 316) = (5^2, 2^3 \times 3 \times 5, 2 \times 5^3, 2^2 \times 79)$.

AQ. Zhang had an interesting way to reject some cases. Suppose that $a = 395 = 5 \times 79$. Then $b + c + d = 316$ and $bcd = 2^6 \times 3^2 \times 5^5$. This gives an arithmetic mean for b, c, d less than 108 and a geometric mean that satisfies

$$\sqrt[3]{2^6 \times 3^2 \times 5^5} = 2^2 \times 5 \times \sqrt[3]{9 \times 25} = 20 \times \sqrt[3]{225} > 20 \times 6 = 120 \ .$$

This is impossible by the arithmetic-geometric means ineqaulity. Similarly, of $a = 375 = 3 \times 5^3$, the arithmetic mean of b, c, d is 112, while the geometric mean is $20 \times \sqrt[3]{237}$ which exceeds 120. Again, this is not possible.

In general $bcd = \frac{10^6 \times 711}{a}$ exceeds 10^6, so that the geometric mean of b, c, d is always at least 100. If $a > 411$, the geometric mean is less than 100. Thus, we eliminate from consideration all multiples of 79 greater than 316 and all multiples of 125 greater than 250.

518. Let I be the incentre of triangle ABC, and let AI, BI, CI, produced, intersect the circumcircle of triangle ABC at the respective points D, E, F. Prove that $EF \perp AD$.

Solution 1. Let $\alpha = \angle BAD = \angle CAD$, $\beta = \angle ABE = \angle CBE$ and $\gamma = \angle ACF = \angle FCB$. Suppose that AI and EF intersect at G. Since $\angle FAB = \angle FCB = \gamma$ and $\angle AFE = \angle ABE = \beta$, it follows that $\angle AGE = \angle FAG + \angle AFG = \alpha + \gamma + \beta = 90°$ and $EF \perp AD$.

Solution 2. Use the same notation as in Solution 1.

$$\angle IGE = \angle IFG + \angle FIG = \angle CFE + \angle AIF = \angle CBE + \angle ACF + \angle IAC = \beta + \gamma + \alpha = 90° \ .$$

Solution 3. Use the same notation as in Solution 1. A rotation with centre F and angle β carries ray FE onto FC. A rotation with centre C and angle γ carries ray CF onto CA. A rotation with centre A and angle α carries AC onto AD. Since all of these rotation have the same sense and the sum of their angles is 90°, it follows that the final position AD of the line EF is perpendicular to EF and the result follows.

Solution 4. Let U be the intersection of AB and CF, P of AB and FE, V of AC and BE and Q of AC and FE. Since $\angle CFE = \angle CBE = \angle EBA$ and $\angle FUA = \angle BUI$, triangles BIU and FPU are similar, so that $\angle BIU = \angle FPU$. Similarly, triangles CIV and EQV are similar, and $\angle CIV = \angle EQV$. Hence, in triangles APG and AQG,

$$\angle APG = \angle FPU = \angle BIU = \angle CIV = \angle EQV = \angle AQG \ .$$

Also, $\angle PAG = \angle QAG$. Therefore, $\angle AGP = \angle AGQ$, and the result follows since P, G, Q are collinear.

519. Let AB be a diameter of a circle and X any point other than A and B on the circumference of the circle. Let t_A, t_B and t_X be the tangents to the circle at the respective points A, B and X. Suppose that AX meets t_B at Z and BX meets t_A at Y. Show that the three lines YZ, t_X and AB are either concurrent (i.e. passing through a common point) or parallel.

Solution. Let t_X intersect t_A and t_B in U and V respectively, and let O be the centre of the circle. If X is the midpoint of the arc AB, then t_X is parallel to AB and the reflection in the diameter of the circle passing through X interchanges A and B, U and V, as well as Y and Z. Hence AB, $UV = t_X$ and YZ, being perpendicular to the diameter, are all parallel.

Henceforth, suppose, say, that X is closer to A than to B. Let $\alpha = \angle XAB$ and $\beta = \angle XBA$. Then, by standard results on isoceles triangles and subtended angles, we have that

$$\alpha = \angle XAB = \angle AXO = \angle AYB = \angle UXY = \angle VBY = \angle VXB$$

and

$$\beta = \angle XBA = \angle OXB = \angle AXU = \angle UAX = \angle BZX = \angle ZXV \ ;$$

also $YU = UX = UA$ and $ZV = XV = BV$.

