OLYMON

VOLUME 3

2002

Problems 127-198

This is the *Mathematical Olympiads Correspondence Program* sponsored by the Canadian Mathematical Society and the University of Toronto Department of Mathematics. The organizer and editor is Edward J. Barbeau of the University of Toronto Department of Mathematics, and the problems and solutions for this volume of Olymon were prepared by **Edward J. Barbeau** of the University of Toronto, **Valeria Pandelieva** in Ottawa and **Mihai Rosu** of Toronto.

Notes. A function $f : A \rightarrow B$ is a *bijection* iff it is one-one and onto; this means that, if $f(u) = f(v)$, then $u = v$, and, if w is some element of B, then A contains an element t for which $f(t) = w$. Such a function has an *inverse* f^{-1} which is determined by the condition

$$f^{-1}(b) = a \Leftrightarrow b = f(a) \ .$$

The sides of a right-angled triangle that are adjacent to the right angle are called *legs*. The *centre of gravity* or *centroid* of a collection of n mass particles is the point where the cumulative mass can be regarded as concentrated so that the motion of this point, when exposed to outside forces such as gravity, is identical to that of the whole collection. To illustrate this point, imagine that the mass particles are connected to a point by rigid non-material sticks (with mass 0) to form a structure. The point where the tip of a needle could be put so that this structure is in a state of balance is its centroid. In addition, there is an intuitive definition of a centroid of a lamina, and of a solid: The centroid of a lamina is the point, which would cause equilibrium (balance) when the tip of a needle is placed underneath to support it. Likewise, the centroid of a solid is the point, at which the solid "balances", *i.e.*, it will not revolve if force is applied. The centroid, G of a set of points is defined vectorially by

$$\overrightarrow{OG} = \frac{\sum_{i=1}^{n} m_i \cdot \overrightarrow{OM}_i}{\sum_{i=1}^{n} m_i}$$

where m_i is the mass of the particle at a position M_i (the summation extending over the whole collection). Problem 181 is related to the centroid of an assembly of three particles placed at the vertices of a given triangle. The *circumcentre* of a triangle is the centre of its circumscribed circle. The *orthocentre* of a triangle is the intersection point of its altitudes. An *unbounded* region in the plane is one not contained in the interior of any circle.

An *isosceles* tetrahedron is one for which the three pairs of oppposite edges are equal. For integers a, b and n, $a \equiv b$, modulo n, iff $a - b$ is a multiple of n.

127. Let

$$A = 2^n + 3^n + 216(2^{n-6} + 3^{n-6})$$

and

$$B = 4^n + 5^n + 8000(4^{n-6} + 5^{n-6})$$

where $n > 6$ is a natural number. Prove that the fraction A/B is reducible.

128. Let n be a positive integer. On a circle, n points are marked. The number 1 is assigned to one of them and 0 is assigned to the others. The following operation is allowed: Choose a point to which 1 is assigned and then assign $(1 - a)$ and $(1 - b)$ to the two adjacent points, where a and b are, respectively,

the numbers assigned to these points before. Is it possible to assign 1 to all points by applying this operation several times if (a) $n = 2001$ and (b) $n = 2002$?

129. For every integer n, a nonnegative integer $f(n)$ is assigned such that

(a) $f(mn) = f(m) + f(n)$ for each pair m, n of natural numbers;

(b) $f(n) = 0$ when the rightmost digit in the decimal representation of the number n is 3; and

(c) $f(10) = 0$.

Prove that $f(n) = 0$ for any natural number n.

130. Let $ABCD$ be a rectangle for which the respective lengths of AB and BC are a and b. Another rectangle is circumscribed around $ABCD$ so that each of its sides passes through one of the vertices of $ABCD$. Consider all such rectangles and, among them, find the one with a maximum area. Express this area in terms of a and b.

131. At a recent winter meeting of the Canadian Mathematical Society, some of the attending mathematicians were friends. It appeared that every two mathematicians, that had the same number of friends among the participants, did not have a common friend. Prove that there was a mathematician who had only one friend.

132. Simplify the expression

$$\sqrt[5]{3\sqrt{2} - 2\sqrt{5}} \cdot \sqrt[10]{\frac{6\sqrt{10} + 19}{2}} \ .$$

133. Prove that, if a, b, c, d are real numbers, $b \neq c$, both sides of the equation are defined, and

$$\frac{ac - b^2}{a - 2b + c} = \frac{bd - c^2}{b - 2c + d} \ ,$$

then each side of the equation is equal to

$$\frac{ad - bc}{a - b - c + d} \ .$$

Give two essentially different examples of quadruples (a, b, c, d), not in geometric progression, for which the conditions are satisfied. What happens when $b = c$?

134. Suppose that

$$a = zb + yc$$

$$b = xc + za$$

$$c = ya + xb \ .$$

Prove that

$$\frac{a^2}{1 - x^2} = \frac{b^2}{1 - y^2} = \frac{c^2}{1 - z^2} \ .$$

Of course, if any of x^2, y^2, z^2 is equal to 1, then the conclusion involves undefined quantities. Give the proper conclusion in this situation. Provide two essentially different numerical examples.

135. For the positive integer n, let $p(n) = k$ if n is divisible by 2^k but not by 2^{k+1}. Let $x_0 = 0$ and define x_n for $n \geq 1$ recursively by

$$\frac{1}{x_n} = 1 + 2p(n) - x_{n-1} \ .$$

Prove that every nonnegative rational number occurs exactly once in the sequence $\{x_0, x_1, x_2, \cdots, x_n, \cdots\}$. ∎

136. Prove that, if in a semicircle of radius 1, five points A, B, C, D, E are taken in consecutive order, then

$$|AB|^2 + |BC|^2 + |CD|^2 + |DE|^2 + |AB||BC||CD| + |BC||CD||DE| < 4 .$$

137. Can an arbitrary convex quadrilateral be decomposed by a polygonal line into two parts, each of whose diameters is less than the diameter of the given quadrilateral?

138. (a) A room contains ten people. Among any three. there are two (mutual) acquaintances. Prove that there are four people, any two of whom are acquainted.

(b) Does the assertion hold if "ten" is replaced by "nine"?

139. Let A, B, C be three pairwise orthogonal faces of a tetrahedran meeting at one of its vertices and having respective areas a, b, c. Let the face D opposite this vertex have area d. Prove that

$$d^2 = a^2 + b^2 + c^2 .$$

140. Angus likes to go to the movies. On Monday, standing in line, he noted that the fraction x of the line was in front of him, while $1/n$ of the line was behind him. On Tuesday, the same fraction x of the line was in front of him, while $1/(n + 1)$ of the line was behind him. On Wednesday, the same fraction x of the line was in front of him, while $1/(n + 2)$ of the line was behind him. Determine a value of n for which this is possible.

141. In how many ways can the rational 2002/2001 be written as the product of two rationals of the form $(n + 1)/n$, where n is a positive integer?

142. Let $x, y > 0$ be such that $x^3 + y^3 \leq x - y$. Prove that $x^2 + y^2 \leq 1$.

143. A sequence whose entries are 0 and 1 has the property that, if each 0 is replaced by 01 and each 1 by 001, then the sequence remains unchanged. Thus, it starts out as $010010101001\cdots$. What is the 2002th term of the sequence?

144. Let a, b, c, d be rational numbers for which $bc \neq ad$. Prove that there are infinitely many rational values of x for which $\sqrt{(a + bx)(c + dx)}$ is rational. Explain the situation when $bc = ad$.

145. Let ABC be a right triangle with $\angle A = 90°$. Let P be a point on the hypotenuse BC, and let Q and R be the respective feet of the perpendiculars from P to AC and AB. For what position of P is the length of QR minimum?

146. Suppose that ABC is an equilateral triangle. Let P and Q be the respective midpoint of AB and AC, and let U and V be points on the side BC with $4BU = 4VC = BC$ and $2UV = BC$. Suppose that PV are joined and that W is the foot of the perpendicular from U to PV and that Z is the foot of the perpendicular from Q to PV.

Explain how that four polygons $APZQ$, $BUWP$, $CQZV$ and UVW can be rearranged to form a rectangle. Is this rectangle a square?

147. Let $a > 0$ and let n be a positive integer. Determine the maximum value of

$$\frac{x_1 x_2 \cdots x_n}{(1 + x_1)(x_1 + x_2) \cdots (x_{n-1} + x_n)(x_n + a^{n+1})}$$

subject to the constraint that $x_1, x_2, \cdots, x_n > 0$.

148. For a given prime number p, find the number of distinct sequences of natural numbers (positive integers) $\{a_0, a_1, \cdots, a_n \cdots\}$ satisfying, for each positive integer n, the equation

$$\frac{a_0}{a_1} + \frac{a_0}{a_2} + \cdots + \frac{a_0}{a_n} + \frac{p}{a_{n+1}} = 1 .$$

149. Consider a cube concentric with a parallelepiped (rectangular box) with sides $a < b < c$ and faces parallel to that of the cube. Find the side length of the cube for which the difference between the volume of the union and the volume of the intersection of the cube and parallelepiped is minimum.

150. The area of the bases of a truncated pyramid are equal to S_1 and S_2 and the total area of the lateral surface is S. Prove that, if there is a plane parallel to each of the bases that partitions the truncated pyramid into two truncated pyramids within each of which a sphere can be inscribed, then

$$S = (\sqrt{S_1} + \sqrt{S_2})(\sqrt[4]{S_1} + \sqrt[4]{S_2})^2 .$$

151. Prove that, for any natural number n, the equation

$$x(x+1)(x+2)\cdots(x+2n-1) + (x+2n+1)(x+2n+2)\cdots(x+4n) = 0$$

does not have real solutions.

152. Andrew and Brenda are playing the following game. Taking turns, they write in a sequence, from left to right, the numbers 0 or 1 until each of them has written 2002 numbers (to produce a 4004-digit number). Brenda is the winner if the sequence of zeros and ones, considered as a binary number (*i.e.*, written to base 2), can be written as the sum of two integer squares. Otherwise, the winner is Andrew. Prove that the second player, Brenda, can always win the game, and explain her winning strategy (*i.e.*, how she must play to ensure winning every game).

153. (a) Prove that, among any 39 consecutive natural numbers, there is one the sum of whose digits (in base 10) is divisible by 11.

(b) Present some generalizations of this problem.

154. (a) Give as neat a proof as you can that, for any natural number n, the sum of the squares of the numbers $1, 2, \cdots, n$ is equal to $n(n+1)(2n+1)/6$.

(b) Find the least natural number n exceeding 1 for which $(1^2 + 2^2 + \cdots + n^2)/n$ is the square of a natural number.

155. Find all real numbers x that satisfy the equation

$$3^{[(1/2)+\log_3(\cos x+\sin x)]} - 2^{\log_2(\cos x - \sin x)} = \sqrt{2} .$$

[The logarithms are taken to bases 3 and 2 respectively.]

156. In the triangle ABC, the point M is from the inside of the angle BAC such that $\angle MAB = \angle MCA$ and $\angle MAC = \angle MBA$. Similarly, the point N is from the inside of the angle ABC such that $\angle NBA = \angle NCB$ and $\angle NBC = \angle NAB$. Also, the point P is from the inside of the angle ACB such that $\angle PCA = \angle PBC$ and $\angle PCB = \angle PAC$. (The points M, N and P each could be inside or outside of the triangle.) Prove that the lines AM, BN and CP are concurrent and that their intersection point belongs to the circumcircle of the triangle MNP.

157. Prove that if the quadratic equation $x^2 + ax + b + 1 = 0$ has nonzero integer solutions, then $a^2 + b^2$ is a composite integer.

158. Let $f(x)$ be a polynomial with real coefficients for which the equation $f(x) = x$ has no real solution. Prove that the equation $f(f(x)) = x$ has no real solution either.

159. Let $0 \leq a \leq 4$. Prove that the area of the bounded region enclosed by the curves with equations

$$y = 1 - |x - 1|$$

and

$$y = |2x - a|$$

cannot exceed $\frac{1}{3}$.

160. Let I be the incentre of the triangle ABC and D be the point of contact of the inscribed circle with the side AB. Suppose that ID is produced outside of the triangle ABC to H so that the length DH is equal to the semi-perimeter of ΔABC. Prove that the quadrilateral $AHBI$ is concyclic if and only if angle C is equal to $90°$.

161. Let a, b, c be positive real numbers for which $a + b + c = 1$. Prove that

$$\frac{a^3}{a^2 + b^2} + \frac{b^3}{b^2 + c^2} + \frac{c^3}{c^2 + a^2} \geq \frac{1}{2} \ .$$

162. Let A and B be fixed points in the plane. Find all positive integers k for which the following assertion holds:

among all triangles ABC with $|AC| = k|BC|$, the one with the largest area is isosceles.

163. Let R_i and r_i re the respective circumradius and inradius of triangle $A_iB_iC_i$ $(i = 1, 2)$. Prove that, if $\angle C_1 = \angle C_2$ and $R_1r_2 = r_1R_2$, then the two triangles are similar.

164. Let n be a positive integer and X a set with n distinct elements. Suppose that there are k distinct subsets of X for which the union of any four contains no more that $n-2$ elements. Prove that $k \leq 2^{n-2}$.

165. Let n be a positive integer. Determine all $n-$tples $\{a_1, a_2, \cdots, a_n\}$ of positive integers for which $a_1 + a_2 + \cdots + a_n = 2n$ and there is no subset of them whose sum is equal to n.

166. Suppose that f is a real-valued function defined on the reals for which

$$f(xy) + f(y - x) \geq f(y + x)$$

for all real x and y. Prove that $f(x) \geq 0$ for all real x.

167. Let $u = (\sqrt{5} - 2)^{1/3} - (\sqrt{5} + 2)^{1/3}$ and $v = (\sqrt{189} - 8)^{1/3} - (\sqrt{189} + 8)^{1/3}$. Prove that, for each positive integer n, $u^n + v^{n+1} = 0$.

168. Determine the value of

$$\cos 5° + \cos 77° + \cos 149° + \cos 221° + \cos 293° \ .$$

169. Prove that, for each positive integer n exceeding 1,

$$\frac{1}{2^n} + \frac{1}{2^{1/n}} < 1 \ .$$

170. Solve, for real x,

$$x \cdot 2^{1/x} + \frac{1}{x} \cdot 2^x = 4 \ .$$

171. Let n be a positive integer. In a round-robin match, n teams compete and each pair of teams plays exactly one game. At the end of the match, the ith team has x_i wins and y_i losses. There are no ties. Prove that

$$x_1^2 + x_2^2 + \cdots + x_n^2 = y_1^2 + y_2^2 + \cdots + y_n^2 \ .$$

172. Let a, b, c, d. e, f be different integers. Prove that

$$(a-b)^2 + (b-c)^2 + (c-d)^2 + (d-e)^2 + (e-f)^2 + (f-a)^2 \geq 18 \; .$$

173. Suppose that a and b are positive real numbers for which $a + b = 1$. Prove that

$$\left(a + \frac{1}{a}\right)^2 + \left(b + \frac{1}{b}\right)^2 \geq \frac{25}{2} \; .$$

Determine when equality holds.

174. For which real value of x is the function

$$(1-x)^5(1+x)(1+2x)^2$$

maximum? Determine its maximum value.

175. ABC is a triangle such that $AB < AC$. The point D is the midpoint of the arc with endpoints B and C of that arc of the circumcircle of ΔABC that contains A. The foot of the perpendicular from D to AC is E. Prove that $AB + AE = EC$.

176. Three noncollinear points A, M and N are given in the plane. Construct the square such that one of its vertices is the point A, and the two sides which do not contain this vertex are on the lines through M and N respectively. [*Note:* In such a problem, your solution should consist of a description of the construction (with straightedge and compasses) and a proof in correct logical order proceeding from what is given to what is desired that the construction is valid. You should deal with the feasibility of the construction.]

177. Let a_1, a_2, $\cdots$, a_n be nonnegative integers such that, whenever $1 \leq i$, $1 \leq j$, $i + j \leq n$, then

$$a_i + a_j \leq a_{i+j} \leq a_i + a_j + 1 \; .$$

(a) Give an example of such a sequence which is not an arithmetic progression.

(b) Prove that there exists a real number x such that $a_k = \lfloor kx \rfloor$ for $1 \leq k \leq n$.

178. Suppose that n is a positive integer and that $x_1, x_2, \cdots, x_n$ are positive real numbers such that $x_1 + x_2 + \cdots + x_n = n$. Prove that

$$\sum_{i=1}^{n} \sqrt[n]{ax_i + b} \leq a + b + n - 1$$

for every pair a, b of real numbers with each $ax_i + b$ nonnegative. Describe the situation when equality occurs.

179. Determine the units digit of the numbers a^2, b^2 and ab (in base 10 numeration), where

$$a = 2^{2002} + 3^{2002} + 4^{2002} + 5^{2002}$$

and

$$b = 3^1 + 3^2 + 3^3 + \cdots + 3^{2002} \; .$$

180. Consider the function f that takes the set of complex numbers into itself defined by $f(z) = 3z + |z|$. Prove that f is a bijection and find its inverse.

181. Consider a regular polygon with n sides, each of length a, and an interior point located at distances a_1, a_2, $\cdots$, a_n from the sides. Prove that

$$a \sum_{i=1}^{n} \frac{1}{a_i} > 2\pi \ .$$

182. Let M be an interior point of the equilateral triangle ABC with each side of unit length. Prove that

$$MA.MB + MB.MC + MC.MA \geq 1 \ .$$

183. Simplify the expression

$$\frac{\sqrt{1+\sqrt{1-x^2}}((1+x)\sqrt{1+x}-(1-x)\sqrt{1-x})}{x(2+\sqrt{1-x^2})} \ ,$$

where $0 < |x| < 1$.

184. Using complex numbers, or otherwise, evaluate

$$\sin 10^\circ \sin 50^\circ \sin 70^\circ \ .$$

185. Find all triples of natural numbers a, b, c, such that all of the following conditions hold: (1) $a < 1974$; (2) b is less than c by 1575; (3) $a^2 + b^2 = c^2$.

186. Find all natural numbers n such that there exists a convex $n-$sided polygon whose diagonals are all of the same length.

187. Suppose that p is a real parameter and that

$$f(x) = x^3 - (p+5)x^2 - 2(p-3)(p-1)x + 4p^2 - 24p + 36 \ .$$

(a) Check that $f(3-p) = 0$.

(b) Find all values of p for which two of the roots of the equation $f(x) = 0$ (expressed in terms of p) can be the lengths of the two legs in a right-angled triangle with a hypotenuse of $4\sqrt{2}$.

188. (a) The measure of the angles of an acute triangle are α, β and γ degrees. Determine (as an expression of α, β, γ) what masses must be placed at each of the triangle's vertices for the centroid (centre of gravity) to coincide with (i) the orthocentre of the triangle; (ii) the circumcentre of the triangle.

(b) The sides of an arbitrary triangle are a, b, c units in length. Determine (as an expression of a, b, c) what masses must be placed at each of the triangle's vertices for the centroid (centre of gravity) to coincide with (i) the centre of the inscribed circle of the triangle; (ii) the intersection point of the three segments joining the vertices of the triangle to the points on the opposite sides where the inscribed circle is tangent (be sure to prove that, indeed, the three segments intersect in a common point).

189. There are n lines in the plane, where n is an integer exceeding 2. No three of them are concurrent and no two of them are parallel. The lines divide the plane into regions; some of them are closed (they have the form of a convex polygon); others are unbounded (their borders are broken lines consisting of segments and rays).

(a) Determine as a function of n the number of unbounded regions.

(b) Suppose that some of the regions are coloured, so that no two coloured regions have a common side (a segment or ray). Prove that the number of regions coloured in this way does not exceed $\frac{1}{3}(n^2 + n)$.

190. Find all integer values of the parameter a for which the equation

$$|2x+1| + |x-2| = a$$

has exactly one integer among its solutions.

191. In **Olymonland** the distances between every two cities is different. When the transportation program of the country was being developed, for each city, the closest of the other cities was chosen and a highway was built to connect them. All highways are line segments. Prove that

(a) no two highways intersect;

(b) every city is connected by a highway to no more than 5 other cities;

(c) there is no closed broken line composed of highways only.

192. Let ABC be a triangle, D be the midpoint of AB and E a point on the side AC for which $AE = 2EC$. Prove that BE bisects the segment CD.

193. Determine the volume of an isosceles tetrahedron for which the pairs of opposite edges have lengths a, b, c. Check your answer independently for a regular tetrahedron.

194. Let ABC be a triangle with incentre I. Let M be the midpoint of BC, U be the intersection of AI produced with BC, D be the foot of the perpendicular from I to BC and P be the foot of the perpendicular from A to BC. Prove that

$$|PD||DM| = |DU||PM| .$$

195. Let $ABCD$ be a convex quadrilateral and let the midpoints of AC and BD be P and Q respectively, Prove that

$$|AB|^2 + |BC|^2 + |CD|^2 + |DA|^2 = |AC|^2 + |BD|^2 + 4|PQ|^2 .$$

196. Determine five values of p for which the polynomial $x^2 + 2002x - 1002p$ has integer roots.

197. Determine all integers x and y that satisfy the equation $x^3 + 9xy + 127 = y^3$.

198. Let p be a prime number and let $f(x)$ be a polynomial of degree d with integer coefficients such that $f(0) = 0$ and $f(1) = 1$ and that, for every positive integer n, $f(n) \equiv 0$ or $f(n) \equiv 1$, modulo p. Prove that $d \geq p-1$. Give an example of such a polynomial.

Solutions and Comments

127. Let

$$A = 2^n + 3^n + 216(2^{n-6} + 3^{n-6})$$

and

$$B = 4^n + 5^n + 8000(4^{n-6} + 5^{n-6})$$

where $n > 6$ is a natural number. Prove that the fraction A/B is reducible.

Solution 1. Observe that

$$x^n + y^n + x^3y^3(x^{n-6} + y^{n-6}) = (x^{n-3} + y^{n-3})(x^3 + y^3) = (x^{n-3} + y^{n-3})(x+y)(x^2 - xy + y^2) .$$

When $(x, y) = (2, 3)$, $x^2 - xy + y^2 = 7$, while, when $(x, y) = (4, 5)$, $x^2 - xy + y^2 = 21 = 7 \times 3$. It follows that, for each $n > 6$, both A and B are divisible by 7 and the result follows.

Solution 2. We consider the values of A and B modulo 7, and use the fact that $a^6 \equiv 1 \pmod 7$, whenever a is not a multiple of 7. Observe that $216 \equiv 8000 \equiv -1 \pmod 7$. Then, modulo 7,

$$A \equiv 2^n + 3^n - 2^{n-6} - 3^{n-6} = 2^{n-6}(2^6 - 1) + 3^{n-6}(3^6 - 1) \equiv 0$$

and

$$A \equiv 4^n + 5^n - 4^{n-6} - 5^{n-6} = 4^{n-6}(2^6 - 1) + 5^{n-6}(3^6 - 1) \equiv 0 \ ,$$

so that both A and B are divisible by 7 and A/B is reducible.

Comment. The same conclusion holds when 216 is replaced by 6 and 8000 replaced by 20.

128. Let n be a positive integer. On a circle, n points are marked. The number 1 is assigned to one of them and 0 is assigned to the others. The following operation is allowed: Choose a point to which 1 is assigned and then assign $(1-a)$ and $(1-b)$ to the two adjacent points, where a and b are, respectively, the numbers assigned to these points before. Is it possible to assign 1 to all points by applying this operation several times if (a) $n = 2001$ and (b) $n = 2002$?

Solution. (a) It is clear that we can obtain a string of 3 ones, with all the other positions marked with a zero. Suppose that we have a string of $2k-1$ ones with all the other postions marked with a zero. Note that three consecutive entries $1, 1, 0$ can be changed to $0, 1, 1$ and that three consecutive entries $0, 1, 0$ can be changed to $1, 1, 1$. By starting at one end of the string of ones and using the first operation to "shift" a zero over two position until we come to the entries $0, 1, 0$ at the other end of the string, we see that we can obtain a string of $2k+1$ ones. We can repeat this for $k = 1, \cdots, 1000$ until we achieve all ones.

(b) Observe that the parity the number of ones does not change. If we start with a single one, then there must be an odd number of ones, so we cannot solve the problem when $n = 2002$.

129. For every integer n, a nonnegative integer $f(n)$ is assigned such that

(a) $f(mn) = f(m) + f(n)$ for each pair m, n of natural numbers;

(b) $f(n) = 0$ when the rightmost digit in the decimal representation of the number n is 3; and

(c) $f(10) = 0$.

Prove that $f(n) = 0$ for any natural number n.

Solution 1. Substituting $m = 1$, we find that $f(1) = 0$. Substituting $m = n = -1$, we find that $f(-1) = 0$, and substituting $m = -1$ yields that $f(n) = f(-n)$ for every integer n. So we suppose henceforth that $n > 0$.

Let $n = 10k + 1$, then $0 = f(30k + 3) = f(3) + f(10k + 1) = 0 + f(n)$. Let $n = 10k + 7$. Then

$$f(n) = f(3) + f(10k + 7) = f((3k + 2)10 + 1) = 0.$$

Finally, let $n = 10k + 9$. Then

$$f(n) = f(3) + f(10k + 9) = f((3k + 2)10 + 7) = 0.$$

If follows that $f(n) = 0$, whenever n and 10 are coprime.

Since $0 = f(10) = f(2) + f(5)$, and $f(2) \geq 0$, $f(5) \geq 0$, if follows that $f(2) = f(5) = 0$. The result now follows by applying (a) to the prime factorization of a given number n.

Solution 2. As in Solution 1, we can show that $f(2) = f(5) = 0$. Consider the prime factorization of n, say, $n = 2^r 5^s b$, where the greatest common divisor of b and 10 is equal to 1. Since $b = 10a \pm 1$, or $b = 10c \pm 3$, we have that $b^4 = 10d + 1$ for some c, so that its rightmost digit is 1 (b, c and d are some integers). Then

$$0 = f(3b^4) = f(3) + f(b^4) = 0 + f(b^4) = 4f(b) \ ,$$

whence $f(n) = rf(2) + sf(5) + f(b) = 0$.