Thus, U and V are the respective midpoints of AY and BZ. Let BA and ZY intersect at W. Since $AY \| BZ$, the dilatation with factor $|WB|/|WA|$ and centre W takes A to B, Y to Z, and the midpoint U of AY to the midpoint V of BZ. Hence W, U and V are collinear and the result follows.

520. The *diameter* of a plane figure is the largest distance between any pair of points in the figure. Given an equilateral triangle of side 1, show how, by a stright cut, one can get two pieces that can be rearranged to form a figure with maximum diameter

(a) if the resulting figure is convex (*i.e.* the line segment joining any two of its points must lie inside the figure);

(b) if the resulting figure is not necessarily convex, but it is connected (*i.e.* any two points in the figure can be connected by a curve lying inside the figure).

Solution. (a) The maximum diameter is $\sqrt{13}/2$.

We first observe that for a convex polygon, the diameter is realized by joining some two of its vertices. To see this, let PQ be any segment contained within the figure and draw two lines l and m perpendicular to PQ through P and Q respectively. Move l in the direction $\overrightarrow{QP}$ to the last position for which it has a nonvoid intersection with the polygon; this intersection must contain a vertex U (it consists of either a side or a vertex of the polygon). Similarly, move m in the direction $\overrightarrow{PQ}$ until it contains a vertex V. Then the distance between U and V must be at least as great as the distance between the lines l and m, which is at least as great as the distance between P and Q.

In cutting the equilateral triangle ABC, there are two possibilities. Either the cut passes through a vertex A and an interior point D of the opposite side BC. Or it passes through an interior point E of a side AB and an interior point F of a side AC.

Suppose first that the cut is AD, through a vertex. For a convex result, we must place triangle ABD against triangle ADC so that one side of one triangle lies along an equal side of the other. There are generally three ways to do this.

(i) Turn ABD over so that A falls on D and D falls on A. Let B fall on U. The diameter of $ACDU$ is equal to the maximum length of the four sides and two diagonals. The lengths of four sides and of the diagonal AD do not exceed 1.

$$|CU| \leq |CA| + |AU| = |CA| + |DB|$$

and

$$|CU| \leq |CD| + |DU| = |CD| + |AB| \ .$$

Hence

$$|CU| \leq 1 + \ \min \ (|DB|, |CD|) \leq \frac{3}{2}$$

. The diameter of this figure does not exceed $3/2$.

(ii) Move triangle ABD so that A stays put, B falls on C and D goes to a point V (this is a rotation about A). Since $|DV| \leq |DC| + |CV| = |DC| + |BD|$, it can be seen that the diameter of this figure does not exceed 1.

(iii) Move triangle ABD so that A falls on C, B falls on A and D falls on W. Since $|DW|$ does not exceed the minimum of $|AD| + |AW| = |AD| + |BD|$ and $|CW| + |DC| = |AD| + |DC|$, we can deduce that the diameter does not exceed $3/2$.

In the case that D is the midpoint of BC, there are two additional possibilities.

(iv) Place triangle ABD alongside triangle ACD that they have the side CD in common to get an obtuse isosceles triangle whose longest side has length $\sqrt{3}$.

(v) Finally place triangle ABD alongside triangle ACD so that B falls on D and D falls on C to get a $150°, 30°$ parallellogram with side lengths 1 and $\sqrt{3}/2$. By the law of cosines, the length of the longer diagonal of this parallelogram is the square root of

$$1 + \frac{3}{4} - \sqrt{3}\cos 150° = 1 + \frac{3}{4} + \frac{3}{2} = \frac{13}{4} \ ,$$

so that the diameter of this figure turns out to be $\sqrt{13}/2$. Note that this exceeds $\sqrt{3}$.

Consider the second possibility in which the cut EF joins a point E in AB to a point F in AC. A side of the triangle AFE must be placed against an equal side of the quadrilateral $BCFE$. No side of triangle AFE can be placed against BC, since BC is longer than any chord of triangle ABC except sides AB and AC. The equality of FE with either AE or AF occurs exactly when $EF \| BC$ and E and F are the respective midpoints of the sides.