130. Let $ABCD$ be a rectangle for which the respective lengths of AB and BC are a and b. Another rectangle is circumscribed around $ABCD$ so that each of its sides passes through one of the vertices of $ABCD$. Consider all such rectangles and, among them, find the one with a maximum area. Express this area in terms of a and b.

Solution. The circumscribed rectangle is the union of the inner rectangle and four right triangles whose hypotenuses are the four sides of the inner rectangle. The apexes of these four right triangles lie on semicircles whose diameters are the four sides of the inner rectangle, so that the altitudes of the right triangles from the hypotenuses cannot exceed the radii of the semi-circles, namely $a/2$ or $b/2$.

We can circumscribe a rectangle each of whose right triangles is isosceles, and whose heights are $a/2$ or $b/2$ and areas are $a^2/4$ or $b^2/4$. Thus, the maximum area of the circumscribed rectangle is

$$2\left(\frac{a^2}{4} + \frac{b^2}{4}\right) + ab = \frac{1}{2}(a+b)^2 \ .$$

131. At a recent winter meeting of the Canadian Mathematical Society, some of the attending mathematicians were friends. It appeared that every two mathematicians, that had the same number of friends among the participants, did not have a common friend. Prove that there was a mathematician who had only one friend.

Comment. Note that the result is false if there is no mathematician who has a friend. Thus, we need to assume that there is at least one pair of friends.

Solution 1. Wolog, we may assume that each person present has at least one friend. We prove the result by induction. It is clearly true when there are only two people. Let S be the set of people in the crowd, each of whom has the minimum number of friends. Suppose if possible, that this minimum exceeds 1. The set S is nonempty. Let T consist of all of the others. Then, looking at all the pairs of friends within T, by an induction hypothesis, there is a person p in T with only one friend in T. Since any two people in S cannot have a friend in common, p can have at most have one friend in S. Since, by hypothesis, p cannot have a single friend, p must have two friends. But then p should have been included in S, and we arrive at a contradiction.

Solution 2. Let q be the person with the maximum number n of friends (or one of several, if there is more than one having the same maximum number of friends). Each of the n friends of q has at least one friend. If any of them has exactly one friend, the the result holds. Assume, if possible, that none of them has exactly one friend. Since n is the maximum possible number of friends, there are $n-1$ possibilities for the number of friends that these n people have. By the Pigeonhole Principle, there must be two of them with the same number of friends. But the given conditions require that two persons with the same number of friends do not have a common friend. But this contradicts q being the common friend of them. Therefore, there must be someone with one friend.

132. Simplify the expression

$$\sqrt[5]{3\sqrt{2} - 2\sqrt{5}} \cdot \sqrt[10]{\frac{6\sqrt{10} + 19}{2}} \ .$$

Solution. Observe that $(3\sqrt{2} - 2\sqrt{5})^2 = 2(19 - 6\sqrt{10})$, and that $3\sqrt{2} - 2\sqrt{5} < 0$. Hence $2\sqrt{5} - 3\sqrt{2} =$

$\sqrt{2(19-6\sqrt{10}}$, and so

$$\begin{aligned}
&\sqrt[5]{3\sqrt{2}-2\sqrt{5}}\cdot\sqrt[10]{\frac{6\sqrt{10}+19}{2}} \\
&\qquad= -\sqrt[5]{2\sqrt{5}-3\sqrt{2}}\cdot\sqrt[10]{\frac{6\sqrt{10}+19}{2}} \\
&\qquad= -\sqrt[10]{(2(19-6\sqrt{10})\cdot\frac{1}{2}(19+6\sqrt{10})} \\
&\qquad= -\sqrt[10]{361-360} = -1\ .
\end{aligned}$$

133. Prove that, if a, b, c, d are real numbers, $b \neq c$, both sides of the equation are defined, and

$$\frac{ac-b^2}{a-2b+c} = \frac{bd-c^2}{b-2c+d}\ ,$$

then each side of the equation is equal to

$$\frac{ad-bc}{a-b-c+d}\ .$$

Give two essentially different examples of quadruples (a,b,c,d), not in geometric progression, for which the conditions are satisfied. What happens when $b=c$?

Solution 1. The given condition is equivalent to

$$\begin{aligned}
0 &= (ac-b^2)(b-2c+d)-(bd-c^2)(a-2b+c) \\
&= (b-c)(ac-b^2+bd-c^2)-[(ac-b^2)(c-d)+(bd-c^2)(a-b)]
\end{aligned}$$

Note that the quantity is square brackets vanishes when $b=c$, so that $(b-c)$ should be a factor of it. Indeed, we have

$$(ac-b^2)(c-d)+(bd-c^2)(a-b) = (b-c)(ad-bc)\ .$$

Since $b \neq c$, we find that the given condition is equivalent to

$$0 = (ac-b^2)+(bd-c^2)-(ad-bc)$$

or

$$(ac-b^2)+(bd-c^2) = ad-bc\ .$$

It follows that

$$\begin{aligned}
\frac{ac-b^2}{a-2b+c} &= \frac{bd-c^2}{b-2c+d} \\
&= \frac{(ac-b^2)+(bd-c^2)}{(a-2b+c)+(b-2c+d)} \\
&= \frac{ad-bc}{a-b-c+d}\ ,
\end{aligned}$$

as desired. Note that, in the event that $a-b-c+d=0$, we must have $ad-bc=0$, as well. (Explain!)

A generic example is $(a,b,c,d) = (r^{k-1}\pm 1, r^k \pm 1, r^{k+1}\pm 1, r^{k+2}\pm 1)$, where $r \neq 0, 1$ and k is arbitrary.

Suppose that $b=c$. If $a \neq b$ and $d \neq b$, then both sides of the datum reduce to b, and the condition is a tautology. However, the final fraction need not be equal to b in this case: an example is $(a,b,c,d)=(2,1,1,3)$. On the other hand, suppose that $a=b=c\neq d$. The one side of the datum is undefined, while we find that

$$\frac{bd-c^2}{b-2c+d} = b \quad \text{and} \quad \frac{ad-bc}{a-b-c+d} = \frac{bd-b^2}{d-b} = b\ .$$

If $a \neq b = c = d$, then we have a similar result. Finally, if $a = b = c = d$, then all fractions are undefined.

Solution 2. [A. Mao] Let

$$k = \frac{ac - b^2}{a - 2b + c} = \frac{bd - c^2}{b - 2c + d} \ .$$

Then

$$(a - k)(c - k) = ac - ak - ck + k^2 = b^2 - 2bk + k^2 = (b - k)^2$$

and

$$(b - k)(d - k) = (c - k)^2 \ .$$

Hence

$$(a - k)(d - k)[(b - k)(c - k)] = [(b - k)(c - k)]^2 \ .$$

Note that $b = k \Leftrightarrow c = k$. Since $b \neq c$, then both b and c must differ from k. Hence

$$(a - k)(d - k) = (b - k)(c - k) \Longrightarrow ad - bc = k(a - b - c + d) \ .$$

If $a - b - c + d = 0$, then $ad - bc = 0$ and the expression in the conclusion is undefined. Otherwise,

$$\frac{ad - bc}{a - b - c + d} = k$$

and the result follows.

The given conditions imply that $a - k$, $b - k$, $c - k$, $d - k$ are in geometric progresion. Conversely, pick u arbitrary and $r \neq 0, 1$, and let $(a, b, c, d) = (k + u, k + ur, k + ur^2, k + ur^3)$ to obtain a generic example.

Comments. Here are some numerical examples provided by various solvers for (a, b, c, d): (1, 2, 5, 14), (1, 5, 3, 4), (2, 6, 12, 21), (5, 9, 1, 17), (8, 24, 12, 21). Note that, when $(a, b, c, d) = (a, b, a, b)$, both sides of the datum equal $\frac{1}{2}(a + b)$, while the fraction in the conclusion is undefined.

The given condition is equivalent to

$$0 = (b - c)(ac + bd + bc - ad - b^2 - c^2) \ ,$$

from which $(ac - b^2) + (bd - c^2) = ad - bc$, and we can proceed as in Solution 1.

Very few full marks were given for this problem, as solvers were not careful about details. Whenever an expression appears in a denominator or you cancel a factor out of a product, you must consider the possibility that it might vanish. Most people ignored this possibility. In addition, the analysis of the situation when $b = c$ was not at all thorough.

134. Suppose that

$$a = zb + yc$$

$$b = xc + za$$

$$c = ya + xb \ .$$

Prove that

$$\frac{a^2}{1 - x^2} = \frac{b^2}{1 - y^2} = \frac{c^2}{1 - z^2} \ .$$

Of course, if any of x^2, y^2, z^2 is equal to 1, then the conclusion involves undefined quantities. Give the proper conclusion in this situation. Provide two essentially different numerical examples.

Solution. Suppose, say, that $x^2 = 1$. Then $x = \pm 1$, whence $za = b \mp c$ and $ya = c \mp b$. Thus $a(y \pm z) = 0$. Suppose $a = 0$; then $b^2 = c^2$. If $b = c = 0$, then $b^2(1 - z^2) = c^2(1 - y^2)$. If $bc \neq 0$, then $|zb| = |yc|$ implies that $z^2 = y^2$, in which case $b^2(1 - z^2) = c^2(1 - y^2)$.

On the other hand, suppose that $x^2 = 1$ and $a \neq 0$. Then $y \pm z = 0$, so that

$$a = z(b \pm c) = az^2$$

whence $z^2 = y^2 = x^2 = 1$. Since $a \neq 0$, it is not possible for both b and c to vanish. Let $b \neq 0$. Then

$$c - xb = ya = yzb + y^2 c = yzb + c$$

so that $x = -yz$. In this case, all the expressions in the conclusion are undefined.

We now suppose that none of x^2, y^2, z^2 is equal to 1. From $b = xc + za$ and $xc = xya + x^2 b$, we deduce that

$$(1 - x^2)b = (xy + z)a \ .$$

Since $x^2 \neq 1$, we have that $xy + z \neq 0$, and so

$$\frac{a^2}{1 - x^2} = \frac{ab}{xy + z} \ .$$

From $c = ya + xb$ and $xb = x^2 c + xza$, we deduce that

$$(1 - x^2)c = (xz + y)a \ ,$$

whence

$$\frac{a^2}{1 - x^2} = \frac{ac}{xz + y} \ .$$

From $a = zb + yc$ and $zb = xzc + z^2 a$, we deduce that

$$(1 - z^2)a = (xz + y)c \ ,$$

whence

$$\frac{c^2}{1 - z^2} = \frac{ac}{xz + y} = \frac{a^2}{1 - x^2} \ .$$

From $a = zb + yc$ and $yc = y^2 a + xyb$, we deduce that

$$(1 - y^2)a = (xy + z)b \ ,$$

whence

$$\frac{b^2}{1 - y^2} = \frac{ab}{xy + z} = \frac{a^2}{1 - x^2} \ .$$

The result now follows.

Comments. Other interesting conclusions can be drawn. Adding and subtracting the first two equations, we have that

$$(a - b)(1 + z) = (y - x)c$$

$$(a + b)(1 - z) = (y + x)c$$

so that

$$(a^2 - b^2)(1 - z^2) = (y^2 - x^2)c^2$$

$$\Longrightarrow \frac{c^2}{1 - z^2} = \frac{a^2 - b^2}{y^2 - x^2} \ .$$

Similarly

$$\frac{b^2}{1 - y^2} = \frac{c^2 - a^2}{x^2 - z^2} \quad \text{and} \quad \frac{a^2}{1 - x^2} = \frac{b^2 - c^2}{z^2 - x^2} \ .$$

Also, since $(xy+z)a = (1-x^2)b$ and $(xz+y)a = (1-x^2)c$, we have that

$$(1-x^2)a = (1-x^2)(zb+yc) = (2xyz+z^2+y^2)a \Longrightarrow a(x^2+y^2+z^2+2xyz) = a \ .$$

Similarly, $b(x^2+y^2+z^2+2xyz) = b$ and $c(x^2+y^2+z^2+2xyz) = c$. For each solution of the given system for which not all of a, b, c vanish, we must have $x^2+y^2+z^2+2xyz = 1$.

Since $xc = b - za$ and $xb = c - ya$, it follows that $b^2 - zab = c^2 - yac$, so that

$$bz - yc = \frac{b^2-c^2}{a} \quad \text{and} \quad bz + yc = a \ .$$

Hence

$$z = \frac{a^2+b^2-c^2}{2ab} \quad \text{and} \quad y = \frac{a^2+c^2-b^2}{2ac} \ .$$

Also,

$$x = \frac{b^2+c^2-a^2}{2bc} \ .$$

If a, b, c are the sides of a triangle ABC, then we have

$$\cos^2 A + \cos^2 B + \cos^2 C + 2\cos A \cos B \cos C = 1 \ .$$

[Can you prove this directly?]

Here are some examples of sextuples $(a,b,c;x,y,z)$: $(a,b,c;\cos A,\cos B,\cos C)$, $(1,3,6;\frac{11}{9},\frac{7}{3},-\frac{13}{3})$, $(2,5,9;\frac{17}{15},\frac{5}{3},-\frac{13}{5})$, $(-5,5,0;4,4,-1)$.

135. For the positive integer n, let $p(n) = k$ if n is divisible by 2^k but not by 2^{k+1}. Let $x_0 = 0$ and define x_n for $n \geq 1$ recursively by

$$\frac{1}{x_n} = 1 + 2p(n) - x_{n-1} \ .$$

Prove that every nonnegative rational number occurs exactly once in the sequence $\{x_0, x_1, x_2, \cdots, x_n, \cdots\}$. ■

Comment. This problem of Donald E. Knuth was posed in a recent issue of the *American Mathematical Monthly*.

Solution. An examination of the table of values leads to the conjectured recursions:

$$x_{2k+1} = x_k + 1 \qquad \text{(for } k \geq 0) \ ,$$

$$\frac{1}{x_{2k}} = \frac{1}{x_k} + 1 \qquad \text{(for } k \geq 1) \ .$$

Since $(x_0, x_1, x_2) = (0, 1, \frac{1}{2})$, this equation holds for the lowest value of k. We use an induction argument. Suppose $k \geq 1$ and that $x_{2k-1} = x_{k-1} + 1$. Then

$$\begin{aligned}\frac{1}{x_{2k}} &= 1 + 2p(2k) - x_{2k-1} \\ &= 1 + 2[1 + p(k)] - (x_{k-1} + 1) \\ &= 1 + (1 + 2p(k) - x_{k-1}) = 1 + \frac{1}{x_k}\end{aligned}$$

and

$$\frac{1}{x_{2k+1}} = 1 - x_{2k} = 1 - \frac{x_k}{x_k+1} = \frac{1}{x_k+1}$$

so that $x_{2k+1} = x_k + 1$. The desired recursions follow.

From the two recursions, we see that $x_n > 1$ when n is odd and exceeds 1 and $x_n < 1$ when n is even and exceeds 0. Thus, the values $x_0 = 0$ and $x_1 = 1$ are assumed only once. Suppose, if possible, that some rational is assumed twice. Let r be the smallest index for which $x_r = x_s$ for some $s > r \geq 2$. Then r and s must have the same parity and it follows from the recursions that $x_{\lfloor r/2 \rfloor} = x_{\lfloor s/2 \rfloor}$, contradicting the minimality of r. Hence, no value of x_n is assumed more than once.

It is straightforward to see that, for each non-negative integer m,

$$x_{2^m - 1} = m \qquad \text{and} \qquad x_{2^m} = \frac{1}{m+1} \ .$$

Also, suppose that $x_u x_v = 1$. Then

$$\frac{1}{x_{2v}} = \frac{1}{x_v} + 1 = x_u + 1 = x_{2u+1}$$

so that $x_{2u+1} x_{2v} = 1$.

Let m be a positive integer. We show that, for $0 \leq k \leq 2^m - 1$, $i = 2^m + k$ and $j = 2^{m+1} - (k+1)$, $x_i x_j = 1$. This is clearly true for $m = 1$, since $x_2 x_3 = 1$. Suppose that it has been established for some particular value of $m \geq 1$. We show that it holds for m replaced by $m + 1$.

Let $0 \leq l \leq 2^{m+1} - 1$, $i = 2^{m+1} + l = 2^{m+2} - (2^{m+1} - l)$ and $j = 2^{m+2} - (l+1) = 2^{m+1} + (2^{m+1} - l - 1)$. Since i and j have opposite parity and can be put into either form, we may suppose wolog that $l = 2r$ is even, whereupon $i = 2v$ is even and $j = 2u + 1$ is odd. Thus $v = 2^m + r$ and $u = 2^m - (r+1)$, so that by the induction hypothesis, $x_u x_v = 1$, whence it follows that $x_i x_j = x_{2v} x_{2u+1} = 1$. The desired result follows by induction.

This shows that the set S of positive rationals of the form x_n contains 0, 1 and is closed under each of the operations $x \longrightarrow x+1$ and $x \longrightarrow 1/x$. Thus S contains all nonnegative integers. Suppose we have shown that S contains all positive rationals with denominators not exceeding s. Consider a rational $p/(s+1)$ with $1 \leq p \leq s$. By the induction hypothesis, $(s+1)/p \in S$, so that $p/(s+1) \in S$. Hence, for each nonnegative integer t,

$$\frac{p}{(s+1)} + t = \frac{p + (s+1)t}{s+1} \in S \ ,$$

and so we can conclude that every positive rational with denominator $s+1$ belongs to S. Hence, by induction, we see that S contains every rational.

136. Prove that, if in a semicircle of radius 1, five points A, B, C, D, E are taken in consecutive order, then

$$|AB|^2 + |BC|^2 + |CD|^2 + |DE|^2 + |AB||BC||CD| + |BC||CD||DE| < 4 \ .$$

Comment. The inequality is strict except in certain degenerate cases in which the points are not all distinct.

Solution 1. If A and E are shifted to the ends of the diameter, then the left side is at least as large. Thus, it suffices to prove the result when AE is a diameter. Let $|AB| = a$, $|BC| = b$, $|CD| = c$, $|DE| = d$, $|AC| = u$ and $|CE| = v$. Observe that $v > c$ and $u > b$ [why?]. Suppose $\alpha = \angle CAE$ and $\beta = \angle CEA$. Then $\angle ABC = 180° - \beta$ and $\angle CDE = 180° - \alpha$. We have $|AE| = 2$, and

$$u^2 = a^2 + b^2 + 2ab\cos\beta = a^2 + b^2 + abv > a^2 + b^2 + abc$$

and

$$v^2 = c^2 + d^2 + 2cd\cos\alpha = c^2 + d^2 + cdu > c^2 + d^2 + bcd \ .$$

Hence

$$a^2 + b^2 + c^2 + d^2 + abc + bcd < u^2 + v^2 = 4 \ .$$

Solution 2. [A. Mao] As in Solution 1, we reduce to the case that AE is a diameter. Use the same notation as in Solution 1, and let $|AD| = p$ and $|BE| = q$. Note that $\angle ABE = \angle ACE = \angle ADE = 90°$. By Ptolemy's Theorem, we have that

$$\begin{aligned}|BC||AE| + |AB||CE| &= |AC||BE| \\ &\Longrightarrow 2b + av = uq = \sqrt{4-v^2}\sqrt{4-a^2} \\ &\Longrightarrow 4b^2 + 4bav = 16 - 4a^2 - 4v^2 \\ &\Longrightarrow a^2 + b^2 + v^2 + abv = 4 \ .\end{aligned}$$

Similarly, $c^2 + d^2 + u^2 + cdu = 4$. Adding and using $u^2 + v^2 = 4$, we obtain that

$$a^2 + b^2 + c^2 + d^2 + abv + cdu = 4 \ .$$

Since triangles ABC and CDE are obtuse with the largest angles at B and D respectively, $c = |CD| < |CE| = v$ and $a < u$, so that

$$a^2 + b^2 + c^2 + d^2 + abc + bcd < 4 \ .$$

137. Can an arbitrary convex quadrilateral be decomposed by a polygonal line into two parts, each of whose diameters is less than the diameter of the given quadrilateral?

Solution. No. Let ABC be an equilateral triangle, and let D be an exterior point in the region bounded by AC and the circle with centre B and radius AB. Then the diameter of the quadrilateral $ABCD$ is $|AB| = |BC| = |CA|$. Any partition of the quadrilateral $ABCD$ into two parts must have one of the parts containing two of the three points A, B, C; the diameter of this part is equal to that of $ABCD$.

138. (a) A room contains ten people. Among any three. there are two (mutual) acquaintances. Prove that there are four people, any two of whom are acquainted.

(b) Does the assertion hold if "ten" is replaced by "nine"?

Solution. Observe that the members of any pair each not acquainted with some third person must be acquainted with each other. It follows that the pairs of any group of people, each not acquainted with a particular outside person, must be mutual acquaintances.

We first show that, if any individual, say A, is acquainted with at least six people, B, C, D, E, F, G, then the conclusion follows. Suppose, B, say, does not know C, D, E, then each pair of A, C, D, E must be acquainted. On the other hand, if B knows each of C, D, E, then, as two of C, D, E know each other, say C and D, then each pair of A, B, C, D must be acquainted. Since any person acquainted with A must either be acquainted with or not be acquainted with three other people acquainted with A, the result follows.

(a) When there are ten people, each person either is not acquainted with four of the others (who then make a set of four each pair of which are acquaintances) or must be acquainted with at least six of the others, in which case we get the result by the previous paragraph.

(b) The assertion holds for nine people. In this case, each person either is not acquainted with four of the others or must be acquainted with at least five of the others. Suppose that each person is acquainted with at least five of the others. Then, adding together the number of acquaintances of each of the nine people, we get a sum of at least $9 \times 5 = 45$. But, each pair of acquaintances is counted twice, so the sum must be even and so be at least 46. But then there must be someone with at least six acquaintances, and we can obtain the desired result.

139. Let A, B, C be three pairwise orthogonal faces of a tetrahedran meeting at one of its vertices and having respective areas a, b, c. Let the face D opposite this vertex have area d. Prove that

$$d^2 = a^2 + b^2 + c^2 \ .$$

Solution 1. Let the tetrahedron be bounded by the three coordinate planes in $\mathbf{R}^3$ and the plane with equation $\frac{x}{u}+\frac{y}{v}+\frac{z}{w}=1$, where u,v,w are positive. The vertices of the tetrahedron are $(0,0,0)$, $(u,0,0)$, $(0,v,0)$, $(0,0,w)$. Let d, a, b, c be the areas of the faces opposite these respective vertices. Then the volume V of the tetrahedron is equal to

$$\frac{1}{3}au=\frac{1}{3}bv=\frac{1}{3}cw=\frac{1}{3}dk \ ,$$

where k is the distance from the origin to its opposite face. The foot of the perpendicular from the origin to this face is located at $((um)^{-1},(vm)^{-1}.(wm)^{-1})$, where $m=u^{-2}+v^{-2}+w^{-2}$, and its distance from the origin is $m^{-1/2}$. Since $a=3Vu^{-1}$, $b=3Vv^{-1}$, $c=3Vw^{-1}$ and $d=3Vm^{1/2}$, the result follows.

Solution 2. [J. Chui] Let edges of lengths x, y, z be common to the respective pairs of faces of areas (b,c), (c,a), (a,b). Then $2a=yz$, $2b=zx$ and $2c=xy$. The fourth face is bounded by sides of length $u=\sqrt{y^2+z^2}$, $v=\sqrt{z^2+x^2}$ and $w=\sqrt{x^2+y^2}$. By Heron's formula, its area d is given by the relation

$$\begin{aligned}
16d^2 &= (u+v+w)(u+v-w)(u-v+w)(-u+v+w)\\
&= [(u+v)^2-w^2][(w^2-(u-v)^2]=[2uv+(u^2+v^2-w^2)][2uv-(u^2+v^2-w^2)]\\
&= 2u^2v^2+2v^2w^2+2w^2u^2-u^4-v^4-w^4\\
&= 2(y^2+z^2)(x^2+z^2)+2(x^2+z^2)(x^2+y^2)+2(x^2+y^2)(x^2+z^2)\\
&\qquad -(y^2+z^2)^2-(x^2+z^2)^2-(x^2+y^2)^2\\
&= 4x^2y^2+4x^2z^2+4y^2z^2=16a^2+16b^2+16c^2 \ ,
\end{aligned}$$

whence the result follows.

Solution 3. Use the notation of Solution 2. There is a plane through the edge bounding the faces of areas a and b perpendicular to the edge bounding the faces of areas c and d. Suppose it cuts the latter faces in altitudes of respective lengths u and v. Then $2c=u\sqrt{x^2+y^2}$, whence $u^2(x^2+y^2)=x^2y^2$. Hence

$$v^2=z^2+u^2=\frac{x^2y^2+x^2z^2+y^2z^2}{x^2+y^2}=\frac{4(a^2+b^2+c^2)}{x^2+y^2} \ ,$$

so that

$$2d=v\sqrt{x^2+y^2}\Longrightarrow 4d^2=4(a^2+b^2+c^2) \ ,$$

as desired.

Solution 4. [R. Ziman] Let $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, $\mathbf{d}$ be vectors orthogonal to the respective faces of areas a, b, c, d that point inwards from these faces and have respective magnitudes a, b, c, d. If the vertices opposite the respective faces are $\mathbf{x}$, $\mathbf{y}$, $\mathbf{z}$, $\mathbf{O}$, then the first three are pairwise orthogonal and $2\mathbf{c}=\mathbf{x}\times\mathbf{y}$, $2\mathbf{b}=\mathbf{z}\times\mathbf{x}$, $2\mathbf{c}=\mathbf{x}\times\mathbf{y}$, and $2\mathbf{d}=(\mathbf{z}-\mathbf{y})\times(\mathbf{z}-\mathbf{x})=-(\mathbf{z}\times\mathbf{x})-(\mathbf{y}\times\mathbf{z})-(\mathbf{x}\times\mathbf{y})$. Hence $\mathbf{d}=-(\mathbf{a}+\mathbf{b}+\mathbf{c})$, so that

$$d^2=\mathbf{d}\cdot\mathbf{d}=(\mathbf{a}+\mathbf{b}+\mathbf{c})\cdot(\mathbf{a}+\mathbf{b}+\mathbf{c})=a^2+b^2+c^2 \ .$$

140. Angus likes to go to the movies. On Monday, standing in line, he noted that the fraction x of the line was in front of him, while $1/n$ of the line was behind him. On Tuesday, the same fraction x of the line was in front of him, while $1/(n+1)$ of the line was behind him. On Wednesday, the same fraction x of the line was in front of him, while $1/(n+2)$ of the line was behind him. Determine a value of n for which this is possible.