We might consider turning triangle AFE over so that E and F are interchanged. If one of the angles, say $\angle AFE$, exceeds the other, $\angle AEF$, then $\angle BEF + \angle AFE > \angle BEF + \angle AEF = 180°$ and we would not get a convex figure. If the two angles $\angle AEF$ and $\angle AFE$ are equal, then we get triangle ABC with diameter 1. Thus, we find that there are essentially six cases.

(i) Suppose that $|EF| = |FC| = x$ and that triangle AEF is moved so that E falls on F, F falls on C and A falls on P. Let $y = |AE| = |FP|$. This gives a pentagon $BCPFE$ whose respective side lengths are 1, $1 - x$, y, x, $1 - y$, none of which exceeds 1. The three diagonals that lie within triangle ABC have length less than 1. Since both angles EBP and EPB are less than $60°$, BEP is the largest angle of triangle BEP and so $|EP| < |BP|$. Finally,

$$|BP| < |BE| + |EF| + |FP| = (1 - y) + x + y = 1 + x$$

and

$$|BP| < |BC| + |CP| = 1 + (1 - x) = 2 - x \ ,$$

it follows that $|BP|$ is less than the minimum of $1 + x$ and $2 - x$, which cannot exceed $3/2$. Thus, the diameter of $BCPFE$ is less than $3/2$.

(ii) Suppose that $|EF| = |FC| = x$ and triangle AEF is moved so that F stays put, E falls on C and A falls on Q. With $y = |AE| = |QC|$, we get a pentagon $BCQFE$ with respective side lengths $1, y, 1-x, x, 1-y$. We find that

$$|BQ| < \min\,(1 + y, (1 - y) + x + (1 - x)) = \min\,(1 + y, 2 - y) < 3/2$$

and that all other sides and diagonals of the pentagon do not exceed 1. Hence the diameter is less than $3/2$.

(iii) Suppose that $|AE| = |FC| = x$, so that $|AF| = 1 - x$. Let triangle AEF be moved so that A falls on C, E falls on F and F falls on R. Let $y = |EF| = |FR|$. The side lengths of pentagon $BCRFE$ are respectively $1, 1 - x, y, y, 1 - x$. Since $|AE| > |AF|$, $x > 1/2$. We have that

$$|ER| < |BR| < |BC| + |RC| = 2 - x < 3/2$$

and so the diameter of the pentagon must be less than $3/2$.

(iv) Suppose that $|AE| = |FC| = x$ and that triangle AEF is moved so that A falls on F, E falls on C and F falls on S. Using the Law of Cosines on triangle AEF, we have that $y = \sqrt{3x^2 - 3x + 1}$. Then

$$|BS| \le |BE| + |EF| + |FS| = 2(1 - x) + \sqrt{3x^2 - 3x + 1}\ .$$

Since $x \ge 1/2$, we have that

$$(4x^2 - 2x + (1/4)) - (3x^2 - 3x + 1) = x^2 + x - (3/4) = (1/4)(2x - 1)(2x + 3) > 0\ .$$

Hence

$$\sqrt{3x^2 - 3x + 1} < \sqrt{4x^2 - 2x + (1/4)} = 2x - (1/2) = (3/2) - 2(1 - x)\ .$$

Therefore $|BS| < 3/2$. Thus, the diameter of the figure obtained is less than $3/2$.

(v) Suppose that $|AF| = |FC|$ (so that F is the midpoint of AC) and that triangle AEF is moved so that F is left in place, A falls on C and E falls on T. Then we get a quadrilateral $BCTE$ and find that

$$|BT| \le |BF| + |FT| \le 2|BF| = \sqrt{3}\ .$$

Thus, the diameter does not exceed $\sqrt{3}$.

(vi) Suppose that $|AF| = |FC|$ and that triangle AEF is moved so that A falls on F, F falls on C and E falls on U. Let H be on FU produced so that $HC \perp AC$. Since $|FC| = 1/2$ and $\angle CFH = 60^\circ$, $BCHF$ is a 150°, 30° parallelogram with side lengths 1 and $\sqrt{3}/2$, so that $|BU| < |BH| = \sqrt{13}/2$. Thus the diameter of $BCUFE$ does not exceed $\sqrt{13}/2$.

Therefore, the maximum diameter of the convex figure formed by the two pieces is $\sqrt{13}/2$.

(b) The diameter of the resulting figure cannot exceed the sum of the diameters of the pieces, and so is at most 2. To get a figure of diameter 2, cut the equilateral triangle into two right triangles by a median, and line them up to have their hypotenuses collinear with only one point in common.

521. On a 8×8 chessboard, either $+1$ or -1 is written in each square cell. Let A_k be the product of all the numbers in the kth row, and B_k the product of all the numbers in the kth column of the board $(k = 1, 2, \cdots, 8)$. Prove that the number

$$A_1 + A_2 + \cdots + A_8 + B_1 + B_2 + \cdots + B_8$$

is a multiple of 4.

Solution 1. it is clear that the value of each A_k and B_k is $+1$ or -1. Assume that p of the eight A_k have the value 1 and $8-p$ have the value -1. Similarly, suppose that q of the eight B_k have the value 1 and $8-q$ have the value -1. The each product is the product of all the entries, the products $A_1A_2\cdots A_8$ and $B_1B_2\cdots B_8$ are equal, so that $(-1)^{8-p}=(-1)^{8-q}$ and p and q have the same parity. We have that

$$A_1+A_2+\cdots+A_8+B_1+B_2+\cdots+B_8=p+(8-p)(-1)+q+(8-q)(-1)=2(p+q)-16\ .$$

Since $p+q$ is even, both terms on the right are divisible by 4 and the result follows.

Solution 2. The proof is by induction on the number of negative entries in the square array. If all of the entries are equal to $+1$, then the sum in the problem is equal to 16, which is divisible by 4. Let n be a positive integer, and suppose that the result holds when there are $n-1$ entries in the array equal to -1. Let an array U be given for which there are exactly n entries equal to -1. Let V be the array obtained from U by changing exactly one of the entries -1 to $+1$, say the entry in the rth row and sth column. Then the numbers A_i and B_j are the same for both arrays when $i\neq r$ and $j\neq s$.

If A_k and B_k denote the row and column products for the matrix V, then the sum of the problem for the array U is obtained from that for the matrix V by the addition of $-2A_r-2B_s=-2(A_r+B_s)$. Since (A_r,B_s) has one of the values $(+1,+1),(+1,-1),(-1,+1),(-1,-1)$, it follows that the sum is altered by a multiple of 4. Since by the induction hypothesis, the sum for U is divisible by 4, then so also must be the sum for V.

522. (a) Prove that, in each scalene triangle, the angle bisector from one of its vertices is always "between" the median and the altitude from the same vertex.

(b) Find the measures of the angles of a triangle if the lengths of the median, the angle bisector and the altitude from one of its vertices are in the ratio $\sqrt{5}:\sqrt{2}:1$.

Solution 1. (a) Let ABC be a triangle and let P, K and M be the respective intersections of the altitude, angle bisector and median from A in the side BC. Suppose, wolog, $AB<AC$. Then (by Pythagoras' Theorem, for example), $BP<CP$, so that $\angle BAP<\angle CAP$ and the bisector AK of angle A falls within the angle CAP. Hence, $BP<BK$. Since $KB:KC=AB:AC$, $KB<KC$ and the midpoint M of BC must lie in the segment KC. The result follows.

(b) Use the same notation as in (a). We may assume that $|AP|=1$, $|AK|=\sqrt{2}$ and $|AM|=\sqrt{5}$. We first note that the altitude from A must lie outside of the triangle. Suppose, on the contrary, that P lies on the side BC. By Pythagoras' Theorem, we have that $|PK|=1$, so that $\angle PAK=45^\circ$. Then

$$\angle BAP+45^\circ=\angle BAK=\angle CAK=\angle CAP-45^\circ\ ,$$

so that

$$\angle CAP=\angle BAP+90^\circ>90^\circ\ ,$$

which is impossible.