Answer. When $x=5/6$, he could have $1/7$ of a line of 42 behind him, $1/8$ of a line of 24 behind him and $1/9$ of a line of 18 behind him. When $x=11/12$, he could have $1/14$ of a line of 84 behind him, $1/15$ of a line of 60 behind him and $1/16$ of a line of 48 behind him. When $x=13/15$, he could have $1/8$ of a line of 120 behind him, $1/9$ of a line of 45 behind him and $1/10$ of a line of 30 behind him.

Solution 1. The strategy in this solution is to try to narrow down the search by considering a special case. Suppose that $x = (u-1)/u$ for some positive integer exceeding 1. Let $1/(u+p)$ be the fraction of the line behind Angus. Then Angus himself represents this fraction of the line:

$$1 - \left(\frac{u-1}{u} + \frac{1}{u+p}\right) = \frac{p}{u(u+p)} ,$$

so that there would be $u(u+p)/p$ people in line. To make this an integer, we can arrange that u is a multiple of p. For $n = u+1$, we want to get an integer for $p = 1, 2, 3$, and so we may take u to be any multiple of 6. Thus, we can arrange that x is any of $5/6$, $11/12$, $17/18$, $23/24$, and so on.

More generally, for $u(u+1)$, $u(u+2)/2$ and $u(u+3)/3$ to all be integers we require that u be a multiple of 6, and so can take $n = 6k+1$. On Monday, there would be $36k^2 + 6k$ people in line with $36k^2 - 1$ in front and $6k$ behind; on Tuesday, $18k^2 + 6k$ with $18k^2 + 3k - 1$ in front and $3k$ behind; on Wednesday, $12k^2 + 6k$ with $12k^2 + 4k - 1$ and $2k$ behind.

Solution 2. [O. Bormashenko] On the three successive days, the total numbers numbers of people in line are un, $v(n+1)$ and $w(n+2)$ for some positive integers u, v and w. The fraction of the line constituted by Angus and those behind him is

$$\frac{1}{un} + \frac{1}{n} = \frac{1}{v(n+1)} + \frac{1}{n+1} = \frac{1}{w(n+2)} + \frac{1}{n+2} .$$

These yield the equations

$$(n-v)(n+1+u) = n(n+1)$$

and

$$(n+1-w)(n+2+v) = (n+1)(n+2) .$$

We need to find an integer v for which $n-v$ divides $n(n+1)$ and $n+2+v$ divides $(n+1)(n+2)$. This is equivalent to determining p, q for which $p+q = 2(n+1)$, $p < n$, p divides $n(n+1)$, $q > n+2$ and q divides $(n+1)(n+2)$. The triple $(n, p, q) = (7, 4, 12)$ works and yields $(u, v, w) = (6, 3, 2)$. In this case, $x = 5/6$.

Comment 1. Solution 1 indicates how we can select x for which the amount of the line behind Angus is represented by any number of consecutive integer reciprocals. For example, in the case of $x = 11/12$, he could also have $1/13$ of a line of 156 behind him. Another strategy might be to look at $x = (u-2)/u$, *i.e.* successively at $x = 3/5, 5/7, 7/9, \cdots$. In this case, we assume that $1/(u-p)$ is the line is behind him, and need to ensure that $u - 2p$ is a positive divisor of $u(u-p)$ for three consecutive values of p. If u is odd, we can achieve this with u any odd multiple of 15, starting with $p = \frac{1}{2}(u-1)$.

Comment 2. With the same fraction in front on two days, suppose that $1/n$ of a line of u people is behind the man on the first day, and $1/(n+1)$ of a line of v people is behind him on the second day. Then

$$\frac{1}{u} + \frac{1}{n} = \frac{1}{v} + \frac{1}{n+1}$$

so that $uv = n(n+1)(u-v)$. This yields both $(n^2+n-v)u = (n^2+n)v$ and $(n^2+n+u)v = (n^2+n)u$, leading to

$$u - v = \frac{u^2}{n^2+n+u} = \frac{v^2}{n^2+n-v} .$$

Two immediate possibilities are $(n, u, v) = (n, n+1, n)$ and $(n, u, v) = (n, n(n+1), \frac{1}{2}n(n+1))$. To get some more, taking $u - v = k$, we get the quadratic equation

$$u^2 - ku - k(n^2+n) = 0$$

with discriminant

$$\Delta = k^2 + 4(n^2+n)k = [k + 2(n^2+n)]^2 - 4(n^2+n)^2 ,$$

a pythagorean relationship when Δ is square and the equation has integer solutions. Select α, β, γ so that $\gamma\alpha\beta = n^2 + n$ and let $k = \gamma(\alpha^2 + \beta^2 - 2\alpha\beta) = \gamma(\alpha - \beta)^2$; this will make the discriminant Δ equal to a square.

Taking $n = 3$, for example, yields the possibilities $(u, v) = (132, 11)$, $(60,\ 10)$, $(36,\ 9)$, $(24,\ 8)$, $(12,\ 6)$, $(6,\ 4)$,$(4,\ 3)$. In general, we find that $(n, u, v) = (n, \gamma\alpha(\alpha - \beta), \gamma\beta(\alpha - \beta))$ when $n^2 + n = \gamma\alpha\beta$ with $\alpha > \beta$. It turns out that $k = u - v = \gamma(\alpha - \beta)^2$.

141. In how many ways can the rational 2002/2001 be written as the product of two rationals of the form $(n + 1)/n$, where n is a positive integer?

Solution 1. We begin by proving a more general result. Let m be a positive integer, and denote by $d(m)$ and $d(m + 1)$, the number of positive divisors of m and $m + 1$ respectively. Suppose that

$$\frac{m+1}{m} = \frac{p+1}{p} \cdot \frac{q+1}{q} \ ,$$

where p and q are positive integers exceeding m. Then $(m + 1)pq = m(p + 1)(q + 1)$, which reduces to $(p - m)(q - m) = m(m + 1)$. It follows that $p = m + u$ and $q = m + v$, where $uv = m(m + 1)$. Hence, every representation of $(m + 1)/m$ corresponds to a factorization of $m(m + 1)$.

On the other hand, observe that, if $uv = m(m + 1)$, then

$$\begin{aligned}
\frac{m+u+1}{m+u} \cdot \frac{m+v+1}{m+v} &= \frac{m^2 + m(u+v+2) + uv + (u+v) + 1}{m^2 + m(u+v) + uv} \\
&= \frac{m^2 + (m+1)(u+v) + m(m+1) + 2m + 1}{m^2 + m(u+v) + m(m+1)} \\
&= \frac{(m+1)^2 + (m+1)(u+v) + m(m+1)}{m^2 + m(u+v) + m(m+1)} \\
&= \frac{(m+1)[(m+1) + (u+v) + m]}{m[m + (u+v) + m + 1]} = \frac{m+1}{m} \ .
\end{aligned}$$

Hence, there is a one-one correspondence between representations and pairs (u, v) of complementary factors of $m(m+1)$. Since m and $m+1$ are coprime, the number of factors of $m(m+1)$ is equal to $d(m)d(m+1)$, and so the number of representations is equal to $\frac{1}{2}d(m)d(m + 1)$.

Now consider the case that $m = 2001$. Since $2001 = 3 \times 23 \times 29$, $d(2001) = 8$; since $2002 = 2 \times 7 \times 11 \times 13$, $d(2002) = 16$. Hence, the desired number of representations is 64.

Solution 2. [R. Ziman] Let m be an arbitrary positive integer. Then, since $(m+1)/m$ is in lowest terms, pq must be a multiple of m. Let $m+1 = uv$ for some positive integers u and v and $m = rs$ for some positive integers r and s, where r is the greatest common divisor of m and p; suppose that $p = br$ and $q = as$, with s being the greatest common divisor of m and q. Then, the representation must have the form

$$\frac{m+1}{m} = \frac{au}{br} \cdot \frac{bv}{as} \ ,$$

where $au = br + 1$ and $bv = as + 1$. Hence

$$bv = \frac{br+1}{u}s + 1 = \frac{brs + s + u}{u} \ ,$$

so that $b = b(uv - rs) = s + u$ and

$$a = \frac{sr + ur + 1}{u} = \frac{m + 1 - ur}{u} = v + r \ .$$

Thus, a and b are uniquely determined. Note that we can get a representation for any pair (u, v) of complementary factors or $m+1$ and (r, s) of complementary factors of m, and there are $d(m+1)d(m)$ of selecting these. However, the selections $\{(u,v),(r,s)\}$ and $\{(v,u),(s,r)\}$ yield the same representation, so that number of representations is $\frac{1}{2}d(m+1)d(m)$. The desired answer can now be found.

142. Let $x, y > 0$ be such that $x^3 + y^3 \leq x - y$. Prove that $x^2 + y^2 \leq 1$.

Solution 1. [R. Barrington Leigh] We have that

$$x - y \geq x^3 + y^3 > x^3 - y^3 \ .$$

Since $x - y \geq x^3 + y^3 > 0$, we can divide this inequality by $x - y$ to obtain

$$1 > x^2 + xy + y^2 > x^2 + y^2 \ .$$

Solution 2. [S.E. Lu]

$$\begin{aligned} x - y \geq x^3 + y^3 > x^3 &> x^3 - [y^3 + xy(x-y)] \\ &= x^3 - x^2y + xy^2 - y^3 = (x^2+y^2)(x-y) \ , \end{aligned}$$

whereupon a division by the positive quantity $x - y$ yields that $1 > x^2 + y^2$.

Solution 3. [O. Bormashenko] Observe that $y < x$ and that $x^3 < x^3 + y^3 \leq x - y < x$, so that $0 < y < x < 1$. It follows that

$$x(x+y) < 2 \Longrightarrow xy(x+y) < 2xy \ . \tag{1}$$

The given condition can be rewritten

$$(x+y)(x^2+y^2) - xy(x+y) \leq x - y \ . \tag{2}$$

Adding inequalities (1) and (2) yields

$$(x+y)(x^2+y^2) < x + y \ ,$$

whence $x^2 + y^2 < 1$.

Solution 4. [R. Furmaniak] We have that

$$\begin{aligned} (x-y)(1-x^2-y^2) &= (x-y) - (x^3 - x^2y + xy^2 - y^3) \\ &\geq (x^3+y^3) - (x^3 - x^2y + xy^2 - y^3) = 2y^3 + x^2y - xy^2 = y(x^2 - xy + 2y^2) \\ &= y[(x - \sqrt{2}y)^2 + (2\sqrt{2} - 1)xy] \geq 0 \ , \end{aligned}$$

from which the result follows upon division by $x - y$.

Solution 5. Let $y = tx$. Since $x > y > 0$, we have that $0 < t < 1$. Then $x^3(1+t^3) \leq x(1-t) \Rightarrow x^2(1+t^3) \leq (1-t)$. Therefore,

$$\begin{aligned} x^2 + y^2 = x^2(1+t^2) &\leq \left(\frac{1-t}{1+t^3}\right)(1+t^2) \\ &= \frac{1 - t + t^2 - t^3}{1+t^3} = 1 - \frac{t(1-t+2t^2)}{1+t^3} \ . \end{aligned}$$

Since $1 - t + 2t^2$, having negative discriminant, is always positive, the desired result follows.

Solution 6. [J. Chui] Suppose, if possible, that $x^2+y^2 = r^2 > 1$. We can write $x = r\sin\theta$ and $y = r\cos\theta$ for $0 \le \theta \le \pi/2$. Then

$$\begin{aligned} x^3 + y^3 - (x-y) &= r^3\sin^3\theta + r^3\cos^3\theta - r\sin\theta + r\cos\theta \\ &> r\sin\theta(\sin^2\theta - 1) + r\cos^3\theta + r\cos\theta \\ &= -r\sin\theta\cos^2\theta + r\cos^3\theta + r\cos\theta \\ &= r\cos^2\theta\left(\cos\theta + \frac{1}{\cos\theta} - \sin\theta\right) \\ &> r\cos^2\theta(2-\sin\theta) > 0 \ , \end{aligned}$$

contrary to hypothesis. The result follows by contradiction.

Solution 7. Let $r > 0$ and $r^2 = x^2 + y^2$. Since $x > y > 0$, we can write $x = r\cos\theta$ and $y = r\sin\theta$, where $0 < \theta < \pi/4$. The given equality is equivalent to

$$r^2 \le \frac{\cos\theta - \sin\theta}{\cos^3\theta + \sin^3\theta} \ ,$$

so it suffices to show that the right side does not exceed 1 to obtain the desired $r^2 \le 1$.

Observe that

$$\begin{aligned} 1 - \frac{\cos\theta - \sin\theta}{\cos^3\theta + \sin^3\theta} &= \frac{(\cos\theta + \sin\theta)(1 - \cos\theta\sin\theta) - (\cos\theta - \sin\theta)}{\cos^3\theta + \sin^3\theta} \\ &= \frac{\sin\theta(2 - \cos\theta\sin\theta - \cos^2\theta)}{\cos^3\theta + \sin^3\theta} > 0 \ , \end{aligned}$$

from which the desired result follows.

Solution 8. Begin as in Solution 7. Then

$$\begin{aligned} \frac{\cos\theta - \sin\theta}{\cos^3\theta + \sin^3\theta} &= \frac{\cos^2\theta - \sin^2\theta}{(\cos\theta + \sin\theta)^2(1 - \cos\theta\sin\theta)} \\ &= \frac{\cos 2\theta}{(1 + \sin 2\theta)(1 - \frac{1}{2}\sin\theta}) = \frac{\cos 2\theta}{1 + \frac{1}{2}\sin 2\theta(1 - \sin 2\theta)} < 1 \ , \end{aligned}$$

from which the result follows.

143. A sequence whose entries are 0 and 1 has the property that, if each 0 is replaced by 01 and each 1 by 001, then the sequence remains unchanged. Thus, it starts out as $010010101001\cdots$. What is the 2002th term of the sequence?

Solution. Let us define finite sequences as follows. Suppose that $S_1 = 0$. Then, for each $k \ge 2$, S_k is obtained by replacing each 0 in S_{k-1} by 01 and each 1 in S_{k-1} by 001. Thus,

$$S_1 = 0; \quad S_2 = 01; \quad S_3 = 01001; \quad S_4 = 010010101001; \quad S_5 = 01001010100101001010010101001; \cdots$$

Each S_{k-1} is a prefix of S_k; in fact, it can be shown that, for each $k \ge 3$,

$$S_k = S_{k-1} * S_{k-2} * S_{k-1} \ ,$$

where $*$ indicates juxtaposition. The respective number of symbols in S_k for $k =$ 1, 2, 3, 4, 5, 6, 7, 8, 9, 10 is equal to 1, 2, 5, 12, 29, 70, 169, 408, 985, 2378.

The 2002th entry in the given infinite sequence is equal to the 2002th entry in S_{10}, which is equal to the $(2002-985-408)$th $=(609)$th entry in S_9. This in turn is equal to the $(609-408-169)$th $=(32)$th entry in S_8, which is equal to the (32)th entry of S_6, or the third entry of S_3. Hence, the desired entry is 0.

Comment. Suppose that $f(n)$ is the position of the nth one, so that $f(1)=2$ and $f(2)=5$. Let $g(n)$ be the number of zeros up to and including the nth position, and so $n-g(n)$ is the number of ones up to and including the nth position. Then we get the two equations

$$f(n)=2g(n)+3(n-g(n))=3n-g(n) \tag{1}$$

$$g(f(n))=f(n)-n\ . \tag{2}$$

These two can be used to determine the positions of the ones by stepping up; for example, we have $f(2)=5$, $g(5)=3$, $f(5)=15-3=12$, $g(12)=f(5)-5=7$, and so on. By messing around, one can arrive at the result, but it would be nice to formulate this approach in a nice clean efficient zeroing in on the answer.

144. Let a, b, c, d be rational numbers for which $bc\neq ad$. Prove that there are infinitely many rational values of x for which $\sqrt{(a+bx)(c+dx)}$ is rational. Explain the situation when $bc=ad$.

Solution 1. We study the possibility of making $c+dx=(a+bx)t^2$ for some rational numbers t. This would require that

$$x=\frac{c-at^2}{bt^2-d}\ .$$

Since the condition $bc\neq ad$ prohibits $b=d=0$, at least one of b and d must fail to vanish. Let us now construct our solution.

Let t be an arbitrary positive rational number for which $bt^2\neq d$. Then $a+bx=(bc-ad)(bt^2-d)^{-1}$ and $c+dx=(bc-ad)t^2(bt^2-d)^{-1}$, whence

$$\sqrt{(a+bx)(c+dx)}=|(bc-ad)t(bt^2-d)^{-1}|$$

is rational.

We need to show that distinct values of t deliver distinct values of x. Let u and v be two values of t for which

$$\frac{c-au^2}{bu^2-d}=\frac{c-av^2}{bv^2-d}\ .$$

Then

$$\begin{aligned}0&=(c-au^2)(bv^2-d)-(c-av^2)(bu^2-d)\\&=bc(v^2-u^2)+ad(u^2-v^2)=(bc-ad)(u^2-v^2)\ ,\end{aligned}$$

so that $u=v$, and the result follows.

Consider the case that $bc=ad$. if both sides equal zero, then one of the possibilities $(a,b,c,d)=(0,0,c,d),(a,b,0,0),(a,0,c,0),(0,b,0,d)$ must hold. In the first two cases, any x will serve. In the third, any value of x will serve provided that ac is a rational square, and in the fourth, provided bd is a rational square; otherwise, no x can be found. Otherwise, let $c/a=d/b=s$, for some nonzero rational s, so that $(a+bx)(c+dx)=s(a+bx)^2$. If s is a rational square, any value of x will do; if s is irrational, then only $x=-a/b=-c/d$ will work.

Solution 2. $(a+bx)(c+dx)=r^2$ for rational r is equivalent to

$$bdx^2+(ad+bc)x+(ac-r^2)=0\ .$$

If $b=d=0$, this is satisfiable by all rational x provided ac is a rational square and $r^2=ac$, and by no rational x otherwise. If exactly one of b and d is zero and $ad+bc\neq 0$, then each positive rational value is assumed by $\sqrt{(a+bx)(c+dx)}$ for a suitable value of x.

Otherwise, let $bd \neq 0$. Then, given r, we have the corresponding

$$x = \frac{-(ad+bc) \pm \sqrt{(ad-bc)^2 + 4bdr^2}}{2bd} \ .$$

If $ad = bc$, then this yields a rational x if and only if bd is a rational square. Let $ad \neq bc$. We wish to make $(ad-bc)^2 + 4bdr^2 = s^2$ for some rational s. This is equivalent to

$$4bdr^2 = (ad-bc)^2 - s^2 = (ad - bc + s)(ad - bc - s) \ .$$

Pick a pair u, v of rationals for which $u + v \neq 0$ and $uv = bd$. We want to make

$$2ur = ad - bc + s \qquad \text{and} \qquad 2vr = ad - bc - s$$

so that $(u+v)r = ad - bc$ and $s = (u-v)r$. Thus, let

$$r = \frac{ad-bc}{u+v} \ .$$

Then

$$\begin{aligned} (ad-bc)^2 + 4bdr^2 &= (ad-bc)^2 + 4uvr^2 \\ &= \left(\frac{ad-bc}{u+v}\right)^2 [(u+v)^2 - 4uv] \\ &= \frac{(ad-bc)^2(u-v)^2}{(u+v)^2} \end{aligned}$$

is a rational square, and so x is rational. There are infinitely many possible ways of choosing u, v and each gives a different sum $u + v$ and so a different value of r and x. The desired result follows.

145. Let ABC be a right triangle with $\angle A = 90°$. Let P be a point on the hypotenuse BC, and let Q and R be the respective feet of the perpendiculars from P to AC and AB. For what position of P is the length of QR minimum?

Solution. $PQAR$, being a quadrilateral with right angles at A, Q and R, is a rectangle. Therefore, its diagonals QR and AP are equal. The length of QR is minimized when the length of AP is minimized, and this occurs when P is the foot of the perpendicular from A to BC.

Comment. P must be chosen so that $PB : PC = AB^2 : AC^2$.

146. Suppose that ABC is an equilateral triangle. Let P and Q be the respective midpoint of AB and AC, and let U and V be points on the side BC with $4BU = 4VC = BC$ and $2UV = BC$. Suppose that PV is joined and that W is the foot of the perpendicular from U to PV and that Z is the foot of the perpendicular from Q to PV.

Explain how that four polygons $APZQ$, $BUWP$, $CQZV$ and UVW can be rearranged to form a rectangle. Is this rectangle a square?

Solution. Consider a $180°$ rotation about Q so that C falls on A, Z falls on Z_1 and V falls on V_1. The quadrilateral $QZVC$ goes to QZ_1V_1A, ZQZ_1 is a line and $\angle QAV_1 = 60°$. Similarly, a $180°$ rotation about P takes quadrilateral $PBUW$ to PAU_1W_1 with WPW_1 a line and $\angle U_1AP = 60°$. Since $\angle U_1AP = \angle PAQ = \angle QAV_1 = 60°$, U_1AV_1 is a line and

$$U_1V_1 = U_1A + AV_1 = UB + CV = \frac{1}{2}BC = UV \ .$$

Translate U and V to fall on U_1 and V_1 respectively; let W fall on W_2. Since

$$\angle W_1U_1W_2 = \angle W_1U_1A + \angle W_2U_1A = \angle WUB + \angle WUV = 180° \ ,$$

$$\angle W_2V_1Z_1 = \angle W_2V_1A + \angle AV_1Z_1 = \angle WVU + \angle CVZ = 180^\circ \ ,$$

and

$$\angle W_2 = \angle W_1 = \angle Z_1 = \angle WZQ = 90^\circ \ ,$$

it follows that $Z_1W_2W_1Z$ is a rectangle composed of isometric images of $APZQ$, $BUWP$, $CQZV$ and UVW.

Since PU and QV are both parallel to the median from A to BC, we have that $PQVU$ is a rectangle for which $PU < PB = PQ$. Thus, $PQVU$ is not a square and so its diagonals PV and QU do not intersect at right angles. It follows that W and Z do not lie on QU and so must be distinct.

Since PZQ and VWU are right triangles with $\angle QPZ = \angle UVW$ and $PQ = VU$, they must be congruent, so that $PZ = VW$, $PW = ZV$ and $UW = QZ$. Since

$$\begin{aligned} W_1W_2 &= W_1U_1 + U_1W_2 = WU + UW = WU + QZ \\ &< UQ = PV = PZ + ZV = PZ + PW = PZ + PW_1 = W_1Z \ , \end{aligned}$$

the adjacent sides of $Z_1W_2W_1Z$ are unequal, and so the rectangle is not square.

Comment. The inequality of the adjacent sides of the rectangle can be obtained also by making measurements. Take 4 as the length of a side of triangle ABC. Then

$$|PU| = \sqrt{3} \ , \quad |PQ| = 2 \ , \quad |QU| = |PV| = \sqrt{7} \ .$$

Since the triangles PUW and PVU are similar, $UW : PU = VU : PV$, whence $|UW| = 2\sqrt{21}/7$. Thus, $|W_1W_2| = 4\sqrt{21}/7 \neq \sqrt{7} = |W_1Z|$.

One can also use the fact that the areas of the triangle and rectangle are equal. The area of the triangle is $4\sqrt{3}$. It just needs to be verified that one of the sides of the rectangle is not equal to the square root of this.

147. Let $a > 0$ and let n be a positive integer. Determine the maximum value of

$$\frac{x_1x_2\cdots x_n}{(1+x_1)(x_1+x_2)\cdots(x_{n-1}+x_n)(x_n+a^{n+1})}$$

subject to the constraint that $x_1, x_2, \cdots, x_n > 0$.

Solution. Let $u_0 = x_1$, $u_i = x_{i+1}/x_i$ for $1 \leq i \leq n-1$ and $u_n = a^{n+1}/x_n$. Observe that $u_0u_1\cdots u_n = a^{n+1}$. The quantity in the problem is the reciprocal of

$$\begin{aligned} &(1+u_0)(1+u_1)(1+u_2)\cdots(1+u_n) \\ &\qquad = 1 + \sum u_i + \sum u_iu_j + \cdots + \sum u_{i_1}u_{i_2}\cdots u_{i_k} + \cdots + u_0u_1\cdots u_n \ . \end{aligned}$$

For $k = 1, 2, \cdots, n$, the sum $\sum u_{i_1}u_{i_2}\cdots u_{i_k}$ adds together all the $\binom{n+1}{k}$ $k-$fold products of the u_i; the product of all the terms in this sum is equal to a^{n+1} raised to the power $\binom{n}{k-1}$, namely, to a raised to the power $k\binom{n+1}{k}$. By the arithmetic-geometric means inequality

$$\sum u_{i_1}u_{i_2}\cdots u_{i_k} \geq \binom{n+1}{k}a^k \ .$$

Hence

$$(1+u_0)(1+u_1)\cdots(1+u_n) \geq 1 + (n+1)a + \cdots + \binom{n+1}{k}a^k + \cdots a^{n+1} = (1+a)^{n+1} \ ,$$

with equality if and only if $u_0 = u_1 = \cdots = u_n = a$. If follows from this that the quantity in the problem has maximum value of $(1+a)^{-(n+1)}$, with equality if and only if $x_i = a_i$ for $1 \leq i \leq n$.

Comment. Some of you tried the following strategy. If any two of the u_i were unequal, they showed that a larger value could be obtained for the given expression by replacing each of these by another value. They then deduced that the maximum occurred when all the u_i were equal. There is a subtle difficulty here. What has really been proved is that, **if** there is a maximum, it can occur only when the u_i are equal. However, it begs the question of the existence of a maximum. To appreciate the point, consider the following argument that 1 is the largest postive integer. We note that, given any integer n exceeding 1, we can find another integer that exceeds n, namely n^2. Thus, no integer exceeding 1 can be the largest positive integer. Therefore, 1 itself must be the largest.

Some of you tried a similar approach with the x_i, and showed that for a maximum, one must have all the x_i equal to 1. However, they neglected to build in the relationship between x_n and a_{n+1}, which of course cannot be equal if all the x_i are 1 and $a \neq 1$. This leaves open the possibility of making the given expression larger by bettering the relationship between the x_i and a and possibly allowing inequalities of the variables.