Hence P must lie on CB produced and B lies in the segment PK. Let $|PB|=x$, so that $|BK|=1-x$, $|PM|=2$ (by Pythagoras' Theorem), $|KM|=1$, $|MC|=2-x$ and $|PC|=4-x$. We have that

$$45^\circ-\angle PAB=\angle BAK=\angle CAK=\angle PAC-45^\circ\ ,$$

so that $\angle PAC=90^\circ-\angle PAB$ and

$$4-x=\tan\angle PAC=\cot\angle PAB=\frac{1}{x}\ .$$

Thus, $x^2-4x+1=0$ and $x=2-\sqrt{3}$. We reject the larger root as it would be the reciprocal of the smaller and so it would be the tangent of $\angle PAC$ which is larger than $\angle PAB$.

Therefore, $\tan \angle PAB = 2 - \sqrt{3}$ and so, from the double angle formula, $\tan 2\angle PAB = 1/\sqrt{3}$ Thus, $\angle PAB = 15°$, $\angle PAC = 75°$ and $\angle BAC = 60°$. Since $\angle PBA = 75°$, it follows that $\angle ABC = 105°$ and $\angle BCA = 15°$.

(It can also be checked that $|AB| = 2\sqrt{2-\sqrt{3}}$, $|AC| = 2\sqrt{2+\sqrt{3}}$ and $|BC| = 2\sqrt{3}$.

Solution 2. (a) can be established as before. For (b), assume wolog that $AC > AB$. We first establish that $\angle ABC$ is obtuse. Let $\angle BAC = \alpha$, $\angle ABC = \beta$ and $\angle ACB = \gamma$. Since $\beta > \gamma$,

$$\angle AKC = \beta + \alpha/2 > \gamma + \alpha/2 = \angle AKB \ ,$$

so that $\angle AKB < 90°$ (which agrees with $\angle AKP = 45°$) and $\angle AKC > 90°$ (more precisely, $\angle AKC = 135°$). Hence $\beta + \alpha/2 = \angle AKC = 135°$, so that $180° - \beta - \gamma = \alpha = 270° - 2\beta$ and $\beta = \gamma + 90° > 90°$.

By Pythagoras's theorem, $|PK| = 1$, $|PM| = 2$ and $|KM| = 1$. Let $|PB| = x$ ($x < 1$), so that $|BK| = 1 - x$, $|BM| = 2 - x$, $|BC| = 2|BM| = 4 - 2x$, and $|PC| = 4 - x$.

The triangles ACP and BAP are similar since both are right and

$$\angle PAB = \angle ABC - 90° = \beta - 90° = \gamma = \angle ACP \ .$$

Therefore $AP : PC = BP : AP$, or, equivalently, $1 : (4-x) = x : 1$. Therefore, x is the smaller of the roots of $x^2 - 4x + 1 = 0$, namely $2 - \sqrt{3}$.

Thus, $\tan \angle PAB = 2 - \sqrt{3}$, so that $\angle PAB = 15°$. (One way to check this is to use the double angle formula to find the tangent of 15°.) Therefore, $\gamma = \angle ACB = \angle PAB = 15°$, $\beta = \angle ABC = \gamma + 90° = 105°$ and $\alpha = \angle BAC = 60°$.

Solution 3. [J. Schneider] Wolog, let $\angle B > \angle C$. we use the notation of the first solution. If B is obtuse, then B lies between P and K. Since $AB < AC$, $BK : KC = AB : AC$, so that $BK < KC$ and M lies between K and C.

Let the angle at B be acute. Then $BP : PC = \tan C : \tan B$, $BK : KC = c : b = \sin C : \sin B$ and $BM : MC = 1 : 1$. Since $\sin C < \sin B$ and $\cos C > \cos B$,

$$\frac{\tan C}{\tan B} = \frac{\cos B}{\cos C} \cdot \frac{\sin C}{\sin B} < \frac{\sin C}{\sin B} < 1 \ ,$$

and the result follows.

(b) Let $x = |MC|$ and coordinatize the situation by $A \sim (0,1)$, $B \sim (0,0)$, $K \sim (1,0)$, $M \sim (2,0)$, $C \sim (2+x, 0)$ and $B \sim (2-x, 0)$. The proportion $AB^2 : AC^2 = AK^2 : KC^2$ leads to the equation

$$\frac{(x+1)^2}{(x-1)^2} = \frac{x^2+4x+5}{x^2-4x+5} \ ,$$

which simplifies to $x(x^2-3) = 0$. Since $vertAK| < |AC|$, we reject $x = -\sqrt{3}$. Hence $x = \sqrt{3}$. Note that this places B to the right of the origin and so angle B is obtuse.

Thus $|AB| = 2\sqrt{2-\sqrt{3}}$, $|AC| = 2\sqrt{2+\sqrt{3}}$ and $|BC| = 2\sqrt{3}$. Angle A can be identified using the Law of Cosines and the remaining angles from their tangents.

523. Let ABC be an isosceles triangle with $AB = AC$. The segments BC and AC are used as hypotenuses to construct three right triangles BCM, BCN and ACP. Prove that, if $\angle ACP + \angle BCM + \angle BCN = 90°$, then the triangle MPN is isosceles.

Solution 1. Clearly, M and N are points on a circle whose diameter is BC. Let O be the midpoint of BC and the centre of this circle, and Q the intersection point of the ray PO and the circle. We have that

$$\angle BCM + \angle BCN = \frac{1}{2}\text{arc } BM + \text{arc } BN) \ . \qquad (1)$$

Observe that, as triangle ABC is isosceles with O the midpoint of its base BC, $AO \perp BC$. Therefore, O and P are on the circle with diameter AC, so that

$$90^\circ - \angle ACP = \angle PAC = \angle POC = \angle BOQ \ . \qquad (2)$$

We are given that $90^\circ - \angle ACP = \angle BCM + \angle BCN$, so that (1) and (2) yield

$$\text{arc } BQ = \angle BOQ = \frac{1}{2}(\text{arc } BM + \text{ arc } BN)$$

or

$$\text{arc } BN \ + \text{arc } NQ \ + \ \text{arc } BQ = \frac{1}{2}(\text{arc } BN \ + \text{arc } MN \ + \ \text{arc } BN)$$

which in turn is equivalent to arc $NQ = \frac{1}{2}$(arc MN). Thus, Q is the midpoint of the arc MN, so that PQ is the right bisector of the segment MN. The result follows.

Solution 2. [J. Schneider] Note that triangle MPN is isosceles with $PM = PN$ if and only if the right bisector of MN passes through P.

Let D be the midpoint of BC. Since ABC is isosceles, $AD \perp BC$ and D lies on the circle with diameter AC. Thus, $APCD$ is concyclic and $\angle ADP = \angle ACP$.

Since D is the centre of the circle with diameter BC that contains M and N, $\angle BDN = 2\angle BCN$ and $\angle BDM = 2\angle BCM$. Let X be the midpoint of MN. Then DX right bisects MN and bisects angle MDN. Hence

$$\angle BDX = \frac{1}{2}(\angle BDM + \angle BDN) = \angle BCN + \angle BCM \ .$$

Suppose that $\angle BCN + \angle BCM + \angle ACP = 90^\circ$, as hypothesized. Then

$$\angle PDC = 90^\circ - \angle ADP = 90^\circ - \angle ACP = \angle BCM + \angle BCN = \angle BDX \ .$$

Hence X, D, P are collinear. But DX is the right bisector of MN, and so is DP. Hence triangle MPN is isosceles.

Comment. The above argument applies when all triangles are external to triangle ABC. It can be adapted to the other cases.

524. Solve the irrational equation

$$\frac{7}{\sqrt{x^2-10x+26}+\sqrt{x^2-10x+29}+\sqrt{x^2-10x+41}} = x^4 - 9x^3 + 16x^2 + 15x + 26 \ .$$

Solution. Observe that

$$x^4 - 9x^3 + 16x^2 + 15x + 26 = (x^2 + x + 1)(x-5)^2 + 1 \ .$$

Since $x^2 + x + 1 = (x + \frac{1}{2})^2 + \frac{3}{4} > 0$ for all x, the quartic on the right side of the equation is never less than 1 and is equal to 1 if and only if $x = 5$.

Since $x^2 - 10x + 25 + k = (x-5)^2 + k$ for $k = 1, 4, 16$, the left side of the equation is never greater than 1 and is equal to 1 if and only if $x = 5$. It follows that $x = 5$ is the only solution of the equation.

525. The circle inscribed in the triangle ABC divides the median from A into three segments of the same length. If the area of ABC is $6\sqrt{14}$, calculate the lengths of its sides.