148. For a given prime number p, find the number of distinct sequences of natural numbers (positive integers) $\{a_0, a_1, \cdots, a_n \cdots\}$ satisfying, for each positive integer n, the equation

$$\frac{a_0}{a_1} + \frac{a_0}{a_2} + \cdots + \frac{a_0}{a_n} + \frac{p}{a_{n+1}} = 1 \ .$$

Solution. For $n \geq 3$ we have that

$$\begin{aligned} 1 = & \frac{a_0}{a_1} + \cdots + \frac{a_0}{a_2} + \cdots + \frac{a_0}{a_{n-2}} + \frac{p}{a_{n-1}} \\ = & \frac{a_0}{a_1} + \frac{a_0}{a_2} + \cdots + \frac{a_0}{a_{n-1}} + \frac{p}{a_n} \end{aligned}$$

whence

$$\frac{p}{a_{n-1}} = \frac{a_0}{a_{n-1}} + \frac{p}{a_n} \ ,$$

so that

$$a_n = \frac{pa_{n-1}}{p - a_0} \ .$$

Thus, for $n \geq 2$, we have that

$$a_n = \frac{p^{n-2}a_2}{(p-a_0)^{n-2}} \ .$$

Since $1 \leq p - a_0 \leq p - 1$, $p - a_0$ and p are coprime. It follows that, either $p - a_0$ must divide a_2 to an arbitrarily high power (impossible!) or $p - a_0 = 1$.

Therefore, $a_0 = p - 1$ and $a_n = p^{n-2}a_2$ for $n \geq 2$. Thus, once a_1 and a_2 are selected, then the rest of the sequence $\{a_n\}$ is determined. The remaining condition that has to be satisfied is

$$1 = \frac{a_0}{a_1} + \frac{p}{a_2} = \frac{p-1}{a_1} + \frac{p}{a_2} \ .$$

This is equivalent to

$$(p-1)a_2 + pa_1 = a_1 a_2 \ ,$$

or

$$[a_1 - (p-1)][a_2 - p] = p(p-1) \ .$$

The factors $a_1 - (p-1)$ and $a_2 - p$ must be both negative or both positive. The former case is excluded by the fact that $(p-1) - a_1$ and $p - a_2$ are respectively less than $p - 1$ and p. Hence, each choice of the pair

(a_1, a_2) corresponds to a choice of a pair of positive divisors of $p(p-1)$. There are $d(p(p-1)) = 2d(p-1)$ such choices, where $d(n)$ is the number of positive divisors of the positive integer n.

Comment. When $p = 5$, for example, the possibilities for (a_1, a_2) are $(5, 25)$, $(6, 15)$, $(8, 10)$, $(9, 9)$, $(14, 7)$, $(24, 6)$. In general, particular choices of sequences that work are

$$\{p-1, p, p^2, p^3, \cdots\}$$

$$\{p-1, 2p-1, 2p-1, p(2p-1), \cdots\}$$
$$\{p-1, p^2-1, p+1, p(p+1), \cdots\} \ .$$

A variant on the argument showing that the a_n from some point on constituted a geometric progression started with the relation $p(a_n - a_{n-1}) = a_0 a_n$ for $n \geq 3$, whence

$$\frac{a_{n-1}}{a_n} = 1 - \frac{a_0}{p} \ .$$

Thus, for $n \geq 3$, $a_{n+1}a_{n-1} = a_n^2$, which forces $\{a_2, a_3, \cdots, \}$ to be a geometric progession. The common ratio must be a positive integer r for which $r = p/(p - a_0)$. This forces $p - a_0$ to be equal to 1.

Quite a few solvers lost points because of poor book-keeping; they did not identify the correct place at which the geometric progression began. It is often a good idea to write out the first few equations of a general relation explicitly in order to avoid this type of confusion. You must learn to pay attention to details and check work carefully; otherwise, you may find yourself settling for a score on a competition less than you really deserve on the basis of ability.

149. Consider a cube concentric with a parallelepiped (rectangular box) with sides $a < b < c$ and faces parallel to that of the cube. Find the side length of the cube for which the difference between the volume of the union and the volume of the intersection of the cube and parallelepiped is minimum.

Solution. Let x be the length of the side of the cube and let $f(x)$ be the difference between the value of the union and the volume of the intersection of the two solids. Then

$$f(x) = \begin{cases} abc - x^3 & (0 \leq x < a) \\ abc + (x-a)x^2 - ax^2 = abc + x^3 - 2ax^2 & (a \leq x < b) \\ x^3 + ab(c-x) - abx = abc + x^3 - 2abx & (b \leq x < c) \\ x^3 - abc & (c \leq x) \end{cases}$$

The function decreases for $0 \leq x \leq a$ and increases for $x \geq c$. For $b \leq x \leq c$,

$$\begin{aligned} f(x) - f(b) &= x^3 - 2abx - b^3 + 2ab^2 \\ &= (x-b)[x^2 + bx + b^2 - 2ab] \\ &= (x-b)[(x^2 - ab) + b(x-a) + b^2] \geq 0 \ , \end{aligned}$$

so that $f(x) \geq f(b)$. Hence, the minimum value of $f(x)$ must be assumed when $a \leq x \leq b$.

For $a \leq x \leq b$, $f'(x) = 3x^2 - 4ax = x[3x - 4a]$, so that $f(x)$ increases for $x \geq 4a/3$ and decreases for $x \leq 4a/3$. When $b \leq 4a/3$, then $f(x)$ is decreasing on the closed interval $[a, b]$ and assumes its minimum for $x = b$. If $b > 4a/3 > a$, then $f(x)$ increases on $[4a/3, b]$ and so achieves its minimum when $x = 4a/3$. Hence, the function $f(x)$ is minimized when $x = \min(b, 4a/3)$.

150. The area of the bases of a truncated pyramid are equal to S_1 and S_2 and the total area of the lateral surface is S. Prove that, if there is a plane parallel to each of the bases that partitions the truncated pyramid into two truncated pyramids within each of which a sphere can be inscribed, then

$$S = (\sqrt{S_1} + \sqrt{S_2})(\sqrt[4]{S_1} + \sqrt[4]{S_2})^2 \ .$$

Solution 1. Let M_1 be the larger base of the truncated pyramid with area S_1, and M_2 the smaller base with area S_2. Let P_1 be the entire pyramid with base M_1 of which the truncated pyramid is a part. Let M_0 be the base parallel to M_1 and M_2 described in the problem, and let its area be S_0. Let P_0 be the pyramid with base M_0 and P_2 the pyramid with base M_2.

The inscribed sphere bounded by M_0 and M_1 is determined by the condition that it touches M_1 and the lateral faces of the pyramid; thus, it is the inscribed sphere of the pyramid P_1 with base M_1; let its radius be R_1. The inscribed sphere bounded by M_2 and M_0 is the inscribed sphere of the pyramid P_0 with base M_0; let its radius be R_0. Finally, let the inscribed sphere of the pyramid P_2 with base M_2 have radius R_2.

Suppose Q_2 is the lateral area of pyramid P_2 and Q_1 the lateral area of pyramid P_1. Thus, $S = Q_1 - Q_2$.

There is a dilation with factor R_0/R_1 that takes pyramid P_1 to P_0; since it takes the inscribed sphere of P_1 to that of P_0, it takes the base M_1 to M_0 and the base M_0 to M_2. Hence, this dilation takes P_0 to P_2. The dilation composed with itself takes P_1 to P_2. Therefore

$$\frac{R_0}{R_1} = \frac{R_2}{R_0} \qquad \text{and} \qquad \frac{Q_2}{Q_1} = \frac{S_2}{S_1} = \frac{R_2^2}{R_1^2} \ .$$

Consider the volume of P_2. Since P_2 is the union of pyramids of height R_2 and with bases the lateral faces of P_2 and M_2, its volume is $(1/3)R_2(Q_2 + S_2)$. However, we can find the volume of P_2 another way. P_2 can be realized as the union of pyramids whose bases are its lateral faces and whose apexes are the centre of the inscribed sphere with radius R_0 with the removal of the pyramid of base M_2 and apex at the centre of the same sphere. Thus, the volume is also equal to $(1/3)R_0(Q_2 - S_2)$.

Hence

$$\frac{Q_2 - S_2}{Q_2 + S_2} = \frac{R_2}{R_0} = \frac{R_2}{\sqrt{R_1R_2}} = \frac{\sqrt{R_2}}{\sqrt{R_1}} = \frac{\sqrt[4]{S_2}}{\sqrt[4]{S_1}}$$
$$\Longrightarrow Q_2(\sqrt[4]{S_1} - \sqrt[4]{S_2}) = S_2(\sqrt[4]{S_1} + \sqrt[4]{S_2}) \ ,$$

so that

$$\begin{aligned} S = Q_1 - Q_2 &= \frac{Q_2}{S_2}(S_1 - S_2) \\ &= \left[\frac{\sqrt[4]{S_1} + \sqrt[4]{S_2}}{\sqrt[4]{S_1} - \sqrt[4]{S_2}}\right][\sqrt{S_1} - \sqrt{S_2}][\sqrt{S_1} + \sqrt{S_2}] \\ &= (\sqrt[4]{S_1} + \sqrt[4]{S_2})^2(\sqrt{S_1} + \sqrt{S_2}) \ . \end{aligned}$$

Solution 2. [S. En Lu] Consider an arbitrary truncated pyramid with bases A_1 and A_2 of respective areas σ_1 and σ_2, in which a sphere Γ of centre O is inscribed. Let the lateral area be σ. Suppose that C is a lateral face and that Γ touches A_1, A_2 and C in the respective points P_1, P_2 and Q.

C is a trapezoid with sides of lengths a_1 and a_2 incident with the respective bases A_1 and A_2; let h_1 and h_2 be the respective lengths of the altitudes of triangles with apexes P_1 and P_2 and bases bordering on C. By similarity (of A_1 and A_2),

$$\frac{h_1}{h_2} = \frac{a_1}{a_2} = \sqrt{\frac{\sigma_1}{\sigma_2}} \ .$$

The plane that contains these altitudes passes through P_1P_2 (a diameter of Γ) as well as Q, the point on C nearest to the centre of Γ. Since the height of C is $a_1 + a_2$ [why?], its area is

$$\begin{aligned} \frac{1}{2}(a_1 + a_2)(h_1 + h_2) &= \frac{1}{2}[a_1h_1 + a_2h_2 + a_1h_2 + a_2h_1] \\ &= \frac{1}{2}[a_1h_1 + a_2h_2 + 2\sqrt{a_1a_2h_1h_2}] \\ &= \frac{1}{2}[a_1h_1 + a_2h_2 + 2a_2h_2\sqrt{\sigma_1/\sigma_2}] \ . \end{aligned}$$

Adding the corresponding equations over all the lateral faces C yields

$$\sigma = \sigma_1 + \sigma_2 + \sqrt{\sigma_1 \sigma_2} = (\sqrt{\sigma_1} + \sqrt{\sigma_2})^2 \ .$$

With S_0 defined as in Solution 1, we have that $S_1/S_0 = S_0/S_2$, so that $S_0 = \sqrt{S_1 S_2}$. Using the results of the first paragraph applied to the truncated pyramids of bases (S_2, S_0) and (S_0, S_1), we obtain that

$$\begin{aligned} S &= (\sqrt{S_1} + \sqrt{S_0})^2 + (\sqrt{S_0} + \sqrt{S_1})^2 \\ &= (\sqrt{S_1} + \sqrt[4]{S_1 S_2})^2 + (\sqrt[4]{S_1 S_2} + \sqrt{S_2})^2 \\ &= (\sqrt{S_1} + \sqrt{S_2})(\sqrt[4]{S_1} + \sqrt[4]{S_2})^2 \ . \end{aligned}$$

151. Prove that, for any natural number n, the equation

$$x(x+1)(x+2)\cdots(x+2n-1) + (x+2n+1)(x+2n+2)\cdots(x+4n) = 0$$

does not have real solutions.

Solution 1. With $y = x + 2n$, the given equation can be rewritten as

$$(y-1)(y-2)\cdots(y-2n) + (y+1)(y+2)\cdots(y+2n) = 0 \ .$$

When the left side is expanded, the coefficients of the odd powers of y cancel out, and we get a sum involving only even powers of y with positive coefficients. Hence the left side must be positive for all real values of y, and so there is not real solution for the equation (and for the given equation in x).

Solution 2. [Y. Yang] Let $A = x(x+1)\cdots(x+2n-1)$ and $B = (x+2n+1)(x+2n+2)\cdots(x+4n)$. It is clear that none of the numbers $0, 1, \cdots, 2n-1, 2n+1, \cdots, 4n$ is a solution. Since A and B cannot simultaneously vanish for any value of x, $A + B = 0$ can occur only if one is positive and the other negative.

Suppose that $A < 0$. Then there must be an odd number of negative factors in A, so that $-2k-1 < x < -2k$ for some integer $k \in [0, n-1]$. In this situation, the factors x, $x+1$, $\cdots$, $x+2k$ are negative and all the other factors involved in A and those in B are positive, so that $B > 0$. We show that the absolute value of A must be less than the absolute value of B by comparing pairs of factors.

For the positive factors of A, we have $x + 2k + 1 < x + 2n + 1$, $x + 2k + 2 < x + 2n + 2$, $\cdots$, $x + 2n - 1 = x + 2k + (2n - 2k - 1) < x + 4n - 2k - 1 = x + 2n + (2n - 2k - 1)$. For the negative factors A, we have that

$$|x| < 2k + 1 = 4k - 2k + 1 \le 4n - 4 - 2k + 1 < 4n - (2k+1) < 4n + x = x + 4n$$

$$|x+1| < 2k < 4n - 1 + x = x + 4n - 1$$

$$\cdots \qquad |x + 2k - 1| < x + 4n - 2k \ .$$

(Draw a diagram of the real line.) Multiplying all the inequalities yields that $|A| < |B| = B$, so that $A + B$ cannot vanish.

Suppose that $B < 0$. Then there are an odd number of negative factors in B, so that, for some positive k not exceeding n, $-2n - 2k < x < -2n - 2k + 1$. Since A has all negative factors, we have that $A > 0$. As in the earlier case, it can be proved that $|A| > |B|$, so that $A + B$ cannot vanish. The desired result follows.

152. Andrew and Brenda are playing the following game. Taking turns, they write in a sequence, from left to right, the numbers 0 or 1 until each of them has written 2002 numbers (to produce a 4004-digit number). Brenda is the winner if the sequence of zeros and ones, considered as a binary number (*i.e.*, written to base 2), **cannot** be written as the sum of two integer squares. Otherwise, the winner is Andrew. Prove

that the second player, Brenda, can always win the game, and explain her winning strategy (*i.e.*, how she must play to ensure winning every game).

Solution. First it will be established that *if the binary representation of a number n has two consecutive ones and an even number of zeros in its rightmost positions, the number cannot be written as the sum of two perfect squares*. For, all such numbers n must have the form $4^k(4s+3)$. Suppose, if possible, that such a number can be written as the sum of two squares: $n = x^2 + y^2$. Using the fact that squares, modulo 4, are congruent to 0 or 1, we see that when n is odd the representation is impossible, while if n is even, both x and y must be even as well and $(n/4) = (x/2)^2 + (y/2)^2$. We can apply the same argument to $(n/4)$ and continue dividing by 4 until we get a representation of $4s+3$. But this yields a contradiction.

We show that the second player, Brenda, has a winning strategy: she should always play so that the resulting 4004 digit number has the form described in the foregoing paragraph.

Case 1: Suppose the first player, Andrew, writes only 0. Then Brenda writes 1 on the first three turns and then keeps writing 0. This will result in the base 2 number $010101000\cdots000 = 21\cdot 4^{1999}$, which by the argument above cannot be written as the sum of two squares.

Case 2: Suppose on the other hand, at some point, Andrew inserts a 1. Brenda then writes 1 on her next term, and copies what Andrew writes thereafter. The result number will have the form of n in the first paragraph and so cannot be written as the sum of two squares. Brenda triumphs again.

Comment. As you will have noticed, the problem solved here differs from the one originally posed which said that Brenda was the winner if the final number *could* be written as the sum of two integer squares. This was our mistake, and, when discovered, was too late to correct. This changed the original problem to a much more difficult one. Nevertheless, some comments on the new problem will be provided as a good example of how advantage can be taken even from mistakes.

There is a simple and clear condition that some numbers cannot be written as a sum of two squares: it is enough to show that they are congruent to 3, modulo 4. The difficulty of the problem arises from the fact that if the number is not congruent to 3, it does not necessarily mean that it can be written as the sum of two squares. A theorem in number theory states that *a positive integer can be written as the sum of two squares if and only if no prime congruent to 3, modulo 4, divides the number to an odd exponent*.

For the problem solved above, it is enough to use the weaker condition and have Brenda ignore all numbers other than those congruent to 3 modulo 4 that are winning numbers for her. To solve the problem actually posed, it is not enough to consider only such numbers. It becomes a problem to connect the binary representation of a number to the number of its prime factors congruent to 3, modulo 4.

Two students, who sent their work on the problem posed, deserve recognition for their excellent ideas and results. Andrew Critch tried to come up with a strategy that eliminates the losing situations for Brenda. He succeeds in eliminating 2^{4003} such situations, out of all the 2^{4004} possible endgame scenarios. Leonid Chindelevitch goes further in exploring the problem, using a different approach. He realizes that the number 2002 in the problem is not sufficient for the algorithm of the game. By experimentation, he concludes that there is a winning strategy for Brenda in the case that every player writes 1, 2 or 4 numbers. (The first two cases simply require Brenda to write a zero every time, while the third one is a bit more complicated.) However, there is no such strategy for Brenda when each player writes three numbers. On the contrary, in this case, there is a winning strategy for Andrew. He starts with 1, and follows with 1 on each of his other two turns if Brenda write 0 on her first move. Otherwise, he follows with 1 on his second move if Brenda's first move is 1, and then imitates Brenda's second move on his third move. This yields one of the results: 101010, 101011, 101110, 101111, 111000, 111110, 111111 (in decimal: 42, 43, 46, 47, 56, 57, 62, 63), none of which is the sum of two integer squares.

In order to investigate the problem further, Leonid has designed an algorithm to verify the existence of a winning strategy with respect to the number n of moves for each player. Computer simulations reveal that, for the $1 \leq n \leq 15$, Brenda has a winning strategy only for $n = 1, 2, 4$. Since the algorithm used is exponential, the time becomes quite large under increase of n; for $n = 15$, more than six hours are required. Leonid also tries to determine the probability for Brenda to lead the game for ending at a winning number, and comes up with the conclusion that the probability is less than 0.31 for large numbers such as 2002. Accordingly, he concludes that it is not likely that there is a winning strategy for such a game with a large number of moves. Isn't it great!

Bonus points will be given to students with reasonable ideas on this problem. For the others, the problem will not be taken into account in compiling their scores.

153. (a) Prove that, among any 39 consecutive natural numbers, there is one the sum of whose digits (in base 10) is divisible by 11.

(b) Present some generalizations of this problem.

Solution. (a) Consider 20 consecutive numbers such that the smallest of them has units digit 0, and tens digit other than 9. Then, if S equals the sum of the digits of the smallest number, then the sums of the digitis of the 20 numbers are, in turn, S, $S+1$, $\cdots$, $S+9$, $S+1$, $\cdots$, $S+10$. These include 11 consecutive numbers, one of which must be divisible by 11.

Among any 39 consecutive integers, we can always find 20 such numbers. All that is required is that at least 20 of the 39 numbers must have the same hundreds digit. If all 39 have the same hundreds digit, then this is so. Otherwise, there are two hundred digits involved, and, by the pigeonhole principle, there must be twenty with the same hundreds digit.

Comment. It is not easy to formulate new problems. This is why those solving part (a) were given a full score, and bonus points assigned to anyone providing a reasonable suggestion for (b). Here are some generalizations:

(1) Prove that, among any $20n+39$ consecutive natural numbers, there is one, the sum of whose digits is divisible by $11+n$. [H. Li]

(2) Prove that, among any $20k-1$ consecutive natural numbers, there is one, the sum of whose digits is divisible by $k+9$ for $1 \leq k \leq 10$. (This is the same as (1) for $n = 2, 3, 4, 5$.)

(3) Prove that, among any $200k-1$ consecutive natural numbers, there is one the sum of whose digits is divisible by $k+18$ $(1 \leq k \leq 5)$.

154. (a) Give as neat a proof as you can that, for any natural number n, the sum of the squares of the numbers $1, 2, \cdots, n$ is equal to $n(n+1)(2n+1)/6$.

(b) Find the least natural number n exceeding 1 for which $(1^2 + 2^2 + \cdots + n^2)/n$ is the square of a natural number.

(a) *Solution.* There are many ways of proving the formula. Two standard ways are to use induction and to make use of the identity $(j+1)^3 = j^3 + 3j^2 + 3j + 1$ summed for $1 \leq j \leq n$. Y.Yang gives a geometric argument. Place unit cubical blocks in a pyramidic array with n^2 blocks at the bottom level, $(n-1)^2$ blocks at the next level, and so on, until there is a single block at the top. Inscribe this in a square-based right pyramid, whose slant edges pass through the upper outer corner of each level of cubes. This pyramid has a base with side length $n+1$ and a height $n+1$. The sum of the first n squares is equal to the volume of the cubes, which in turn is equal to the volume of the circumscribed pyramid less the volume of the space within the pyramid not occupied by the cubes.

The volume of the circumscribing pyramid is $\frac{1}{3}(n+1)^2(n+1) = \frac{1}{3}(n+1)^3$. Now look at the left-over part. It consists of a small pyramid at the top, and at each level, 4 square corner pyramids and 4 triangular prisms. Calling the top level the zeroth, we find that at the kth level down, the volume of the left-over part is equal to $4 \cdot \frac{1}{3} \cdot (1/2)^2 + 4 \cdot \frac{1}{2}(1/2)k = (1/3) + k$. Summing this for $0 \leq k \leq n$ yields $(1/3)(n+1) + (1/2)n(n+1)$, so that the total volume of all the cubes is

$$\frac{1}{3}(n+1)^3 - \frac{1}{3}(n+1) - \frac{1}{2}n(n+1) = \frac{1}{6}(n+1)[2(n+1)^2 - 2 - 3n] = \frac{1}{6}(n+1)(2n^2+n) = \frac{1}{6}n(n+1)(2n+1) \ .$$

(b) *Solution 1.* The condition to be satisfied is $(n+1)(2n+1) = 6m^2$ for some integer m. Since the right side is even, n be an odd number, say, $2k-1$. The equation reduces to $k(4k-1) = 3m^2$. Suppose, to begin with, k is a multiple of 3: $k = 3s$. Then $s(12s-1) = m^2$. Since s and $12s-1$ are coprime, both must be squares, *i.e.*, $s = p^2$ and $12s-1 = q^2$ for some natural numbers p and q. But $12s-1 \equiv 3 \pmod 4$ cannot be the square of a natural number. Thus, there is no solution in this case.

The other possibility is that $4k-1$ is a multiple of 3; equivalently, $k-1$ is a multiple of 3 (why?), so that $k = 3s+1$. Thus, $(3s+1)(4s+1) = m^2$. Since $3s+1$ and $4s+1$ are coprime (why?), $3s+1 = p^2$ and $4s+1 = q^2$ for some natural numbers p and q. We now check in turn all the odd squares of the form $4s+1$ until we come to one for which $3s+1$ is square. This leads us to $4s+1 = 15^2$ and $3s+1 = 13^2$, so that $s = 56$, $k = 169$, $n = 337$.

Solution 2. It is required to find the smallest natural number n for which $(n+1)(2n+1) = 6m^2$, for some positive integer m. We manipulate the equation to make use of the technique of completion of the square. The equation is equivalent to $16n^2 + 24n + 8 = 48m^2$ or $(4n+3)^2 - 1 = 3(4m)^2$. Setting $x = 4n+3$ and $y = 4m$, we see that we need to find solutions of the equation $x^2 - 3y^2 = 1$ where $x \equiv 3$ and $y \equiv 0$ (modulo 4).

Noting that $x^2 - 3y^2 = (x+\sqrt{3})(x-\sqrt{3})$, and that $2^2 - 3 \cdot 1^2 = 1$, we can deduce that the equation $x^2 - 3y^2 = 1$ is satisfied by $(x, y) = (x_n, y_n)$ where $x_n + \sqrt{3}y_n = (2+\sqrt{3})^n$ for each nonnegative integer n. These solutions can be given through the recursion

$$x_1 = 2\ , \qquad y_1 = 1$$

$$x_{n+1} = 2x_n + 3y_n\ , \qquad y_{n+1} = x_n + 2y_n\ .$$

The smallest solutions of this equation are $(x, y) = (1, 0), (2, 1), (7, 4), (26, 15), (97, 56), (362, 209), (1351, 780)$. Of these, only $(7, 4)$ and $(1351, 780)$ have the desired properties, and these correspond to $(n, m) = (1, 1)$ and $(337, 195)$. The first gives the trivial case, and the second gives the minimum value 337 of n.

155. Find all real numbers x that satisfy the equation

$$3^{[(1/2)+\log_3(\cos x + \sin x)]} - 2^{\log_2(\cos x - \sin x)} = \sqrt{2}\ .$$

[The logarithms are taken to bases 3 and 2 respectively.]

Solution. By the definition and properties of the logarithmic function, it follows that (1) the domain of the equation is the set of all x for which $\cos x + \sin x > 0$ and $\cos x - \sin x > 0$, and that (2) $3^{\log_3 M} = M$ when $M > 0$.

The given equation is equivalent to

$$\sqrt{3}(\cos x + \sin x) - (\cos x - \sin x) = \sqrt{2}$$

$$\Longleftrightarrow (\sqrt{3} - 1)\cos x + (\sqrt{3} + 1)\sin x = \sqrt{2}$$

$$\Longleftrightarrow \frac{\sqrt{6} - \sqrt{2}}{4}\cos x + \frac{\sqrt{6} + \sqrt{2}}{4}\sin x = \frac{1}{2}\ .$$

Noting that $(\sqrt{6} - \sqrt{2})/4 = \sin 15^\circ$ and $(\sqrt{6} + \sqrt{2})/4 = \cos 15^\circ$, we see that the equation is equivalent to $\sin(x + 15^\circ) = \frac{1}{2}$, whose solutions are $x = 15^\circ + k360^\circ$ and $x = 165^\circ + k360^\circ$ for integers k, The latter family are extraneous, so only the former family constitute the solutions to the given problem.

Comment. Most participants attempted this problem and dealt well with the trigonometric transformations. However, it is important to check all purported solutions against the given equation, because the derived equation becomes equivalent to the equation solved in the solution only in the presence of the restrictions on the domain. Many students neglected this.

156. In the triangle ABC, the point M is from the inside of the angle BAC such that $\angle MAB = \angle MCA$ and $\angle MAC = \angle MBA$. Similarly, the point N is from the inside of the angle ABC such that $\angle NBA = \angle NCB$ and $\angle NBC = \angle NAB$. Also, the point P is from the inside of the angle ACB such that $\angle PCA = \angle PBC$ and $\angle PCB = \angle PAC$. (The points M, N and P each could be inside or outside of the triangle.) Prove that the lines AM, BN and CP are concurrent and that their intersection point belongs to the circumcircle of the triangle MNP.

Solution. First, note that the point M is inside the triangle ABC when $\angle A$ is acute and outside the triangle when $\angle A$ is obtuse. Indeed, $\angle A = \angle MAB + \angle MAC = \angle MCA + \angle MBA < 90°$ implies that in the quadrilateral $AMBC$, $\angle BMC > 180°$, which means that the point M is inside the triangle. Similarly, $\angle A > 90°$ implies that, in the quadrilateral $ABMC$, $\angle BMC < 180°$; thus, the point M is outside the triangle. The same reasoning can be applied to determine whether the points N and P are inside or outside of the triangle ABC. It is straightforward to verify that when one of the angles is right, the respective point is on the side of the triangle opposite to the right angle.

Wolog, consider $\angle C$. *Case 1*: $\angle C < 90°$. Let CP instersect the segment AB at C'. Since PC' is the angle bisector of $\angle APB$ ($\angle APC' = \angle BPC' = \angle ACB$), it follows that $AC' : C'B = AP : BP$. The triangles APC and BPC are similar, so that $AP : CP = AC : BC$ and $BP : CP = BC : AC \Longrightarrow AP : BP = AC^2 : CB^2 \Longrightarrow AC' : C'B = AC^2 : CB^2$. Similarly, if AM intersects BC at M' and BN instersects AC at N", then

$$BA' : A'C = BA^2 : AC^2$$

and $CB' : B'A = CB^2 : BA^2$. Apply Ceva's Theorem to the lines AM, BN, CP to deduce from

$$\frac{AC'}{C'B} \cdot \frac{BA'}{A'C} \cdot \frac{CB'}{B'A} = \frac{AC^2}{CB^2} \cdot \frac{BA^2}{AC^2} \cdot \frac{CB^2}{BA^2} = 1$$

that the lines AM, BN and CP are concurrent. Denote by K the intersection point of these lines and by O the circumcentre of the triangle ABC.

Since $\angle APB = 2\angle C = \angle AOB$, the quadrilateral $APOB$ is concyclic. Also, if the line PC' intersects the circumcircle of $APOB$ at the point C", then this point C" is the midpoint of that arc AB not containing O (as PC' is the angle bisector of $\angle APB$), and O is the midpoint of the other arc AB. Thus, OC" is the diameter of this circle and $\angle OPC" = \angle OPC = 90°$. Since the point K is on the line PC, $\angle OPK = 90°$, too; thus P is a point on the circle whose diamter is OK. Similarly, the points M and N are on the same circle. This circle contains all the points M, N, P, K, O, so that the intersection point of AM, BN and CP is on the circumcircle of MNP.

Case 2: $\angle C = 90°$. In this case, the point P is on the hypotenuse AB (thus, it coincides with the point C' of Case 1), and is also the point where the altitude from C toAB intersects AB. By the similarity of the triangles ABC and APC, $AP : AC = AC : AB$, while, by the similarity of the triangles, BPC and ABC, it follows that $PC : BC = BC : AB$. Then $AP : PC = AC^2 : BC^2$, which replaces the equation $AC' : C'B = AC^2 : BA^2$ of Case 1.The other two angles are acute, so that the remaining parts of the solution can be pursued as in Case 1.

Case 3: $\angle C > 90°$ Now the point C is outside of the triangle ABC, as well as the circumcentre O. The proof is similar to that of Case 1, and follows the same steps.

Sources: Problem 151 is from the Bulgarian magazine *Matematika Plus*, 2001. Problems 152, 154, 155, 156 are from the contest papers of the 2001 Winter and Spring National Mathematics Contests for High School students in Bulgaria.

157. Prove that if the quadratic equation $x^2 + ax + b + 1 = 0$ has nonzero integer solutions, then $a^2 + b^2$ is a composite integer.

Solution. Suppose that the integer roots are u and v. Then $u + v = -a$ and $uv = b + 1$, whence

$$\begin{aligned} a^2 + b^2 &= (u+v)^2 + (uv-1)^2 = u^2 + v^2 + u^2v^2 + 1 \\ &= (u^2+1)(v^2+1) \ . \end{aligned}$$

Since each factor exceeds 1, $a^2 + b^2$ is composite.

Comment. You should make sure that you are familiar with the relationship between the coeffients and roots of polynomials; a poor way to solve this problem is to use the quadratic formula. Note that, if, say, $u = 0$, then $a^2 + b^2$ can be prime; for example, you get the value 5 when $(u, v) = (0, 2)$.

158. Let $f(x)$ be a polynomial with real coefficients for which the equation $f(x) = x$ has no real solution. Prove that the equation $f(f(x)) = x$ has no real solution either.

Solution 1. Let $g(x) = f(x) - x$. Then, $g(x)$ is a polynomial that never vanishes. We argue that it must always have the same sign. Suppose if possible that $g(a) < 0 < g(b)$ for some reals a and b. Since $g(x)$, being a polynomial, is continuous, the Intermediate Value Theorem applies and there must be a number c between a and b for which $g(c) = 0$, yielding a contradition.

Thus, either $g(x) > 0$ for all x or else $g(x) < 0$ for all x. Then

$$\begin{aligned} f(f(x)) - x &= f(f(x)) - f(x) + f(x) - x \\ &= g(f(x)) + g(x) \end{aligned}$$

for all real x. Since g never changes sign, both $g(x)$ and $g(f(x))$ have the same sign (either positive or negative) and so their sum cannot vanish. Hence $f(f(x)) \neq x$ for any real x.

Solution 2. Suppose, if possible, that $f(f(a)) = a$. Let $b = f(a)$. Then $f(b) = a$. By hypothesis, $b \neq a$. Wolog, suppose that $a < b$. Then $f(a) - a > 0$ and $f(b) - b < 0$. Since $f(x) - x$ is a polynomial, it is continuous and so the Intermediate Value Theorem applies on the closed interval $[a, b]$. As it has opposite signs at the endpoints of $[a, b]$, it must vanish somewhere in the interior of the interval. But then this contradicts the hypothesis. Hence $f(f(x)) = x$ can have no real solutions.

Comment. It is important to highlight that $f(x)$ is a continuous function, as this is key to the result. (Can you construct a counterexample where it is not assumed that f is a polynomial or continuous?) Several of you tried to give a geometric argument for this, and apart from mangling the terminology (*e.g.*, not distinguishing between a function and its graph, points and values), operated at too intuitive a level. Notice that the statement "$f(x) > 0$ or $f(x) < 0$ for all x" lacks a certain precision, as it could be interpreted to mean that at any individual point x, one of the two alternatives occurs (the function does not vanish), but that different alternative might occur at different points.

159. Let $0 \leq a \leq 4$. Prove that the area of the bounded region enclosed by the curves with equations

$$y = 1 - |x - 1|$$

and

$$y = |2x - a|$$

cannot exceed $\frac{1}{3}$.

Solution. In the situation that $0 \leq a \leq 1$, the two curves intersect in the points $(a/3, a/3)$ and (a, a), and the bounded region is the triangle with these two vertices and the vertex $(a/2, 0)$. This triangle is contained in the triangle with vertices $(0, 0)$, $(1/2, 0)$ and $(1, 1)$ with area $1/4$. Hence, when $0 \leq a \leq 1$, the area of the bounded region cannot exceed $1/4$.

Let $1 \leq a \leq 3$. In this case, the bounded region is a quadrilateral with the four vertices $(a/3, a/3)$, $(a/2, 0)$, $((a+2)/3, (4-a)/3)$ and $(1, 1)$. Noting that this quadrilateral is the result of removing two smaller triangles from a larger one (draw a diagram!), we find that its area is

$$1-\frac{1}{2}\cdot\frac{a}{3}\cdot\frac{a}{2}-\frac{1}{2}\cdot\frac{(4-a)}{3}\cdot\left(2-\frac{a}{2}\right)$$
$$=1-\frac{a^2}{12}-\frac{1}{12}(4-a)^2$$
$$=-\frac{a^2-4a+2}{6}=\frac{1}{3}-\frac{(a-2)^2}{6}\ ,$$

whence we find that the area does not exceed $1/3$ and is equal to $1/3$ exactly when $a=2$.

The case $3\leq a\leq 4$ is the symmetric image of the case $0\leq a\leq 1$ and we find that the area of the bounded region cannot exceed $1/4$.

Comment. This is not a difficult problem, but it does require a lot of care in keeping the situation straight and the computations sound. Always keep in mind completion of the square when it is a matter of optimizing a quadratic. Use of calculus is more cumbersome in such situations, and begs the question as to whether the optimum is a maximum or a minimum. (You can lose points for not explicitly resolving this issue.) R. Barrington Leigh solved the problem by looking at the area between the graph of $y=1-|x-1|-|2x-a|$ and the $x-$axis. The equation of this curve with the absolute value signs is equal to $y=3x-a$ for $x\leq a/2$, to $y=a-x$ when $a/2\leq x\leq 1$ and $a+2-3x$ when $x\leq 1$, with suitable modification when $a\geq 2$. Because at each value of x, the difference in the ordinates remains the same as in the original problem, the area here is the same as the area between the two graphs of the problem. You might want to try the problem in this way.

160. Let I be the incentre of the triangle ABC and D be the point of contact of the inscribed circle with the side AB. Suppose that ID is produced outside of the triangle ABC to H so that the length DH is equal to the semi-perimeter of ΔABC. Prove that the quadrilateral $AHBI$ is concyclic if and only if angle C is equal to $90°$.

Note. In the solutions that follow, we use the standard notation that a, b, c are the respective lengths of the sides BC, CA, AB, r is the inradius of the triangle and s is its semi-perimeter (so that $2s=a+b+c$). We note that the area of the triangle is rs, that the distances from the vertices A, B, C to the tangent points of the incircle are respectively $s-a$, $s-b$, $s-c$, and that $r=(s-c)\tan(C/2)$. If you are not familiar with these relationships, then regard them as exercises.

Solution 1. The quadrilateral $AHBI$ is concyclic if and only if $DI:DH=DB:DA$, or equivalently,

$$rs=(s-a)(s-b)=s^2-s(a+b)+ab=s^2-s(2s-c)+ab=s(c-s)+ab\ .$$

The area of the triangle is $rs=\frac{1}{2}ab\sin C=abt(1+t^2)^{-1}$ and $r=(s-c)t$, where $t=\tan C/2$. Then the concylic condition is equivalent to

$$rs=-\frac{rs}{t}+\frac{rs(1+t^2)}{t}=rst\ ,$$

or $t=1$. This is equivalent to $C/2=45°$, and the result follows.

Solution 2. The concyclic condition is equivalent to

$$rs=(s-a)(s-b)\Longleftrightarrow 4rs=(c+b-a)(c+a-b)=c^2-b^2-a^2+2ab\ .$$

We have, for any triangle, that $2ab\sin C=4rs$ (four times the area) and $c^2=a^2+b^2-2ab\cos C$. Hence the concyclic condition is equivalent to

$$2ab\sin C=2ab(1-\cos C)\Longleftrightarrow\sin C+\cos C=1\ .$$

If $\sin C + \cos C = 1$, then, by squaring, $1 + 2\sin C \cos C = 1 \Longrightarrow \sin 2C = 0$ so that $C = 90°$. On the other hand, if $C = 90°$, then $\sin C + \cos C = 1$. The result follows.

Comment. We can end the solution, keeping the equivalences to the end, by noting that

$$\sin C + \cos C = 1 \Leftrightarrow \sin(C + 45°) = \sqrt{2} \Leftrightarrow C = 90° \ .$$

Solution 3. [A. Feizmohammadi]

$$\angle AIB = 180° - (\angle BAI + \angle ABI) = 180° - \frac{\angle A + \angle B}{2} = 90° + \frac{\angle C}{2} \ .$$

Also,

$$\tan \angle AHB = \frac{(s-a)/s + (s-b)/s}{1 - (s-a)(s-b)/s^2} = \frac{(2s-a-b)s}{(a+b)s - ab} = \frac{cs}{(2s-c)s - ab} = \frac{cs}{2s^2 - cs - ab} \ .$$

Suppose that $\angle C$ is right. Then $\angle AIB = 135°$ and

$$c^2 = a^2 + b^2 \Longrightarrow (a+b)^2 - c^2 = 2ab \Longrightarrow (a+b-c)(a+b+c) = 2ab \Longrightarrow ab = 2s(s-c) \ ,$$

so that

$$\tan \angle AHB = \frac{cs}{2s^2 - cs - 2s^2 + 2cs} = 1 \Longrightarrow \angle AHB = 45° \ .$$

Since angles AHB and AIB are supplementary, $AHBI$ is concyclic.

On the other hand, suppose that $AHBI$ is concyclic. Then

$$\begin{aligned}\sqrt{s(s-a)(s-b)(s-c)} = (s-a)(s-b) = rs &\Longrightarrow (s-a)(s-b) = s(s-c) \\ &\Longrightarrow ab = (a+b-c)s \Longrightarrow 2ab = (a+b)^2 - c^2 \\ &\Longrightarrow a^2 + b^2 = c^2 \ ,\end{aligned}$$

so that $\angle ACB$ is right.

161. Let a, b, c be positive real numbers for which $a + b + c = 1$. Prove that

$$\frac{a^3}{a^2+b^2} + \frac{b^3}{b^2+c^2} + \frac{c^3}{c^2+a^2} \geq \frac{1}{2} \ .$$

Solution. Observe that, by the arithmetic-geometric means inequality $ab \leq \frac{1}{2}(a^2 + b^2)$, so that

$$\frac{a^3}{a^2+b^2} = a - b\frac{ab}{a^2+b^2} \geq a - \frac{b}{2}$$

with a similar inequality for the other two terms on the left side. Adding these inequalities and using $a + b + c = 1$ yields the desired result.

Comment. R. Furmaniak noted that

$$\frac{2a^3}{a^2+b^2} - 2a + b = \frac{b(b-a)^2}{a^2+b^2} \geq 0 \ ,$$

which again yields the result by concluding as in the solution.

162. Let A and B be fixed points in the plane. Find all positive integers k for which the following assertion holds:

among all triangles ABC with $|AC| = k|BC|$, the one with the largest area is isosceles.

Solution 1. In the case that $k = 1$, all triangles satisfying the condition are isosceles with $AC = BC$, the locus of C is the right bisector of AB and there is no triangle with largest area (the area can be arbitrarily large). Hence $k \geq 2$. Since, by the triangle inequality,

$$|AB| + |BC| > |AC| = k|BC| \geq 2|BC| \ ,$$

we must have that $|AB| > |BC|$. Therefore, the only way a triangle satisfying the condition can be isosceles is for $|AB| = |AC| = k|BC|$. Since $|BC| < |AC|$, $\angle CAB$ cannot exceed $60°$, so that when $k \geq 2$, there is no isosceles triangle in the set.

Solution 2. As in Solution 1, we need consider only the case that $k \geq 1$, and we can dispose of the case $k = 1$. Let D be the foot of the perpendicular from C to AB (possibly produced; note that D and B lie on the same side of A) and let the respective lengths of AB, AD and CD be c, x and v. We wish to maximize v. The given condition implies that

$$x^2 + v^2 = k((x - c)^2 + v^2)$$

so that

$$v^2 = -x^2 + \frac{2k^2c}{k^2 - 1}x - \frac{k^2c^2}{k^2 - 1} \ .$$

The maximum value of v is equal to

$$\frac{kc}{k^2 - 1}$$

assumed when $x = \frac{k^2c}{k^2-1} > c$. If this maximum value of v is assumed when $|AC| = |AB|$, then we have $x^2 + v^2 = c^2$ which leads to $k = 1/\sqrt{3}$. If the maximum is assumed when $|BC| = |AB|$, then $(x - c)^2 + v^2 = c^2$, which leads to $k = \sqrt{3}$. As both solutions are extraneous, there is no positive integer value of k for which the area is maximized when the triangle is isosceles.

Solution 3. As before, we can deal with the $k = 1$ case. Let $k \neq 1$. Let the sides of the triangle be (a, ka, c) where c is the length of AB; let C be the angle opposite c. Then

$$c^2 = (k^2 + 1)a^2 - 2ka^2 \cos C$$

whence $a^2 = c^2(k^2 + 1 - 2k \cos C)^{-1}$. The area of the triangle is equal to

$$\frac{1}{2}ka^2 \sin C = \frac{1}{2}kc^2(k^2 + 1 - 2k \cos C)^{-1} \sin C \ .$$

The derivative of $\sin C(k^2 + 1 - 2k \cos C)^{-1}$ with respect to C is equal to $[(\cos C)(k^2 + 1) - 2k][k^2 + 1 - 2k \cos C]^{-2}$, from which we can read off that area is maximized when $\cos C = 2k(k^2+1)^{-1}$. At this maximum value, $(k^2 + 1)c^2 = (k^2 - 1)^2a^2$. The triangle maximizing the area can be isosceles in two ways. The case $c = a$ occurs when $k = \sqrt{3}$, and the case $c = ka$ occurs when $k = 1/\sqrt{3}$. Neither of these is an integer.

Alternatively, the case $c = a$ corresponds to $\cos C = k/2$, and equating $k/2$ and $2k/(k^2 + 1)$ leads to $k^2 = 3$. The case $c = ka$ corresponds to $\cos C = 1/2k$, and this leads to $k^2 = 1/3$. We conclude as before.

Solution 4. Dispose of the $k = 1$ case as before. Let $k \neq 1$. Let $c = |AB|$ be fixed, and let the other two sides have length a and la. Sixteen times the square of the area of the triangle is, by Heron's rule,

$$\begin{aligned}
&[(1 + k)a + c][(1 + k)a - c][c + (1 - k)a][c - (1 - k)a] \\
&\qquad = [(1 + k)^2a^2 - c^2][c^2 - (1 - k)^2a^2] = 2a^2c^2(1 + k^2) - (k^2 - 1)^2a^4 - c^4 \\
&\qquad = \left[\frac{(k^2 + 1)^2}{(k^2 - 1)^2}c^4 - c^4\right] - \left[(k^2 - 1)a^2 - \frac{(k^2 + 1)}{(k^2 - 1)}c^2\right]^2 \\
&\qquad = \frac{4k^2c^4}{(k^2 - 1)^2} - (k^2 - 1)\left[a^2 - \frac{k^2 + 1}{(k^2 - 1)^2}c^2\right] \ .
\end{aligned}$$

The maximum area occurs when $(k^2-1)^2a^2 = (k^2+1)c^2$. This occurs when $a = c$ exactly when $(k^2-1)^2 = k^2+1$, so $k^2 = 3$, and occurs when $ka = c$ exactly when $(k^2-1)^2 = k^4+k^2$, so $3k^2 = 1$. We conclude as before.

Comments. Several students committed the fallacy of setting, for example, $AC = AB$, noting that the locus of C in this case was a circle with diameter containing AB, claiming that the area was maximized when $\angle C = 90°$ and so BC was equal to $\sqrt{2}AC$. They deduced that $\sqrt{2}$ and $1/\sqrt{2}$ were the only values of k that allowed the area to be maximized by an isosceles triangle. However, this puts things the wrong way around. The value k is fixed, and then we look at all appropriate triangles. All the solvers did was to maximize the area over all isosceles triangles for which $AC = AB$; as C traces out the set of points satisfying these conditions, of course, the ratio of the lengths of AC and BC vary. Those solvers who made this mistake are invited to investigate the case $k = \sqrt{2}$ in more detail, and this will help them understand where they went wrong.

If we coordinatize the situation with $A \sim (0,0)$ and $B \sim (1,0)$, we see that the locus of C is given by

$$(k^2-1)(x^2+y^2) - 2k^2x + k^2 = 0 \ .$$

This is the equation of a circle (of Apollonius) with centre at $(k^2/(k^2-1), 0)$ and with radius $k(k^2-1)^2$. Thus, the area is maximized when C is located at the point $(k^2(k^2-1)^{-1}, k(k^2-1)^{-1})$. Checking when this yields an isosceles triangle leads to the two values of k that we have already seen: $\sqrt{3}$ and $1/\sqrt{3}$

163. Let R_i and r_i re the respective circumradius and inradius of triangle $A_iB_iC_i$ $(i = 1,2)$. Prove that, if $\angle C_1 = \angle C_2$ and $R_1r_2 = r_1R_2$, then the two triangles are similar.

Solution. Let Γ be the circumcircle of $\Delta A_1B_1C_1$. Scale $\Delta A_2B_2C_2$ so that $A_2 = A_1 = A$, $B_2 = B_1 = A$ and C_2 and C_1 lie on the same side of AB. Then Γ is the circumcircle of ABC_2, so that $R_1 = R_2$ and $r_1 = r_2$. Using the conventional notation, we have that

$$(s_1 - c_1)\cot(C_1/2) = r_1 = r_2 = (s_2 - c_2)\cot(C_2/2)$$

whence, as $c_1 = c_2$ and $C_1 = C_2$, $s_1 = s_2$. Therefore $a_1 + b_1 = a_2 + b_2$. Since $r_1s_1 = r_2s_2$, the two triangles have the same area, so that $a_1b_1 \sin C_1 = a_2b_2 \sin C_2$ and thus $a_1b_1 = a_2b_2$. Since the pairs (a_1, b_1) and (a_2, b_2) have the same sum and product, they are roots of the same quadratic equation, and so we must have that $(a_1, b_1) = (a_2, b_2)$ or $(a_1, b_1) = (b_2, a_2)$. In either case, the triangles are congruent, so that triangle $A_1B_1C_1$ is a scaled version of the original triangle $A_2B_2C_2$. The result follows.

Comment. There were many variations using various trigonometric identities.

164. Let n be a positive integer and X a set with n distinct elements. Suppose that there are k distinct subsets of X for which the union of any four contains no more that $n-2$ elements. Prove that $k \leq 2^{n-2}$.

Solution 1. [R. Furmaniak] We may assume that $n \geq 2$. We give a proof by contradiction. Suppose the S is a family of k subsets of X, where $k > 2^{n-2}$. Let $X = \{x_1, x_2, \cdots, x_n\}$ where the first element is selected so that x_1 belongs to some member of S but not to all. (Why can you do this?) For each i, $1 \leq i \leq n-1$, let A_i be the class of all subsets of $\{x_1, x_2, \cdots, x_i\}$ which are contained in some subset of S, and B_i be the class of all subsets of $\{x_{i+1}, \cdots, x_n\}$ which are contained in some subset of S.

Note that $2^{n-2} < k \leq |A_i||B_i|$, since each set of S is the union of a set in A_i and a set in B_i. ($|\cdot|$ refers to the number of elements.) Therefore, either $|A_i| > 2^{i-1}$ or $|B_i| > 2^{n-i-1}$. In the former case, A_i must contain a complementary pair of subsets (use the pigeonhole principle on complementary pairs of sets) of $\{x_1, \cdots, x_i\}$; the the latter, B_i must contain a complementary pair of sets of $\{x_{i+1}, \cdots, x_n\}$.

By our choice of x_1, we see that $A_1 = \{\emptyset, x_1\}$, so that $|A_1| = 2$. Therefore there are values of the index i for which $|A_i| > 2^{i-1}$. If it turns out that $|A_{n-1}| > 2^{n-2}$, then A_{n-1} contains two sets whose union is $\{x_1, \cdots, x_{n-1}\}$ and so S contains two sets whose union has at least $n-1$ elements. In this case, S fails to satisfy the hypotheses of the problem.

Suppose that $|A_{n-1}| \leq 2^{n-2}$. Then select the smallest index k for which $|A_k| \leq 2^{k-1}$. Then $k \geq 2$ and $|A_{k-1}| > 2^{k-2}$ while $|B_k| > 2^{n-k-1}$. Therefore, A_{k-1} contains two sets whose union is $\{x_1, x_2, \cdots, x_{k-1}\}$ and B_k contains two sets whose union is $\{x_{k+1}, \cdots, x_n\}$. We conclude that S must contain four sets whose union contains the $n-1$ elements of X apart from x_k, and so S again fails to satisfy the hypothesis of the problem.

We conclude that any family of more than 2^{n-2} sets fails so satisfy the hypothesis of the problem, and so that every family that does satisfy the hypothesis must have at more 2^{n-2} elements.

Solution 2. Let P be a class of k subsets for which the union of no four contains more than $n-2$ elements. Suppose that A_1 and A_2 are two members for which the cardinality $m = |A_1 \cup A_2|$ of the union of a pair is maximum. Since $m < n$, we can construct a set $Y = A_1 \cup A_2 \cup \{y\}$, where $y \notin A_1 \cup A_2$; thus $|Y| = m + 1$.

Let $Q = \{Y \cap A : A \in P\}$. No two subsets in Q are complementary (*i.e.* have union equal to Y), so that $|Q| \leq 2^m$. Let $Z = X \setminus Y$ and $R = \{Z \cap A : A \in P\}$. Then $|Z| = n - (m+1) = n - m - 1$. We prove that R has no two sets that are complementary, *i.e.*, whose union is Z. For, otherwise, suppose that $(A_3 \cap Z) \cup (A_4 \cap Z) = Z$, for two sets A_3 and A_4 in P. Then $Z = (A_3 \cup A_4) \cap Z$, so that

$$Z \subseteq A_3 \cup A_4 \Longrightarrow A_1 \cup A_2 \cup A_3 \cup A_4 \supseteq (Y \cup Z) \setminus \{y\}$$

, whence we see that $|A_1 \cup A_2 \cup A_3 \cup A_4| = n - 1$, contrary to hypothesis. Hence $|R| \leq 2^{n-m-2}$.