Solution. Let the median from A meet the side BC at M. Let a, b, c denote the side lengths of ABC as usual, and let the length of the median AM be $3u$. Suppose that the incircle of triangle ABC touches sides BC, CA, AB at U, V, W, respectively. Suppose, wolog, that $AB < AC$, so that U lies between B and M.

By the power of a point, we have that $|AV|^2 = 2u^2 = |MU|^2$, so that

$$(1/2)(b+c-a) = |AV| = |MU| = (1/2)a - (1/2)(a+c-b) = (1/2)(b-c) \ ,$$

and $8u^2 = b^2 - 2bc + c^2$. Hence $b+c-a = b-c$, whence $a = 2c$ and $|BM| = |MC| = |AB| = c$. By the Law of Cosines applied to triangles ABM and AMC, with $\alpha = \angle AMB$,

$$c^2 = c^2 + (3u)^2 - 6uc\cos\alpha$$

and

$$b^2 = c^2 + (3u)^2 + 6uc\cos\alpha \ ,$$

whence

$$b^2 = c^2 + 18u^2 = (9/4)(b^2 - 2bc + c^2) \ .$$

This simplifies to

$$0 = 5b^2 - 18bc + 13c^2 = (b-c)(5b-13c) \ .$$

Since $b \neq c$ (otherwise, the median from A would be the angle bisector of A and the incircle would touch BC at M), we must have $b = 13c/5$. Hence $(a,b,c) = (2c, 13c/5, c)$, the semiperimeter of the triangle is $14c/5$ and the square of its area is $(1/5^4)(14c)(4c)c(9c) = (c^4/5^4)(14)(36)$. Since we are given that the square of the area is $(14)(36)$, $c = 5$ and the dimensions of the triangle are $(10, 5, 13)$.

Comment. All triangles described in the first sentence of the problem have a common property, in that their sides are in the ratio $10 : 5 : 13$. This is, in fact, the essence of the problem. There are many modifications with the same core idea; for example, instead of giving the area of the triangle, we could give the length of the altitude from B, of the angle bisector from C or of the median from A. Recall that these last three quantities are given respectively by

$$h_b = \frac{2}{b}\sqrt{s(s-a)(s-b)(s-c)}$$

$$l_c = \frac{2}{a+b}\sqrt{abs(s-c)}$$

$$m_a = \frac{1}{2}\sqrt{2b^2 + 2c^2 - a^2}$$

where $s = \frac{1}{2}(a+b+c)$ is the semiperimeter of the triangle.

526. For the non-negative numbers a, b, c, prove the inequality

$$4(a+b+c) \geq 3(a + \sqrt{ab} + \sqrt[3]{abc}) \ .$$

When does equality hold?

Solution 1. Equality holds when $a = b = c = 0$. The inequality clearly holds when $a = 0$ and when $b = 0$, so henceforth we will assume that $ab \neq 0$. Define the nonnegative numbers u and v by

$$u^2 = \frac{b}{a} \qquad \text{and} \qquad v^3 = \frac{bc}{a^2} \ .$$

Dividing the inequality through by a, we see that it is equivalent to

$$4\left(1 + u^2 + \frac{v^3}{u^2}\right) \geq 3(1 + u + v)$$

or

$$4(u^2 + u^4 + v^3) \geq 3(u^2 + u^3 + u^2 v) \ .$$

The difference between the two members of the last inequality is

$$4u^4 - 3u^3 + u^2 + 4v^3 - 3u^2v = u^2(2u-1)^2 + (2v-u)^2(v+u) \ .$$

Because of the square terms, it is always nonnegative, and it is equal to zero if and only if $(u, v) = (1/2, 1/4)$. This is achieved when $a : b : c = 16 : 4 : 1$. Therefore, the inequality always holds and equality occurs when $(a, b, c) = (16t, 4t, t)$ for some nonnegative value of t.