Since each set in P is uniquely determined by its intersections with Y and Z, we have that

$$k = |P| \leq |Q||R| \leq 2^m 2^{n-m-2} = 2^n - 2 \ ,$$

as desired.

Comment. Several solvers pointed out that if you took all the subsets of $n-2$ of the elements of X, then we got an "extreme" case of a family of sets that satisfied the hypothesis and the conclusion, and essentially said that if we tried to stuff in any more sets, we would contradict the hypothesis. However, this begs the question as to whether you could drop some of these sets and replace than by a larger number of sets while still keeping the hypothesis.

165. Let n be a positive integer. Determine all $n-$tples $\{a_1, a_2, \cdots, a_n\}$ of positive integers for which $a_1 + a_2 + \cdots + a_n = 2n$ and there is no subset of them whose sum is equal to n.

Solution 1. Let $s_k = a_1 + a_2 + \cdots + a_k$ for $1 \leq k \leq n$, By hypothesis, all of these are incongruent modulo n. Now let $t_1 = a_2$, $t_k = s_k$ for $2 \leq k \leq n$. Again, the hypothesis forces all to be incongruent. Hence the sets $\{s_k\}$ and $\{t_k\}$ both consitute a complete set of residues, modulo n, so it must happen that $s_1 \equiv t_1 \equiv a_1 \pmod{n}$. A similar argument can be marshalled when the pair (a_1, a_2) is replaced by any other pair of elements. Hence, each of the terms in the set $\{a_i\}$ is congruent to some number $m \pmod{n}$, where $0 \leq m \leq n-1$. Now

$$mn \leq a_1 + a_2 + \cdots + a_n = 2n$$

from which either $m = 2$, n is odd, and the n-tple is $\{2, 2, 2, \cdots, 2\}$ or else $m = 1$ and the n-tple is $\{1, 1, 1, \cdots, 1, n+1\}$.

Solution 2. [J.Y. Jin] Wolog, suppose that $a_1 \leq a_2 \leq \cdots \leq a_n$. Note that a_1 cannot exceed 2. If $a_1 = 2$, then each a_i is equal to 2 and n is odd. Suppose that $a_1 = 1$. Then $a_n \geq 3$. Let $s_k = a_1 + \cdots + a_k$ for $1 \leq k \leq n$. Then $s_{n-1} = 2n - a_n \leq 2n - 3$. By the hypothesis, we see that the set of the s_k with $1 \leq k \leq n-1$ contains at most one member from each of the $n-1$ sets:

$$\{1, n+1\}, \{2, n+2\}, \{3, n+3\}, \cdots, \{n-3, 2n-3\}, \{n-2\}, \{n-1\}$$

so it must contain *exactly* one from each set. In particular, the values $n-2$ and $n-1$ are assumed, so that for some i, $s_i = n-2$ and $s_{i+1} = n-1$. Thus $a_{i+1} = 1$, and $a_1 = a_2 = \cdots = a_{i+1} = 1$. But this means that $i = n-2$, so that $a_{n-1} = 1$ and $a_n = n+1$.

Comment. Some of the solvers tried an induction argument. However, the process was faulty, since they generally started with an n-tple and then tried to build from that an $(n+1)$-tple. However, for an induction argument to work, you need to start with a general $(n+1)$-tple that satisfies the condition, and then indicate how to reduce it to the result for a smaller set.

166. Suppose that f is a real-valued function defined on the reals for which

$$f(xy) + f(y-x) \geq f(y+x)$$

for all real x and y. Prove that $f(x) \geq 0$ for all real x.

Solution. Suppose $x \neq 1$; let $y = x(x-1)^{-1}$. Then $xy = y + x$, so that

$$f\left(\frac{x^2 - 2x}{1-x}\right) = f(y-x) \geq f(x+y) - f(xy) = 0 \ .$$

For each real z, the equation $z = (x^2 - 2x)(1-x)^{-1}$ is equivalent to $x^2 + (z-2)x - z = 0$. This quadratic equation (in x) has a discriminant $z^2 + 4$ which is positive and therefore it is solvable for all real z. (Note that in no case is $x = 1$ a solution.) Hence $f(z) \geq 0$ for all real z.

Comment. O. Bormashenko noted that when $2x = \sqrt{z^2+4} + 2 - z$ and $2y = \sqrt{z^2+4} + 2 + z$, then $xy = x + y$ and $y - x = z$.

167. Let $u = (\sqrt{5}-2)^{1/3} - (\sqrt{5}+2)^{1/3}$ and $v = (\sqrt{189}-8)^{1/3} - (\sqrt{189}+8)^{1/3}$. Prove that, for each positive integer n, $u^n + v^{n+1} = 0$.

Solution. Observe that, if $c = a - b$, then $c^3 = a^3 - b^3 - 3abc$, so that $c^3 + 3abc - (a^3 - b^3) = 0$. Applying this to u and v in place of c, we obtain that

$$0 = u^3 + 3u + 4 = (u-1)(u^2 - u + 4) = (u+1)\left[\left(u - \frac{1}{2}\right)^2 - \frac{15}{4}\right]$$

and

$$0 = v^3 + 15v + 16 = (v+1)(v^2 - v + 16) = (v+1)\left[\left(v - \frac{1}{2}\right)^2 - \frac{63}{4}\right] \ .$$

Since the expresseions in square brackets are negative, we must have that $u = v = -1$, from which the result follows.

Comment. Some students observed that

$$\left(\frac{\sqrt{5} \pm 1}{2}\right)^3 = \sqrt{5} \pm 2$$

and

$$\left(\frac{\sqrt{21} \pm 1}{2}\right)^3 = \sqrt{189} \pm 8 \ ,$$

and this immediately leads to the result.

168. Determine the value of

$$\cos 5^\circ + \cos 77^\circ + \cos 149^\circ + \cos 221^\circ + \cos 293^\circ \ .$$

Preliminary work. Before getting into the solution, we will discuss how to obtain the trigonometric ratios of certain angles related to 36°. It is useful for you to know some of these techniques, as these angles tend to come up in problems, and to be on the safe side in a contest, you should try to include a justification for assertions that you make about these angles. Here is one way to evaluate $t = \cos 36^\circ$. Observe that

$$\begin{aligned} t &= -\cos 144^\circ = 1 - 2\cos^2 72^\circ \\ &= 1 - 2(2t^2 - 1)^2 = -8t^4 + 8t^2 - 1 \end{aligned}$$

from which we see that

$$0 = 8t^4 - 8t^2 + t + 1 = (2t - 1)(t + 1)(4t^2 - 2t - 1) \ .$$

Since t is equal to neither 1 nor $\frac{1}{2}$, we must have that $4t^2 = 2t + 1$. Solving this equation will give you an actual numerical value (can you justify your choice of root?).

A very useful relation is $4\cos 36^\circ \cos 72^\circ = 1$. This can be checked geometrically. Let PQS be a triangle for which $\angle P = \angle S = 36^\circ$ and $\angle PQS = 108^\circ$. Let R be a point on the side PS for which $\angle PQR = 72^\circ$ and $\angle SQR = 36^\circ$. Then $PQ = PR$, $PQ = QS$ and $QR = RS$; let r be the common length of PQ, PR, QS and let s be the common length of QR and RS. Then $\cos 72^\circ = s/2r$ and $\cos 36^\circ = r/2s$ and the desired result follows. An algebraic derivation of this result can also be given.

$$\begin{aligned} 4\cos 36^\circ \cos 72^\circ &= \frac{4\sin 36^\circ \cos 36^\circ \cos 72^\circ}{\sin 36^\circ} \\ &= \frac{2\sin 72^\circ \cos 72^\circ}{\sin 36^\circ} \\ &= \frac{\sin 144^\circ}{\sin 36^\circ} = 1 \ . \end{aligned}$$

We can make use of complex numbers. Let $\zeta = \cos 72^\circ + i \sin 72^\circ$ so that ζ is a nonreal root of

$$0 = x^5 - 1 = (x - 1)(x^4 + x^3 + x^2 + x + 1) \ .$$

Hence $1 + \zeta + \zeta^2 + \zeta^3 + \zeta^4 = 0$; using de Moivre's theorem, and taking the real part of this equation, we find that

$$1 + \cos 72^\circ + \cos 144^\circ + \cos 216^\circ + \cos 288^\circ = 1 + 2\cos 72^\circ - 2\cos 36^\circ = 0 \ .$$

(Note that taking the imaginary part yields a triviality.) Another way to look at this result is to note that the vectors represented by the five roots of unity sum bound a closed regular pentagon and so sum to zero.

Solution 1. Let $\cos 36^\circ = t$. Then

$$\begin{aligned} &\cos 5^\circ + \cos 77^\circ + \cos 149^\circ + \cos 221^\circ + \cos 293^\circ \\ &\quad = [\cos 5^\circ + \cos 293^\circ] + [\cos 77^\circ + \cos 221^\circ] + \cos 149^\circ \\ &\quad = \cos 149^\circ [2\cos 144^\circ + 2\cos 72^\circ + 1] \\ &\quad = \cos 149^\circ [-2\cos 36^\circ + 2\cos 72^\circ + 1] = 0 \ . \end{aligned}$$

Alternatively, this is seen to be equal to

$$\cos 149^\circ [-2t + 2(2t^2 - 1) + 1] = \cos 149^\circ [-2t + 4t^2 - 1] = 0 \ .$$

Solution 2. [C. Huang]

$$\begin{aligned} &\cos 5^\circ + \cos 77^\circ + \cos 149^\circ + \cos 221^\circ + \cos 293^\circ \\ &\quad = \cos 5^\circ + 2\cos 185^\circ \cos 108^\circ + 2\cos 185^\circ \cos 36^\circ \\ &\quad = \cos 5^\circ [1 + 2(\cos 72^\circ - \cos 36^\circ)] = \cos 5^\circ [1 - 4\sin 18^\circ \sin 54^\circ] \\ &\quad = \cos 5^\circ [1 - 4\cos 72^\circ \cos 36^\circ] = 0 \ . \end{aligned}$$

Comment. There were other solutions with similar manipulations. A quick solution can also be realized using complex numbers. The given expression is equal to the real part of

$$(\cos\theta + i\sin\theta)(1 + \zeta + \zeta^2 + \zeta^3 + \zeta^4) = 0 \ ,$$

where ζ is an imaginary fifth root of unity and θ is the angle whose degree measure is $5°$.

169. Prove that, for each positive integer n exceeding 1,

$$\frac{1}{2^n} + \frac{1}{2^{1/n}} < 1 \ .$$

Solution 1. For positive integer $n > 1$ and $-1 < x$, $x \neq 0$, we have that $(1+x)^n > 1+nx$ (use induction). Hence, for $n \geq 2$,

$$\left(1 - \frac{1}{2^n}\right)^n > 1 - \frac{n}{2^n} \geq \frac{1}{2} \ .$$

(The latter inequality is left as an an easy induction exercise.) Hence

$$1 - \frac{1}{2^n} > \frac{1}{2^{1/n}}$$

and the result follows.

Solution 2. [R. Furmaniak] By the arithmetic-geometric means inequality, we have that

$$1 - \frac{1}{2^n} = \frac{1}{2} \cdot 1 + \frac{1}{4} \cdot 1 + \cdots + \frac{1}{2^{n-1}} \cdot 1 + \frac{1}{2^{n-1}} \cdot \frac{1}{2} > \left(\frac{1}{2}\right)^{1/(2^{n-1})} \ .$$

Now, $2^{n-1} = 1 + 1 + 2 + \cdots + 2^{n-2} \leq 1 + 1 + 1 + \cdots + 1 = n$, so that $1/(2^{n-1}) < 1/n$ and $(1/2)^{1/(2^{n-1})} > (1/2)^{1/n} = 1/(2^{1/n})$. The result follows.

Solution 3. [S. Wong] By the arithmetic-geometric means inequality

$$\frac{1}{2^{1/n}} = \left(\frac{1}{2} \cdot 1 \cdot 1 \cdots 1\right)^{1/n} < \frac{n - 1 + \frac{1}{2}}{n} = 1 - \frac{1}{2n}$$

$$\Longrightarrow \frac{1}{2^n} + \frac{1}{2^{1/n}} < 1 + \frac{1}{2^n} - \frac{1}{2n} \ .$$

When $n = 2$, $2n = 2^n$, while if $n \geq 3$,

$$2^n = (1+1)^n = 1 + n + \cdots + \binom{n}{n-1} + 1 > 2n$$

The desired inequality follows.

170. Solve, for real x,

$$x \cdot 2^{1/x} + \frac{1}{x} \cdot 2^x = 4 \ .$$

Solution 1. If $x < 0$, the left side is negative and so there is no negative value of x which satisfies the equation. If $x = 0$, then the left side is undefined. Suppose that $x > 0$. Then by the arithmetic-geometric means inequality, we have that

$$4 = x \cdot 2^{1/x} + \frac{1}{x} \cdot 2^x \geq 2\sqrt{2^{(1/x)+x}} \geq 2\sqrt{2^2} = 4 \ ,$$

since $(1/x)+x \geq 2$. Since the outside members of this inequality are equal, we must have equality everywhere, and this occurs if and only if $x = 1$. Hence, $x = 1$ is the sole solution of the equation.

Solution 2. As before, we see that x must be positive. The equation is equivalent to

$$\begin{aligned} 0 &= x^2 2^{1/x} - 4x + 2^x \\ &= (x \cdot 2^{1/(2x)} - 2 \cdot 2^{-1/(2x)})^2 - (4 \cdot 2^{-(1/x)} - 2^x) \; . \end{aligned}$$

Since the first member of the right side is nonnegative, we must have that $4 \cdot 2^{-1/x} \geq 2^x$, which is equivalent to $4 \geq 2^{x+(1/x)}$ or $x + (1/x) \leq 2$. But the last can occur if and only if $x = 1$. Indeed, $x = 1$ satisfies the equation.

171. Let n be a positive integer. In a round-robin match, n teams compete and each pair of teams plays exactly one game. At the end of the match, the ith team has x_i wins and y_i losses. There are no ties. Prove that

$$x_1^2 + x_2^2 + \cdots + x_n^2 = y_1^2 + y_2^2 + \cdots + y_n^2 \; .$$

Solution 1. Each game results in both a win and a loss, so the total number of wins is equal to the total number of losses. Thus $\sum x_i = \sum y_i$. For each team, the total number of its wins and losses is equal to the number of games it plays. Thus $x_i + y_i = n - 1$ for each i. Accordingly,

$$0 = (n-1)\sum_{i=1}^{n}(x_i - y_i) = \sum_{i=1}^{n}(x_i + y_i)(x_i - y_i) = \sum_{i=1}^{n}(x_i^2 - y_i^2)$$

from which the desired result follows.

Solution 2. Since $x_i + y_i = n - 1$ for each i and $x_1 + x_2 + \cdots + x_n = y_1 + y_2 + \cdots + y_n = \binom{n}{2}$ (the number of games played), we find that

$$\begin{aligned} \sum_{i=1}^{n} x_i^2 &= \sum_{i=1}^{n}[(n-1) - y_i]^2 \\ &= \sum_{i=1}^{n} y_i^2 - 2(n-1)\sum_{i=1}^{n} y_i + \sum_{i=1}^{n}(n-1)^2 \\ &= \sum_{i=1}^{n} y_i^2 - n(n-1)^2 - n(n-1)^2 = \sum_{i=1}^{n} y_i^2 \; . \end{aligned}$$

Solution 3. [J. Zhao] Consider the special case where team i vanquishes team j if and only if $i < j$. The the sum of the squares of the x_i and the sum of the squares of the y_i are both equal to $1^2 + 2^2 + \cdots + (n-1)^2$. The general situation can be obtained from this special case by a finite sequence of switches of outcomes of single games. Suppose that we know for some configuration of results that $x_1^2 + \cdots + x_n^2 = y_1^2 + \cdots + y_n^2$. Then if the results of the (i,j)th game are swtiched, so that, say, team i wins $x_i + 1$ and loses $y_i - 1$ games and team j wins $x_j - 1$ and loses $y_j + 1$ games, then, since $x_i - x_j = (n - 1 - y_i) - (n - 1 - y_j) = y_j - y_i$,

$$\begin{aligned} & x_1^2 + \cdots + (x_i + 1)^2 + \cdots + (x_j - 1)^2 + \cdots + x_n^2 \\ &\quad = (x_1^2 + \cdots + x_n^2) + 2(x_i - x_j) + 2 \\ &\quad = (y_1^2 + \cdots + y_n^2) + 2(y_j - y_i) + 2 \\ &\quad = y_1^2 + \cdots + (y_i - 1)^2 + \cdots + (y_j + 1)^2 + \cdots + y_n^2 \; . \end{aligned}$$

The result now follows by induction on the number of switches required to obtain the general case from the special case.

172. Let a, b, c, d. e, f be different integers. Prove that

$$(a-b)^2 + (b-c)^2 + (c-d)^2 + (d-e)^2 + (e-f)^2 + (f-a)^2 \geq 18 \ .$$

Solution 1. Since the sum of the differences is 0, an even number, there must be an even number of odd differences, and therefore an even number of odd squares. If the sum of the squares is less than 18, then this sum must be one of the numbers 6, 8, 10, 12, 14, 16. The only possibilities for expressing any of these numbers as the sum of six nonzero squares is

$$6 = 1^2 + 1^2 + 1^2 + 1^2 + 1^2 + 1^2$$

$$12 = 2^2 + 2^2 + 1^2 + 1^2 + 1^2 + 1^2$$

$$14 = 3^3 + 1^2 + 1^2 + 1^2 + 1^2 + 1^2 \ .$$

Taking note that the sum of the differences is zero, the possible sets of differences (up to order and sign) are $\{1, 1, 1, -1, -1, -1\}$, $\{3, 1, -1, -1, -1, -1\}$, $\{2, 2, -1, -1, -1, -1\}$. Since the numbers are distinct, the difference between the largest and smallest is at least 5. This difference must be the sum of differences between adjacent numbers; but checking proves that in each case, an addition of adjacent differences must be less than 5. Hence, it is not possible to achieve a sum of squares less than 18. The sum 18 can be found with the set $\{0, 1, 3, 5, 4, 2\}$.

Solution 2. [A. Critch] We prove a more general result. Let n be a positive integer exceeding 1. Let $(t_1, t_2, \cdots, t_{n-1}, t_n)$ be an $n-$tple of distinct integers, and suppose that the smallest of these is t_n. Define $t_0 = t_n$, and wolog suppose that $t_0 = t_n = 0$. Suppose that, for $1 \leq i \leq n$, $s_i = |t_i - t_{i-1}|$. Let the largest integer be t_r; since the integers are distinct, we must have

$$\begin{aligned} n - 1 \leq t_r = t_r - t_0 &= |t_r - t_0| \\ &\leq |t_1 - t_0| + \cdots + |t_r - t_{r-1}| \\ &= s_1 + s_2 + \cdots + s_r \end{aligned}$$

and

$$\begin{aligned} n - 1 \leq t_r - t_0 &= |t_n - t_r| \\ &\leq |t_{r+1} - t_r| + \cdots + |t_n - t_{n-1}| = s_{r+1} + \cdots + s_n \ . \end{aligned}$$

Hence.

$$s_1 + s_2 + \cdots + s_n \geq 2n - 2 \ .$$

By the root-mean-square, arithmetic mean (RMS-AM) inequality, we have that

$$\left(\frac{s_1^2 + s_2^2 + \cdots + s_n^2}{n}\right)^{1/2} \geq \frac{s_1 + s_2 + \cdots + s_n}{n} \geq \frac{2n-2}{n} \ ,$$

so that

$$s_1^2 + s_2^2 + \cdots + s_n^2 \geq \frac{4n^2 - 8n + 4}{n} = 4n - 8 + \frac{4}{n} \ .$$

Thus,

$$\sum_{i=1}^n s_i^2 \geq 4n - 8 + \lceil 4/n \rceil \ .$$

Since the sum of the differences of consecutive t_i is zero, and so even, the sum of the squares is even. Since $4n - 8$ is even and $n \geq 2$, and since the sum exceeds $4n - 8$, we see that

$$\sum_{i=1}^n s_i^2 \geq 4n - 8 + 2 = 4n - 6 \ .$$

How can this lower bound be achieved? Since it is equal to $2^2(n-1)+1^2+1^2$, we can have $n-2$ differences equal to 2 and 2 differences equal to 1. Thus, we can start by going up the odd integers, and then come down via the even integers to 0. In the case of $n=6$, this yields the 6−tple $(1,3,5,4,2,0)$.

Solution 3. [R. Barrington Leigh] Let a and m be the minimum and the maximum of the numbers a,b,c,d,e,f. Since the six numbers are distinct, $m-a\geq 5$. Thus,

$$10\leq 2(m-a)=|a-m|+|m-a|\leq |a-b|+|b-c|+\cdots+|f-a|\ .$$

Observe that, for every integer t, we have that $t^2-3t+2=(t-2)(t-1)\geq 0$, with equality if and only if $t=1$ or $t=2$. It follows that

$$\begin{aligned}(a-b)^2+(b-c)^2+(c-d)^2+(d-e)^2+(e-f)^2+(f-a)^2\\ &\geq (3|a-b|-2)+(3|b-c|-2)+(3|c-d|-2)+(3|d-e|-2)+\\ &\qquad (3|e-f|-2)+(3|f-a|-2)\\ &=3(|a-b|+|b-c|+\cdots+|f-a|)-12\geq 30-12=18\ ,\end{aligned}$$

with equality if and only if all consecutive pairs differ by 1 or 2. This can actually occur, with $(a,b,c,d,e,f)=(0,1,3,5,4,2)$.

Solution 4. [M. Guay-Paquet] Since we want the sum to be as small as possible, we can assume that the integers are consecutive and that the smallest one is 0. For each positive integer n, let $f(n)$ be the smallest sum $\sum_{i=0}^{n-1}(a_{i+1}-a_i)^2$ (with $a_n=a_0$), where the $(a_0,\cdots,a_{n-1})$ is a permutation of $(0,1,\cdots,n-1)$. Suppose that $(a_0,a_1,\cdots,a_m)$ is a finite sequence that realizes the sum $f(m+1)$ and that $a_j=m$. Then

$$\begin{aligned}f(m)&\leq (a_1-a_0)^2+\cdots+(a_{j-1}-a_{j-2})^2+(a_{j+1}-a_{j-1})^2+\cdots+(a_0-a_m)^2\\ &=\sum_{i=0}^{m}(a_{i+1}-a_i)^2+[(a_{j+1}-a_{j-1})^2-(n-a_{j-1})^2-(a_{j+1}-n)^2]\\ &=f(m+1)-2[(n-a_{j-1})(n-a_{j+1})]\leq f(m+1)-4\ ,\end{aligned}$$

since one of $n-a_{j-1}$ and $n-a_{j+1}$ is at least 1 and the other at least 2. Hence $f(m+1)\geq f(m)+4$ for every postive integer $n\geq 2$. In particular, $f(6)\geq f(5)+4\geq f(4)+8\geq f(3)+12\geq f(2)+16$. Since, $f(2)=2$, the result follows. Note that equality can occur as in the previous solutions.

Solution 5. Suppose that we have selected six distinct integers for which the sum of the squares is minimum. Wolog, we may assume that the consecutive integers $0,1,2,3,4,5$ are involved, that $a=0$ and $b<f$. [If the integers are not consecutive, we can subtract from the larger integers to fill in the gaps and decrease the differences. Since only differences are involved, we may assume that the smallest number is 0. If $b>f$, we can reverse the order of the numbers to get the same sum of squares.] Since the case $(a,b,c,d,e,f)=(0,1,3,5,4,2)$ yields a square sum of 18, we need only consider choices that yield sums at most equal to 18. Thus, we can exclude cases where any consecutive pair differ by at least 4, or at least two consecutive pairs differ by at least 3. Thus, both b and f cannot exceed 3. Now we examine cases:

Suppose that $f=3$. If $b=2$, then one of c,d,e is equal to 1 and the other two equal to 4 and 5; all possibilities yield, along with (f,a) a second pair of consecutive numbers with difference at least 3. Thus, we must have $b=1$. Since $f-a=3$, we must have $c-b\leq 2$, so that $c=2$ and $d=4$. Thus, $(a,b,c,d,e,f)=(0,1,2,4,5,3)$, which yields a square sum of 20.

The only remaining possibility is that $b=1$ and $f=2$. Then c is equal to either 3 or 4, and we have the possibilities $(a,b,c,d,e,f)=(0,1,3,4,5,2)$, $(0,1,3,5,4,2)$, $(0,1,4,3,5,2)$, $(0,1,4,5,3,2)$, yielding the respective square sums 20, 18, 28, 20. Hence the minimum possible square sum is 18.

173. Suppose that a and b are positive real numbers for which $a+b=1$. Prove that

$$\left(a+\frac{1}{a}\right)^2+\left(b+\frac{1}{b}\right)^2\geq\frac{25}{2}\ .$$

Determine when equality holds.

Remark. Before starting, we note that when $a+b=1$, $a,b>0$, then $ab \leq \frac{1}{4}$. This is an immediate consequence of the arithmetic-geometric means inequality.

Solution 1. By the root-mean-square, arithmetic mean inequality, we have that

$$\begin{aligned}\frac{1}{2}\left[\left(a+\frac{1}{a}\right)^2+\left(b+\frac{1}{b}\right)^2\right] &\geq \frac{1}{4}\left[\left(a+\frac{1}{a}\right)+\left(b+\frac{1}{b}\right)\right]^2 \\ &= \frac{1}{4}\left(1+\frac{1}{a}+\frac{1}{b}\right)^2 = \frac{1}{4}\left(1+\frac{1}{ab}\right)^2 \geq \frac{1}{4}\cdot 5^2\end{aligned}$$

as desired.

Solution 2. By the RMS-AM inequality and the harmonic-arithmetic means inequality, we have that

$$\begin{aligned}a^2+b^2+(1/a)^2+(1/b)^2 &\geq \frac{1}{2}(a+b)^2+\frac{1}{2}\left(\frac{1}{a}+\frac{1}{b}\right)^2 \\ &= \frac{1}{2}+2\left[\frac{(1/a)+(1/b)}{2}\right]^2 \\ &\geq \frac{1}{2}+2\cdot\frac{4}{(a+b)^2}=\frac{17}{2}\ ,\end{aligned}$$

from which the result follows.

Solution 3.

$$\begin{aligned}&\left(a+\frac{1}{a}\right)^2+\left(b+\frac{1}{b}\right)^2 \\ &\quad= a^2+b^2+\frac{a^2+b^2}{a^2b^2}+4 \\ &\quad= (a+b)^2-2ab+\frac{(a+b)^2-2ab}{a^2b^2}+4 \\ &\quad= 5-2ab+\frac{1}{(ab)^2}-\frac{2}{ab} \\ &\quad= 4-2ab+\left(\frac{1}{ab}-1\right)^2 \geq 4-2\left(\frac{1}{4}\right)+(4-1)^3=\frac{25}{2}\ .\end{aligned}$$

Comment. The left side can also be manipulated to $(1-2ab)(1+(ab)^{-2})+4 \geq (1-\frac{1}{2})(1+4^2)+4 = 25/2$.

Solution 4. [F. Feng] From the Cauchy-Schwarz and arithmetic-geometric means inequalities, we find that

$$\begin{aligned}&\left[\left(a+\frac{1}{a}\right)^2+\left(b+\frac{1}{b}\right)^2\right][1^2+1^2] \\ &\quad= \left[\left(a+\frac{1}{a}\right)^2+\left(b+\frac{1}{b}\right)^2\right][(a+b)^2+(a+b)^2] \\ &\quad\geq \left[\left(a+\frac{1}{a}\right)(a+b)+\left(b+\frac{1}{b}\right)(a+b)\right]^2 \\ &\quad= \left[(a+b)^2+2+\left(\frac{a}{b}+\frac{b}{a}\right)\right]^2 \\ &\quad\geq [1+2+2]^2 = 25\ .\end{aligned}$$

The desired result follows.

Solution 5. [K. Kawaji] First, we note that the Cauchy-Schwarz Inequality applied to the vectors $(u_i/\sqrt{v_i})$ and $(\sqrt{v_i})$ yields the result

$$\sum_{i=1}^{m} \frac{u_i^2}{v_i} \geq \frac{(\sum_{i=1}^m u_i)^2}{\sum_{i=1}^m v_i} \ .$$

This will be employed several times in the following.

$$\begin{aligned}\left(a+\frac{1}{a}\right)^2 + \left(b+\frac{1}{b}\right)^2 &\geq \frac{(a+(1/a)+b+(1/b))^2}{1+1} = \frac{1}{2}\left(1+\frac{1}{a}+\frac{1}{b}\right)^2 \\ &= \frac{1}{2}\left[1+2\left(\frac{1}{a}+\frac{1}{b}\right)+\left(\frac{1}{a^2}+\frac{1}{ab}+\frac{1}{ab}+\frac{1}{b^2}\right)\right] \\ &\geq \frac{1}{2}\left[1+2\left(\frac{4}{a+b}\right)+\left(\frac{4^2}{(a+b)^2}\right)\right] \\ &= \frac{1}{2}[1+8+16] = \frac{25}{2} \ ,\end{aligned}$$

with equality if and only if $a = b = 2$.

174. For which real value of x is the function

$$(1-x)^5(1+x)(1+2x)^2$$

maximum? Determine its maximum value.

Solution 1. The function assumes negative values when $x < -1$ and $x > 1$. Accordingly, we need only consider its values on the interval $[-1, 1]$. Suppose, first, that $-1/2 \leq x \leq 1$, in which case all factors of the function are nonnegative. Then we can note, by the arithmetic-geometric means inequality, that

$$(1-x)^5(1+x)(1+2x) \leq [(5/8)(1-x)+(1/8)(1+x)+(2/8)(1+2x)]^8 = 1$$

with equality if and only if $x = 0$. Thus, on the interval $[-1/2, 1]$, the function takes its maximum value of 1 when $x = 0$.

We adopt the same strategy to consider the situation when $-1 \leq x \leq -1/2$. For convenience, let $u = -x$, so that the want to maximize

$$(1+u)^5(1-u)(2u-1)^2$$

for $1/2 \leq u \leq 1$. In fact, we are going to maximize

$$[\alpha(1+u)]^5[\beta(1-u)][\gamma(2u-1)]^2$$

where α, β and γ are positive integers to be chosen to make $5\alpha - \beta + 4\gamma = 0$ (so that when we calculate the airthmetic mean of the eight factors, the coefficient of u will vanish), and $\alpha(1+u) = \beta(1-u) = \gamma(2u-1)$ (so that we can actually find a case where equality will occur). The latter conditions forces

$$\frac{\beta-\alpha}{\beta+\alpha} = \frac{\beta+\gamma}{2\gamma+\beta}$$

or $\beta\gamma = 3\alpha\gamma + 2\alpha\beta$. Plugging $\beta = 5\alpha + 4\gamma$ into this yields that

$$0 = 2(3\alpha\gamma + 5\alpha^2 - 2\gamma^2) = 2(5\alpha - 2\gamma)(\alpha + \gamma) \ .$$

Thus, let us take $(\alpha, \beta, \gamma) = (2, 30, 5)$. Then, by the arithmetic-geometric means inequality, we obtain that

$$\begin{aligned}&[2(1+u)]^{5/8}[30(1-u)]^{1/8}[5(2u-1)]^{1/4} \\ &\qquad\leq \frac{5}{4}(1+u) + \frac{15}{4}(1-u) + \frac{5}{4}(2u-1) = \frac{15}{4} \ ,\end{aligned}$$

with equality if and only if $2(1+u) = 30(1-u) = 5(2u-1)$, *i.e.*, $u = 7/8$. Hence,

$$\begin{aligned}(1-x)^5(1+x)(1+2x)^5 &= (1+u)^5(1-u)(1-2u)^2 \\ &\leq \left(\frac{15}{8}\right)^8 2^{-5/8}30^{-1}5^{-2} \\ &= \frac{3^7 \times 5^5}{2^{22}} = \left(\frac{15}{16}\right)^5 \left(\frac{9}{4}\right) ,\end{aligned}$$

with equality if and only if $x = -7/8$.

It remains to show whether the value of the function when $x = -7/8$ exceeds its value when $x = 0$. Recall the Bernoulli inequality: $(1+x)^n > 1 + nx$ for $-1 < x, x \neq 0$ and positive integer n. This can be established by induction (do it!). Using this, we find that

$$\left(\frac{15}{16}\right)^5 \left(\frac{9}{4}\right) > \left(1 - \frac{5}{16}\right) \times 2 = \frac{22}{16} > 1 \ .$$

Thus, the function assumes its maximum value of $3^7 \times 5^5 \times 2^{-22}$ when $x = -7/8$.

Solution 2. Let $f(x) = (1-x)^5(1+x)(1+2x)^2$. Then $f'(x) = -2(1-x)^4(1+2x)x(7+8x)$. We see that $f'(x) > 0$ if and only if $x < -7/8$ or $-1/2 < x < 0$. Hence $f(x)$ has relative maxima only when $x = -7/8$ and $x = 0$. Checking these two candidates tells us that the absolute maximum of $3^7 \times 5^5 \times 2^{-22}$ occurs when $x = -7/8$.

*Comments.*In checking for the maximum, one must be careful to exclude the possibility that the function might become positively unbounded at $x \to \pm\infty$. This can be done either by noting that the leading coefficients is negative while the degree is positive, or by analyzing the sign of the first derivative. O. Bormashenko made the interesting observation that, for $-1/2 \leq x < 1$ and $x \neq 0$,

$$\begin{aligned}f(x^2) &= (1-x^2)^5(1+x^2)(1+2x^2)^2 \\ &= (1+x)^5(1-x)^5(1+x^2)(1+2x^2)^2 \\ &> (1+2x+x^2)^2(1+x)(1-x)^5 \cdot 1 \cdot 1 \\ &> (1+2x)^2(1-x)^5(1+x) = f(x) \ ,\end{aligned}$$

from which we see that the maximum on $[-1/2, 1]$ can occur only at $x = 0$.

175. ABC is a triangle such that $AB < AC$. The point D is the midpoint of the arc with endpoints B and C of that arc of the circumcircle of ΔABC that contains A. The foot of the perpendicular from D to AC is E. Prove that $AB + AE = EC$.

Solution 1. Draw a line through D parallel to AC that intersects the circumcircle again at F. Let G be the foot of the perpendicular from F to AC. Then $DEGF$ is a rectangle. Since arc BD is equal to arc DC, and since arc AD is equal to arc FC (why?), arc BA is equal to arc DF. Therefore the chords of these arcs are equal, so that $AB = DF = EG$. Hence $AB + AE = EG + GC = EC$.

Solution 2. Note that the length of the shorter arc AD is less than the length of the shorter arc DC. Locate a point H on the chord AC so that $AD = HD$. Consider triangles ABD and HCD. We have that $AD = DH$, $BD = CD$ and $\angle ABD = \angle HCD$. This is a case of SSA congruence, the ambiguous case. Since angles BAD and DHC are both obtuse (why?), they must be equal rather the supplementary, and the triangles ABD and HCD are congruent. (Congruence can also be established using the Law of Sines.) In particular, $AB = HC$. Since triangle ADH is isosceles, E is the midpoint of AH, so that $AB + AE = HC + EH = EC$.

Solution 3. Let DM be a diameter of the circumcircle, so that M is the midpoint of one of the arcs BC. Let H be that point on the chord AC for which $DA = DH$ and let DH be produced to meet the circle again in K. Since $\angle MAD = \angle DEA = 90°$, it follows that $\angle MAC = 90° - \angle EAD = \angle ADE$. Since AM and DE are both angle bisectors, $\angle BAC = \angle ADH$.

Because $ADCK$ is concyclic, triangles ADH and KCH are similar, so that $HC = CK$. From the equality of angles BAC and ACK, we deduce the equality of the arcs BAC and ACK, and so the equality of the arcs BA and CK. Hence $AB = CK = HC$. Therefore $AB + AE = HC + EH = EC$.

Solution 4. [R. Shapiro] Since A lies on the short arc BD, $\angle BAD$ is obtuse. Hence the foot of the perpendicular from D to BA produced is outside of the circumcircle of triangle ABC. In triangles KBD and ECD, $\angle BKD = \angle DEC = 90°$, $\angle KBD = \angle ABD = \angle ACD$ and $BD = CD$. Hence the triangles KBD and ECD are congruent, so that $DK = DE$ and $BK = EC$. Since the triangle ADK and ADE are right with a common hypotenuse AD and equal legs DK and DE, they are congruent and $AK = AE$. Hence $EC = BK = BA + AK = BA + AE$, as desired.

Solution 5. [F. Wang] Produce BA to P so that $AP = AB$. Since $BADC$ is concyclic, $\angle BDC = \angle BAC = \angle BPA + \angle BAP = 2\angle BPC$. In the circumcircle of ΔBPC, D is a point on the right bisector of BC at which BC subtends an angle double that at the circumference (at P). Hence D is the centre of the circumcircle of BPC. Therefore, the perpendicular DE to AC must bisect the chord PC, and we find that $EC = EP = AE + AP = AE + AB$, as desired.

176. Three noncollinear points A, M and N are given in the plane. Construct the square such that one of its vertices is the point A, and the two sides which do not contain this vertex are on the lines through M and N respectively. [*Note:* In such a problem, your solution should consist of a description of the construction (with straightedge and compasses) and a proof in correct logical order proceeding from what is given to what is desired that the construction is valid. You should deal with the feasibility of the construction.]

Solution 1. `Construction.` Draw the circle with diameter MN and centre O. This circle must contain the point C, as MC and NC are to be perpendicular. Let the right bisector of MN meet the circle in K and L. Join AK and, if necessary, produce it to meet the circle at C. Now draw the circle with diameter AC and let it meet the right bisector of AC at B and D. Then $ABCD$ is the required rectangle. There are two options, depending how we label the right bisector KL.

However, the construction does not work if A actually lies on the circle with diameter MN. In this case, A and C would coincide and the situation degenerates. If A lies on the right bisector of MN, then C can be the other point where the right bisector intersects the circle, and M and N can be the other two vertices of the square. If A is not on the right bisector, then there is no square; all of the points A, M, C, N would have to be on the circle, and AM and AN would have to subtend angles of 45° at C, which is not possible.

`Proof.` If C and O are on the same side of KN, then $\angle KCN = \frac{1}{2}\angle KON = 45°$, so that CN makes an angle of 45° with AC produced, and so CN produced contains a side of the square. Similarly, CM produced contains a side of the square. If C and O are on opposite sides of KN, then $\angle KCN = 135°$, and CN still makes an angle of 45° with AC produced; the argument can be completed as before.

Solution 2. `Construction.` Construct circle of diameter MN. Draw $AM = MR$ (with the segment MR intersecting the interior of the circle) and $AM \perp MR$. Construct the circle AMR. Let this circle intersect the given circle at C. Then construct the square with diagonal AC. If A lies on the circle, then the candidates for C are A and M. We cannot take $C = A$, as the situation degererates; if we take $C = M$, then the angle ACM and segment CM degenerate. We can complete the analysis as in the first solution.

`Proof.` Since ΔARM is right isosceles, $\angle ARM = 45°$. Hence the circle is the locus of points at which AM subtends an angle equal to 45° or 135°. Hence the lines AC and CM intersect at an angle of 45°. Since $\angle MCN = 90°$, the lines AC and CN also intersect at 45°. It follows that the remaining points on the square with diagonal AC must lie on the lines CM and CN.

177. Let a_1, a_2, $\cdots$, a_n be nonnegative integers such that, whenever $1 \leq i$, $1 \leq j$, $i+j \leq n$, then

$$a_i + a_j \leq a_{i+j} \leq a_i + a_j + 1 \ .$$

(a) Give an example of such a sequence which is not an arithmetic progression.

(b) Prove that there exists a real number x such that $a_k = \lfloor kx \rfloor$ for $1 \leq k \leq n$.

(a) *Solution.* [R. Marinov] For positive integers n, let $a_n = k-1$ when $n = 2k$ and $a_n = k$ when $n = 2k+1$, so that the sequence is $\{0, 1, 1, 2, 2, 3, 3, \cdots\}$. Observe that

$$a_{(2p+1)+(2q+1)} = a_{2(p+q+1)} = p + q = a_{2p+1} + a_{2q+1}$$

$$a_{2p+2q} = a_{2(p+q)} = p + q - 1 = (p-1) + (q-1) + 1 = a_{2p} + a_{2q} + 1$$

and

$$a_{2p+(2q+1)} = a_{2(p+q)+1} = p + q = (p-1) + q + 1 = a_{2p} + a_{2q+1}$$

for positive integers p and q, whence we see that this sequence satisfies the condition. The corresponding value of x is $1/2$.

Solution 1. [A. Critch] The assertion to be proved is that all the semi-closed intervals $[a_k/k, a_{k+1}/k)$ have a point in common. Suppose, if possible, that this fails. Then there must be a pair (p, q) of necessarily distinct integers for which $a_q/q \geq (a_p + 1)/p$. This is equivalent to $pa_q \geq qa_p + q$. Suppose that the sum of these two indices is as small as possible.

Suppose that $p > q$, so that $p = q + r$ for some positive r. Then

$$(q+r)a_q \geq qa_p + q = qa_{q+r} + q \geq qa_q + qa_r + q$$

whence $ra_q \geq qa_r + q$ and $a_q/q \geq (a_r + 1)/r$. Thus p and r have the property of p and q and we get a contradiction of the minimality condition.

Suppose that $p < q$, so that $q = p + s$ for some positive s. Then

$$p + pa_p + pa_s \geq pa_{p+s} \geq (p+s)a_p + q = pa_p + sa_p + q$$

so that $p + pa_s \geq sa_p + q > sa_p + p + s$, $pa_s \geq sa_p + s$ and $a_s/s \geq (a_p + 1)/p$, once again contradicting the minimality condition.

Solution 2. [O. Bormashenko] Define $a_0 = 0$. Let k be selected so that $a_i/i \leq a_k/k$ for $1 \leq i \leq n$, with strict inequality for $i < k$. This is equaivalent to $ka_i \leq ia_k$. We wish to prove that

$$a_i \leq \frac{ia_k}{k} < a_i + 1 \ ,$$

from which we see that $a = a_k/k$ will satisfy the requirement.

Suppose that $i < k$, so that $k = ir + s$ for $0 \leq s < i$. Repeated application of the given condition yields that

$$\begin{aligned} a_k &\leq a_{ir} + a_s + 1 \leq a_i + a_{i(r-1)} + a_s + 2 \leq \cdots \leq ra_i + a_s + r \\ &< ra_i + (s/k)a_k + r = ra_i + (1 - (ir/k))a_k + r \ , \end{aligned}$$

whence $(ir/k)a_k < ra_i + r$, so that $ia_k/k < a_i + 1$.

Suppose that $k < i$, so that $i = ku + v$ for $0 \leq v < k$. Then

$$\begin{aligned} a_i &\geq a_k + a_{i-k} \geq a_k + a_k + a_{i-2k} \geq \cdots \geq ua_k + a_v \\ &> ua_k + (v/k)a_k - 1 = (i/k)a_k - 1 \ , \end{aligned}$$

so that $a_i + 1 > (i/k)a_k$, as desired. The result follows.

178. Suppose that n is a positive integer and that $x_1, x_2, \cdots, x_n$ are positive real numbers such that $x_1 + x_2 + \cdots + x_n = n$. Prove that

$$\sum_{i=1}^{n} \sqrt[n]{ax_i + b} \leq a + b + n - 1$$

for every pair a, b of real numbers with each $ax_i + b$ nonnegative. Describe the situation when equality occurs.

Solution. Regarding $ax_i + b$ as a product with $n - 1$ ones, we use the arithmetic-geometric means inequality to obtain that

$$\sqrt[n]{ax_i + b} \leq \frac{(ax_i + b) + 1 + \cdots + 1}{n}$$

for $1 \leq i \leq n$, with equality if and only if $x_i = (1-b)/a$. Adding these n inequalities yields the desired result. Since the sum of the x_i is equal to n, the condition for equality is that $x_i = 1$ $(1 \leq i \leq n)$ and $a + b = 1$.

179. Determine the units digit of the numbers a^2, b^2 and ab (in base 10 numeration), where

$$a = 2^{2002} + 3^{2002} + 4^{2002} + 5^{2002}$$

and

$$b = 3^1 + 3^2 + 3^3 + \cdots + 3^{2002} \ .$$

Solution. Observe that, for positive integer k, $2^{4k} \equiv 6$ and $3^{4k} \equiv 1$, modulo 10, so that $2^{2002} \equiv 6 \cdot 4 \equiv 4$, $3^{2002} \equiv 9$ and $4^{2002} \equiv 6$, modulo 10. Hence $a \equiv 4 + 9 + 6 + 5 \equiv 4$ and $a^2 \equiv 6$, modulo 10. Note that $b = (1/2)(3^{2003} - 3)$, and that $3^{2003} - 3 \equiv 7 - 3 = 4$, modulo 10. Since b is the sum of evenly many factors, it is even, and so $b \equiv 2$ and $b^2 \equiv 4$, modulo 10. Finally, $ab \equiv 4 \cdot 2 = 8$, modulo 10. Hence the units digits of a^2, b^2 and ab are respectively 6, 4 and 8.

180. Consider the function f that takes the set of complex numbers into itself defined by $f(z) = 3z + |z|$. Prove that f is a bijection and find its inverse.

Solution 1. Injection (one-one). Suppose that $z = x + yi$ and $w = u + vi$, and that $f(z) = f(w)$. Then

$$3x + 3yi + \sqrt{x^2 + y^2} = 3u + 3vi + \sqrt{u^2 + v^2} \ .$$

Equating imaginary parts yields that $y = v$, so that

$$3(x - u) = \sqrt{u^2 + y^2} - \sqrt{x^2 + y^2} = (u^2 - x^2)/(\sqrt{u^2 + y^2} + \sqrt{x^2 + y^2}) \ .$$

Suppose, if possible, that $u \neq x$. Then

$$3(\sqrt{x^2 + y^2} + x) = -[3(\sqrt{u^2 + y^2} + u] \ .$$

Since $\sqrt{x^2 + y^2} \geq |x|$, and $\sqrt{u^2 + y^2} \geq |u|$, we see that, unless $x = y = u = v = 0$, this equation is impossible as the left side is positive and the right is negative Thus, $x = u$.

Surjection (onto). Let $a + bi$ be an arbitrary complex number, and suppose that $f(x + yi) = a + bi$. It is straightforward to see that $f(z) = 0$ implies that $z = 0$, so we may assume that $a^2 + b^2 > 0$. We must have that

$$3x + \sqrt{x^2 + y^2} = a$$

and

$$3y = b \ .$$

Substituting $y = b/3$ into the first equation yields

$$\sqrt{9x^2 + b^2} = 3a - 9x \ .$$

For this equation to be solvable, it is necessary that $3x \leq a$. Squaring both sides of the equation leads to

$$72x^2 - 54ax + 9a^2 - b^2 = 0 \ .$$

When $x = a/3$ is substituted into the left side of the equation, we obtain $8a^2 - 18a^2 + 9a^2 - b^2 = -(a^2 + b^2) < 0$. This means that the two roots of the equation straddle $a/3$, so that exactly one of the roots satisfies the necessary condition $3x \leq a$. Hence, we must have

$$(x, y) = \left(\frac{9a - \sqrt{9a^2 + 8b^2}}{24}, \frac{b}{3} \right) \ .$$

Thus, the function is injective and surjective, and so it is a bijection.

Solution 2. [R. Barrington Leigh] We have that

$$f(x + yi) = (3x + \sqrt{x^2 + y^2}) + 3yi$$

for all real pairs (x, y). For each real pair (x, y), let

$$g(x, y) = \frac{3x - \sqrt{x^2 + (8y^2/9)}}{8} + \frac{1}{3}yi \ .$$

Then

$$\begin{aligned} f(g(x + yi)) &= \frac{1}{8}\left(9x - \sqrt{9x^2 + 8y^2} + \sqrt{9x^2 + 8y^2 - 2x\sqrt{9x^2 + 8y^2} + x^2} \right) + yi \\ &= \frac{1}{8}[(9x - \sqrt{9x^2 + 8y^2}) + (\sqrt{9x^2 + 8y^2} - x)] + yi = x + yi \ , \end{aligned}$$

and

$$\begin{aligned} g(f(x, y)) &= \frac{1}{8}\left(9x + 3\sqrt{x^2 + y^2} - \sqrt{9x^2 + 9y^2 + 6x\sqrt{x^2 + y^2} + x^2} \right) + yi \\ &= \frac{1}{8}[(9x + 3\sqrt{x^2 + y^2}) - (3\sqrt{x^2 + y^2} + x)] + yi = x + yi \ . \end{aligned}$$

Hence, g is the inverse of f, so that $f(u) = f(v) \Leftrightarrow u = g(f(u)) = g(f(v)) = v$ and $f(z) = w$ is satisfied by $z = g(w)$. The result follows.

181. Consider a regular polygon with n sides, each of length a, and an interior point located at distances a_1, a_2, $\cdots$, a_n from the sides. Prove that

$$a \sum_{i=1}^{n} \frac{1}{a_i} > 2\pi \ .$$

Solution 1. By constructing triangles from bases along the sides of the polygons to the point A, we see that the area of the polygon is equal to

$$\frac{aa_1}{2} + \frac{aa_2}{2} + \cdots + \frac{aa_n}{2} = \frac{a}{2}\sum_{i=1}^{n} a_i \ .$$

However, by constructing triangles whose bases are the sides of the polygons and whose apexes are at the centre of the polygon, we see that the area of the polygon is equal to $\frac{1}{4}na^2 \cot(\pi/n)$. Making use of the arithmetic-harmonic means inequality, we find that

$$\frac{a}{2} \cot \frac{\pi}{n} = \frac{1}{n} \sum_{i=1}^{n} a_i \geq \frac{n}{\sum_{i=1}^{n} 1/a_i} \ ,$$

from which

$$\sum_{i=1}^{n} \frac{1}{a_i} \geq \frac{2n \cdot \tan(\pi/n)}{a} \ .$$

Since $\tan x > x$ for $0 < x < \pi/2$, we have that $\tan(\pi/n) > (\pi/n)$, we obtain that

$$\sum_{i=1}^{n} \frac{1}{a_i} > \frac{2\pi}{a} \ .$$

Solution 2. Let r be the radius of the inscribed circle of the polygon. Then the area of the polygon is equal to both $(a/2)\sum_{i=1}^{n} a_i$ and $na/2$. Hence

$$\frac{1}{n}\sum_{i=1}^{n} a_i = r \ .$$

By the harmonic-arithmetic means inequality

$$n\left(\sum_{i=1}^{n} a_i^{-1}\right)^{-1} \leq \frac{1}{n}\sum_{i=1}^{n} a_i \ .$$

Since the perimeter of the polygon exceeds that of its inscribed circle, $2\pi r < na$. Putting these three facts together yields the result.

Solution 3. [O. Ivrii] Let S be the area and P the perimeter of the polygon; observe that $2S = Pr$, where r is the inradius of the polygon. Then $a\sum_{i=1}^{n} a_i = 2S$, so that, by the Cauchy-Schwarz Inequality,

$$\frac{2S}{a}\sum_{i=1}^{n} \frac{1}{a_i} = \sum_{i=1}^{n} a_i \sum_{i=1}^{n} \frac{1}{a_i} \geq n^2$$

and so

$$a\sum_{i=1}^{n} \frac{1}{a_i} \geq \frac{a^2 n^2}{2S} = \frac{P^2}{2S} = \frac{P}{r} > 2\pi \ ,$$

from which the result follows.