Comment. Since the genesis of the solution is far from obvious, it might be worth commenting on how it was arrived at. It is straightforward to dispose of the cases in which any of the variables vanish, so we may as well suppose that all are positive. We observe that the left and right sides of the inequality are homogeneous of degree 1, so that any scalar mutiple of a solution vector is also a solution. Thus, we might as well assume that $a = 1$. The next step is to get rid of the radicals, which we can do by assuming the quantity under the square root sign is u^2 and under the cube root sign is v^3; it is now a matter of backtracking to define these in terms of a, b and c. Some manipulation gives an equivalent polynomial inequality in terms of u and v. We now look at the difference between the two sides and investigate the possibility of getting some representation of this difference in terms of squares and things known to be positive. However, all these machinations can be avoided by a little insight, as we shall see in the next solution.

$4u^4 - 3u^3 + u^2$ is almost a square, so we might as well complete it by subtracting u^3 and adding it to the rest of the expression to get $(2u^2 - u)^2 + (4v^3 - 3u^2v + u^3)$. We notice that the expression in the second parentheses vanished when $v = -u$, which makes $v + u$ a factor of it. The remaining factor turns out to be $(2v - u)^2$ and we are finished.

Solution 2. [J. Schneider] Let $a = u$, $b = v/4$ and $c = w/16$. The inequality is equivalent to

$$4\left(u + \frac{v}{4} + \frac{w}{16}\right) \geq 3\left(u + \frac{1}{2}\sqrt{uv} + \frac{1}{4}\sqrt[3]{uvw}\right) \ .$$

Since $3u \geq 3u$, $(3/4)(u + v) \geq (3/2)\sqrt{uv}$ and $(1/4)(u + v + w) \geq 3\sqrt[3]{uvw}$ (the last two by the arithmetic-geometric means inequality), the desired inequality follows. Equality occurs if and only if $u = v = w$, or $a = 4b = 16c$.

Solution 3. The left side of the inequality can be rewritten

$$4(a + b + c) = 3a + \frac{3}{4}(a + 4b) + \frac{1}{4}(a + 4b + 16c) \ .$$

Using the arithmetic-geometric means inequality, we have that

$$a + 4b \geq 2\sqrt{a(4b)} = 4\sqrt{ab}$$

and

$$a + 4b + 16c \geq 3\sqrt[3]{a(4b)(16c)} = 12\sqrt[3]{abc} \ ,$$

from which the desired result follows. Equality occurs if and only if $a = 4b = 16c$.

527. Consider the set A of the $2n-$digit natural numbers, with 1 and 2 each occurring n times as a digit, and the set B of the $n-$digit numbers all of whose digits are 1, 2, 3, 4 with the digits 1 and 2 occurring with equal frequency. Show that A and B contain the same number of elements (*i.e.*, have the same cardinality).

Solution 1. We show that A and B have the same number of elements by pairing off the elements of one set with elements of the other. Suppose that a number in A is given; separate it into n consecutive pairs of digits; these pairs will be one of 11, 12, 21, 22. Observe that, since the digits 1 and 2 occur equally frequently, the pairs 11 and 22 must occur equally frequently. Moving from left to right, we construct an $n-$ digit number by replacing each pair 11 by the digit 1, 22 by the digit 2, 12 by the digit 3 and 21 by the digit 4. Thus, for example, the number 1222112112112212 corresponds to 32143123. Because the number in A has equally many pairs 11 and 22, the corresponding number will have 1 and 2 occurring equally often and will lie in B.

Conversely, given an $n-$digits number in B, construct a $2n-$digit number by replacing each 1 by 11, 2 by 22, 3 by 12 and 4 by 21. Because 1 and 2 occur equally often, the pairs 11 and 22 will occur equally often in the resulting number, which will then belong to A. Thus the correspondence is one-one and the result follows.

Comment. The number of elements in A is $\binom{2n}{n}$, the number of ways of selecting the places for the n ones. To select numbers in B with $r \leq n/2$ digits equal to 1, we can choose the places for the ones in $\binom{n}{r}$ ways, the places for the twos in $\binom{n-r}{r}$ ways. This leaves $n - 2r$ places left over, which can be filled with either threes or fours in 2^{2n-r} ways. Thus, the number of elements in B is

$$\sum_{r=0}^{\lfloor n/2 \rfloor} \binom{n}{r} \binom{n-r}{r} 2^{2n-2} .$$

The current problem provides a combinatorial way of verifying the equality of these two expressions. Can you find an alegebraic demonstration?