182. Let M be an interior point of the equilateral triangle ABC with each side of unit length. Prove that

$$MA.MB + MB.MC + MC.MA \geq 1 \ .$$

Solution. Let the respective lengths of MA, MB and MC be x, y and z, and let the respective angles BMC, CMA and AMB be α, β and γ. Then $\alpha + \beta + \gamma = 2\pi$. Now

$$\begin{aligned}
\cos\alpha + \cos\beta + \cos\gamma &= 2\cos\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2} + 2\cos^2\frac{\gamma}{2} - 1 \\
&= -2\cos\frac{\gamma}{2}\cos\frac{\alpha-\beta}{2} + 2\cos^2\frac{\gamma}{2} - 1 \\
&= \frac{1}{2}\left[2\cos\frac{\gamma}{2} - \cos\frac{\alpha-\beta}{2}\right]^2 + \frac{1}{2}\sin^2\frac{\alpha-\beta}{2} - \frac{3}{2} \geq -\frac{3}{2} \ .
\end{aligned}$$

From the Law of Cosines applied to the triangles MBC, MCA and MAB, we convert this equation to

$$\frac{y^2+z^2-1}{2yz} + \frac{x^2+z^2-1}{2xz} + \frac{y^2+x^2-1}{2xy} \geq -\frac{3}{2} \ .$$

This simplifies to $(x+y+z)(xy+xz+yz)-(x+y+z)\geq 0$. Since $x+y+z\neq 0$, the result follows.

Comment. Ali Feiz Mohammadi noted that the more general result

$$\frac{PA\cdot PB}{CA\cdot CB}+\frac{PA\cdot PC}{BA\cdot BC}+\frac{PB\cdot PC}{AB\cdot AC}\geq 1$$

holds for any triangle ABC and any interior point P. This can be obtained by using vectors with the origin at the point P, the Cauchy Inequality and the identity

$$\frac{ab}{(c-a)(c-b)}+\frac{ac}{(b-a)(c-a)}+\frac{bc}{(b-a)(a-c)}=1\ .$$

To obtain this identity, note that the polynomial

$$\frac{(x-a)(x-b)}{(c-a)(c-b)}+\frac{(x-a)(x-c)}{(b-a)(b-c)}+\frac{(x-b)(x-c)}{(a-b)(a-c)}$$

is a polynomial of degree not exceeding 2 that takes the value 1 at $x=a,b,c$. But this makes is identically equal to the constant polynomial 1. Equating constant coefficients gives the desired result.

183. Simplify the expression

$$\frac{\sqrt{1+\sqrt{1-x^2}}((1+x)\sqrt{1+x}-(1-x)\sqrt{1-x})}{x(2+\sqrt{1-x^2})}\ ,$$

where $0<|x|<1$.

Solution. Observe that

$$\begin{aligned}\sqrt{1+\sqrt{1-x^2}}&=\sqrt{\frac{1+x+2\sqrt{1-x^2}+1-x}{2}}\\&=\sqrt{\frac{(\sqrt{1+x}+\sqrt{1-x})^2}{2}}\\&=\frac{\sqrt{1+x}+\sqrt{1-x}}{\sqrt{2}}\ .\end{aligned}$$

Then, using the formula $a^3-b^3=(a-b)(a^2+ab+b^2)$, we find that the expression given in the problem is equal to

$$\begin{aligned}&\frac{(\sqrt{1+x}+\sqrt{1-x})(\sqrt{1+x}^3-\sqrt{1-x}^3)}{x\sqrt{2}(2+\sqrt{1-x^2})}\\&=\frac{(\sqrt{1+x}+\sqrt{1-x})(\sqrt{1+x}-\sqrt{1-x})(1+x+\sqrt{1-x^2}+1-x)}{x\sqrt{2}(2+\sqrt{1-x^2})}\\&=\frac{(1+x-1+x)(2+\sqrt{1-x^2})}{x\sqrt{2}(2+\sqrt{1-x^2})}\\&=\frac{2x}{x\sqrt{2}}=\sqrt{2}\ .\end{aligned}$$

184. Using complex numbers, or otherwise, evaluate

$$\sin 10^\circ\sin 50^\circ\sin 70^\circ\ .$$

Solution 1. Let $z = \cos 20° + i \sin 20°$, so that $1/z = \cos 20° - i \sin 20°$. Then, by De Moivre's Theorem, $z^9 = -1$. Now,

$$\sin 70° = \cos 20° = \frac{1}{2}(z + \frac{1}{z}) = \frac{z^2+1}{2z} \ ,$$

$$\sin 50° = \cos 40° = \frac{1}{2}(z^2 + \frac{1}{z^2}) = \frac{z^4+1}{2z^2} \ ,$$

and

$$\sin 10° = \cos 80° = \frac{1}{2}(z^4 + \frac{1}{z^4}) = \frac{z^8+1}{2z^4} \ .$$

Hence

$$\begin{aligned}
\sin 10° \sin 50° \sin 70° &= \frac{z^2+1}{2z} \cdot \frac{z^4+1}{2z^2} \cdot \frac{z^8+1}{2z^4} \\
&= \frac{1 + z^2 + z^4 + z^6 + z^8 + z^{10} + z^{12} + z^{14}}{8z^7} \\
&= \frac{1 - z^{16}}{8z^7(1-z^2)} \\
&= \frac{1 - z^7 z^9}{8(z^7 - z^9)} \\
&= \frac{1 + z^7}{8(z^7+1)} = \frac{1}{8} \ .
\end{aligned}$$

Solution 2. We have that

$$\begin{aligned}
\sin 10° \sin 50° \sin 70° &= \frac{1}{2}[\cos 40° - \cos 60°] \sin 70° \\
&= \frac{1}{2}[\cos 40° \sin 70° - \frac{1}{4} \sin 70° \\
&= \frac{1}{4}[\sin 110° + \sin 30°] - \frac{1}{4} \sin 70° \\
&= \frac{1}{4}[\sin 110° - \sin 70°] + \frac{1}{8} = \frac{1}{8} \ .
\end{aligned}$$

Solution 3. Observe that

$$\begin{aligned}
\sin 20° \sin 70° \sin 50° \sin 10° &= \sin 20° \cos 20° \cos 40° \cos 80° \\
&= \frac{1}{2} \sin 40° \cos 40° \cos 80° = \frac{1}{4} \sin 80° \cos 80° \\
&= \frac{1}{8} \sin 160° = \frac{1}{8} \sin 20° \ .
\end{aligned}$$

Since $\sin 20° \neq 0$, we can cancel this factor from both sides to obtain that $\sin 10° \sin 50° \sin 70° = 1/8$.

Solution 4. [O. Ivrii]

$$\begin{aligned}
\sin 10° \sin 50° \sin 70° &= \frac{1}{2} \sin 10° [\cos 20° - \cos 120°] \\
&= \frac{1}{2} \sin 10° \cos 20° + \frac{1}{4} \sin 10° \\
&= \frac{1}{4}[\sin 30° - \sin 10°] + \frac{1}{4} \sin 10° = \frac{1}{4} \sin 30° = \frac{1}{8} \ .
\end{aligned}$$

Solution 5. [L. Chindelevitch] Observe that $\sin 10° \sin 50° \sin 70° = \cos 20° \cos 40° \cos 80°$ and that $\cos 6\theta = -1/2$ is satisfied by $\theta = \pm 20°$, $\pm 40°$, $\pm 80°$, $\pm 100°$, $\pm 140°$ and $\pm 160°$. Now, $\cos 6\theta$ is equal to the real part of $(\cos\theta + i \sin\theta)^6$, namely $32\cos^6\theta - 48\cos^4\theta + 18\cos^2\theta - 1$.

Thus, the sextic equation $32x^6 - 48x^4 + 18x^2 - 1 = -1/2$ is satisfied by $x = \pm\cos 20°$, $\pm\cos 40°$. $\pm\cos 60°$. As the equation can be rewritten

$$x^6 - \frac{3}{2}x^4 + \frac{9}{16}x^2 - \frac{1}{64} = 0 \ ,$$

the product of its six roots is $-1/64$. Thus

$$-(\cos 20° \cos 40° \cos 80°)^2 = -1/64$$

and the result follows.

Solution 6. [A. Critch] Observe that $\sin 10° \sin 50° \sin 70° = -\cos 40° \cos 80° \cos 160°$, and that $\theta = 40°, 80°, 160°$ satisfy $\cos 3\theta = -1/2$. Thus the cosines of $40°$, $80°$ and $160°$ are the roots of the cubic equation $4x^3 - 3x + \frac{1}{2} = 0$. The result follows, since the product of the roots of this equation is $-1/8$.

185. Find all triples of natural numbers a, b, c, such that all of the following conditions hold: (1) $a < 1974$; (2) b is less than c by 1575; (3) $a^2 + b^2 = c^2$.

Solution. Conditions (2) and (3) can be written as $c - b = 1575$ and $a^2 = c^2 - b^2$. Hence $a^2 = 1575(c + b) = 3^2 \cdot 5^2 \cdot 7 \cdot (b + c)$, so that $a = 3 \cdot 5 \cdot 7 \cdot k = 105k$. By (1), $k \leq 18$.

From $105^2k^2 = 1575(c + b)$, it follows that $7k^2 = c + b$. Putting this with $1575 = c - b$ yields that $2c = 7k^2 + 1575$ and $2b = 7k^2 - 1575$. Since b is a natural number, $7k^2 > 1575$, whence $k > 15$. Since k is also odd (why?), the only possibility is for k to be equal to 17. This works and we obtain the triple $(a, b, c) = (1785, 224, 1799)$.

Comment. This was solved by most participants. Some solutions used the fact that all *primitive* pythagorean triples can be parametrized by $(a, b, c) = (m^2 - n^2, 2mn, m^2 + n^2)$, where m and n are natural numbers, $m > n$ and the greatest common divisor of m and n is 1. Note that this representation is not adequate to get all non-primitive triples, such as $(15, 36, 39)$ and the triple of this problem. The general form is given by $(a, b, c) = (k(m^2 - n^2), 2kmn, k(m^2 + n^2))$, where k is a positive constant. When you build your solution on such a "well-known" fact, make sure that you recall it correctly and respect all restrictions and specifics. Otherwise, you risk applying it in a situation that does not satisfy all the requirements or you find only a subset of the possible solutions.

186. Find all natural numbers n such that there exists a convex $n-$sided polygon whose diagonals are all of the same length.

Solution 1. [T. Yue] First, we prove that $n < 6$. Suppose that $n \geq 6$ and that there exists a convex $n-$sided polygon with vertices A_1, A_2, $\cdots$, A_n whose diagonals are all of the same length. Since the diagonals are all of the same length, we have that $A_1A_5 = A_2A_5 = A_1A_4 = A_2A_4$, so that $A_1A_2A_5$ and $A_1A_2A_4$ are isosceles triangles and A_4 and A_5 both lie on the right bisector of A_1A_2. Since the polynomial is convex, both A_4 and A_5 must lie on the same side of A_1A_2. But then they could not be distinct, yielding a contradiction.

In the case $n = 3$, there are no diagonals, so the result is vacuously true. For $n = 4$, all squares, rectangles and isosceles trapezoids satisfy the condition. When $n = 5$, the regular pentagon is an example. (Is it the only example?)

Solution 2. Suppose that $n \geq 6$, and the vertices of the polygon be A_1, A_2, $\cdots$, A_n. Suppose that A_1A_{n-2} and A_2A_{n-1} intersect at O. Then, by the triangle inequality, $A_1O + OA_{n-1} > A_1A_{n-1}$ and $A_2O + OA_{n-2} > A_2A_{n-2}$, so that

$$A_1A_{n-2} + A_2A_{n-1} = A_1O + OA_{n-2} + A_2O + OA_{n-1} > A_1A_{n-1} + A_2A_{n-2}$$

and so the diagonals A_iA_j are not all of the same length. Hence, $n \leq 5$ and we can conclude as before.

187. Suppose that p is a real parameter and that

$$f(x) = x^3 - (p+5)x^2 - 2(p-3)(p-1)x + 4p^2 - 24p + 36 \ .$$

(a) Check that $f(3-p) = 0$.

(b) Find all values of p for which two of the roots of the equation $f(x) = 0$ (expressed in terms of p) can be the lengths of the two legs in a right-angled triangle with a hypotenuse of $4\sqrt{2}$.

Solution. (a) Observe that

$$f(x) = [(x - (3-p))][x^2 - 2(p+1)x + 4(p-3)] \ .$$

(b) [Y. Sun] From the factorization in (a), we can identify the three roots: $x_1 = 3 - p$,

$$x_2 = (p+1) + \sqrt{(p-1)^2 + 12} \ ,$$

$$x_3 = (p+1) - \sqrt{(p-1)^2 + 12} \ .$$

Note that $x_2 - x_3 = 2\sqrt{(p-1)^2 + 12} \geq 2\sqrt{12} = 4\sqrt{3} > 4\sqrt{2}$, so that, by the triangle inequality, x_2 and x_3 cannot be the legs of a right triangle with hypotenuse $4\sqrt{2}$. On the other hand,

$$x_1 + x_3 = 4 - \sqrt{(p-1)^2 + 12} \leq 4 - \sqrt{12} = 2(2 - \sqrt{3}) < 4\sqrt{2} \ ,$$

so that, by the triangle inequality, x_1, x_3 and $4\sqrt{2}$ cannot be the sides of a triangle. Thus, the only possibility is that $x_1^2 + x_2^2 = 32$.

Thus, we must have

$$(3-p)^2 + [(p+1) + \sqrt{p^2 - 2p + 13}]^2 = 32$$

$$\Longrightarrow 2(p+1)\sqrt{p^2 - 2p + 13} = -3p^2 + 6p + 9 = 3(p+1)(3-p) \ .$$

Therefore, either $p = -1$ or $2\sqrt{p^2 - 2p + 13} = 3(3-p)$. The latter possibility leads to

$$4(p^2 - 2p + 13) = 9(p-3)^2 \Rightarrow 5p^2 - 46p + 29 = 0 \ .$$

Since $3(3-p)$ must be positive, we reject one root of this quadratic for p, and so have only the additional possibility $(23 - 8\sqrt{6})/5$.

188. (a) The measure of the angles of an acute triangle are α, β and γ degrees. Determine (as an expression of α, β, γ) what masses must be placed at each of the triangle's vertices for the centroid (centre of gravity) to coincide with (i) the orthocentre of the triangle; (ii) the circumcentre of the triangle.

(b) The sides of an arbitrary triangle are a, b, c units in length. Determine (as an expression of a, b, c) what masses must be placed at each of the triangle's vertices for the centroid (centre of gravity) to coincide with (i) the centre of the inscribed circle of the triangle; (ii) the intersection point of the three segments joining the vertices of the triangle to the points on the opposite sides where the inscribed circle is tangent (be sure to prove that, indeed, the three segments intersect in a common point).

Solution. [R. Barrington Leigh] Suppose that a, b, c are the respective lengths of BC, CA and AB of triangle ABC and that $\alpha = \angle CAB$, $\beta = \angle ABC$ and $\gamma = \angle BCA$. Let masses m_a, m_b, m_c be masses placed at the respective vertices A, B, C. We use the following lemma, which will be established in the Comments:

Lemma 1. *The centroid G of two combined collections of mass particles is on the line joining the centroids P and Q of the two collections, and the following ratio holds: $PG : GQ = m_Q : m_P$, where m_Q and m_P are the totals of the masses in the two mass collections at Q and P respectively.*

It follows that, if X is the centroid of the masses at the vertices of ΔABC and Y is the intersection point of AX and BC, then Y is the centroid of the masses at the vertices B and C. Furthermore, let h_b and h_c be the perpendicular distances from B and C respectively to the line AY. By the lemma and the similar right triangles formed by the drawn perpendiculars,

$$\frac{m_b}{m_c} = \frac{YC}{YB} = \frac{h_c}{h_b} \qquad (*)$$

(a) (i) Let Y be the foot of the altitude from A to BC, so that the orthocentre of the triangle is on AY. By $(*)$,

$$\frac{m_b}{m_c} = \frac{AY/\tan\gamma}{AY/\tan\beta} = \frac{\tan\beta}{\tan\gamma} \ .$$

By symmetry, we see that

$$m_a : m_b : m_c = \tan\alpha : \tan\beta : \tan\gamma \ .$$

Since the triangle is acute, the elements of the ratio are positive and it is well-defined.

(a) (ii) Let $X = O$ be the circumcentre of ΔABC. From $(*)$,

$$\frac{m_b}{m_c} = \frac{h_c}{h_b} = \frac{OC \cdot \sin\angle COA}{OB \cdot \sin\angle BOA} = \frac{\sin 2\beta}{\sin 2\gamma} \ .$$

By symmetry, we have that

$$m_a : m_b : m_c = \sin 2\alpha : \sin 2\beta : \sin 2\gamma \ .$$

(b) (i) Let $X = I$, the incentre of the triangle ABC; this is the intersection of the angle bisectors. Then $\angle BAI = \angle CAI$, and, by $(*)$,

$$\frac{m_b}{m_c} = \frac{h_c}{h_b} = \frac{b \cdot \sin\angle CAI}{c \cdot \sin\angle BAI} = \frac{b}{c} \ .$$

By symmetry, we find that $m_a : m_b : m_c = a : b : c$.

(b) (ii) Let the incircle of ABC meet BC at D, CA at E and AB at F. Let x, y, z be the respective lengths of $AE = AF$, $BD = BF$, $CD = CE$, so that $y + z = a$, $z + x = b$, $x + y = c$. Then $2x = b + c - a$, $2y = c + a - b$ and $2z = a + b - c$. The semi-perimeter s of the triangle is equal to $(a + b + c)/2$, so that $x = s - a$, $y = s - b$ and $z = s - c$. Using Ceva's Theorem, we see from

$$\frac{AF}{FB} \cdot \frac{BD}{DC} \cdot \frac{CE}{EA} = \frac{x}{y} \cdot \frac{y}{z} \cdot \frac{z}{x} = 1$$

that AD, BE and CF concur at a point X. From $(*)$, we have that

$$\frac{m_b}{m_c} = \frac{CD}{BD} = \frac{z}{y} = \frac{1/(s-b)}{1/(s-c)}$$

so that, from symmetry,

$$m_a : m_b : m_c = \frac{1}{s-a} : \frac{1}{s-b} : \frac{1}{s-c} \ .$$

Comment. A couple of correct solutions were received, most based on the lemma and the derived statement $(*)$. This solution was chosen because of its excellently organized and clear presentation. Some students came up with other expressions in the ratios, equivalent after some simple trigonometric transformations to the expressions here. Two of the students, A. Feiz Mohammadi and J.Y. Zhoa used an interesting an helpful fact:

Lemma 2. *Let X be the centroid of triangle ABC and denote by S_a, S_b and S_c the respective areas of the triangles XBC, XCA and XAB. Then $m_a : m_b : m_c = S_a : S_b : S_c$.*

Lemma 2 can be used to obtain a beautiful and elegant solution to each of the four parts of the problem. Let us turn to the proofs of these lemmata.

Proof of Lemma 1. We statement can be replaced by the equivalent form: given two point masses m_P and m_Q at the respective points P and Q, their centroid is on the line segment PQ at the point G for which $PG : GQ = m_Q : m_P$. From the definition of a centroid, if O is any point selected as the origin of vectors,

$$\overrightarrow{OG} = \frac{m_P\overrightarrow{OP} + m_Q\overrightarrow{OQ}}{m_P + m_Q} = \frac{m_P}{m_P + m_Q}\overrightarrow{OP} + \frac{m_Q}{m_P + m_Q}\overrightarrow{OQ} \ . \tag{1}$$

In general, consider a point K on the segment PQ for which $PK : KQ = k : (1-k)$, for $0 < k < 1$. Then

$$\overrightarrow{OK} = \overrightarrow{OP} + \overrightarrow{PK} = \overrightarrow{OP} + k\overrightarrow{PQ} = \overrightarrow{OP} + k(\overrightarrow{OQ} - \overrightarrow{OP}) = (1-k)\overrightarrow{OP} + k\overrightarrow{OQ} \ . \tag{2}$$

Now choose $k = m_Q/(m_P + m_Q)$, so that $1 - k = m_P/(m_P + m_Q)$. From (1) and (2), we have that $G = K$ and we can complete the proof of the lemma.

Proof of Lemma 2. We prove that $m_a : m_b = S_a : S_b$, whence the rest follows from symmetry. Let K be the intersection of CX and AB. Then $m_a : m_b = BK : AK$. Since triangles AKC and BKC have the same altitudes from C, $[AKC] : [BKC] = AK : BK$. Similarly, $[AXK] : [BXK] = AK : BK$. Now $S_a = [BKC] - [BXK]$ and $S_b = [AKC] - [AXK]$, so that $S_a : S_b = BK : AK$ and the result is proven.

189. There are n lines in the plane, where n is an integer exceeding 2. No three of them are concurrent and no two of them are parallel. The lines divide the plane into regions; some of them are closed (they have the form of a convex polygon); others are unbounded (their borders are broken lines consisting of segments and rays).

(a) Determine as a function of n the number of unbounded regions.

(b) Suppose that some of the regions are coloured, so that no two coloured regions have a common side (a segment or ray). Prove that the number of regions coloured in this way does not exceed $\frac{1}{3}(n^2 + n)$.

Solution 1. (a) Draw a circle big enough to contain in its interior all the intersection points of the n lines. Since there are n lines and each of them intersects the circle in two points, they divide the circle into $2n$ arcs. Each of the arcs corresponds to one unbounded region, so that there are $2n$ unbounded regions.

(b) [M. Butler] Let k regions be coloured. Construct a circle around all the bounded regions, big enough to contain part of each of the unbounded regions as well as the intersection points of the lines. Regions that were unbounded before are now bounded (in part by an arc of the circle). Denote the k coloured regions by R_1, R_2, $\cdots$, R_k, and let R_i have v_i vertices (including the intersections of the lines and the circles). Every region has at least three vertices, so that $v_i \geq 3$, and $v_1 + v_2 + \cdots + v_k \geq 3k$ (1).

On the other hand, we can tabulate $v_1 + v_2 + \cdots + v_k$ by adding up the numer of coloured regions meeting at an intersection, for all intersection points. Since no three lines have a common point, there are $\binom{n}{2}$ intersection points among the lines and each of them is a vertex of at most two coloured regions. So there are at most $2\binom{n}{2}$ coloured regions counted so far. In addition, there are $2n$ intersections between the circles and the lines with at most one coloured region at each of them. Taking (1) into consideration, we deduce that

$$2\binom{n}{2} + 2n \geq v_1 + v_2 + \cdots + v_k \geq 3k$$

so that $n^2 + n \geq 3k$ and the result follows.

Solution 2. (b) Let m_2, m_3, $\cdots$ m_k denote the number of coloured regions with respectively $2, 3, \cdots, k$ sides (rays or segments). Regions with two sides are angles, and hence unbounded. Since no two adjecent regions are coloured, $m_2 \leq (1/2)2n = n$. On the other hand, each line is divided by the other $n-1$ lines into n parts, so that the number of parts of all the lines is n^2. Therefore

$$2m_2 + 3m_3 + 4m_4 + \cdots + km_k \leq n^2 \ .$$

The total number k of all the coloured regions satisfies

$$\begin{aligned} k = m_1 + m_2 + \cdots + m_k &= (1/3)m_2 + (2/3)m_2 + m_3 + \cdots + m_k \\ &\leq (1/3)m_2 + (1/3)(2m_2 + 3m_3 + 4m_4 + \cdots + km_k \\ &\leq (1/3)(n + n^2) \ , \end{aligned}$$

as desired.

190. Find all integer values of the parameter a for which the equation

$$|2x + 1| + |x - 2| = a$$

has exactly one integer among its solutions.

Solution 1. [H. Li] To deal with the absolute values, we consider the equation for different ranges of x. If $x < -1/2$, the equation becomes $-3x + 1 = a$ (E_1). If $-1/2 \leq x < 2$, the equation is $x + 3 = a$ (E_2). If $x \geq 2$, the equation is $3x - 1 = a$ (E_3). Thus, the given equation is the union of three linear equations. The graph of a as a function of x is a broken line consisting of two rays u_1 and u_3 (corresponding to E_1 and E_3) and a segment u_2 (corresponding to E_2); please graph it before continuing to read. The minimum possible value of a is 2.5. For an integer a to admit more than one integer solution to the equation, the horizontal line $y = a$ must intersect the graph in at least two lattice points.

When $x < -1/2$, $a > 2.5$ and when $-1/2 \leq x < 2$, then $2.5 \leq a < 5$. The only lattice points on segment u_2 are $(0, 3)$ and $(1, 4)$ and the second one has a corresponding lattice point on u_1. So, for $a = 4$, there are two integer solutions, -1 and 1, to the equation. Let x be an integer. When $x \geq 2$, $a \geq 5$ and we want to consider lines $y = a$ intersecting rays u_1 and u_3. When $x < -1/2$, then $-3x + 1 \equiv 1 \pmod 3$, while if $x \geq 2$, $3x - 1 \equiv 2 \pmod 3$, so that none of the horizontal lines can intersect both of them at a lattice point.

Hence, in conclusion, the given equation has exactly one integer solution x for the following values of a: $a = 3$, $a \geq 5$ and $a \equiv 1 \pmod 3$ and $a \geq 5$ and $a \equiv 2 \pmod 3$.

Solution 2. First, solve the equation by considering the three cases:

- $x \geq 2$: Then $x = \frac{a+1}{3}$ and the equation is solvable in this range $\Leftrightarrow a \geq 5$;
- $-1/2 < x < 2$: Then $x = a - 3$ and the equation is solvable in this range $\Leftrightarrow 5/2 < a < 5$;
- $x \leq 1/2$: Then $x = (1 - a)/3$ and the equation is solvable in this range $\Leftrightarrow a \geq 5/2$.

Thus, summing up, we see that the equation has

- no solution when $a < 5/2$,
- the unique solution $x = -1/2$ when $a = 5/2$,
- two solutions $x = a - 3$ and $x = (1 - a)/3$ when $5/2 < a < 5$, of which both are integers when $a = 4$ and one is an integer when $a = 3$,
- two solutions $x = 2$ and $x = -4/3$ when $a = 5$, and
- two solutions $x = (1 \pm a)/3$ when $a > 5$.

When $a > 5$ we can check that there is no solution when $a \equiv 0$, and exactly one solution when $a \not\equiv 0 \pmod 3$, and we obtain the set in the first solution.

191. In **Olymonland** the distances between every two cities is different. When the transportation program of the country was being developed, for each city, the closest of the other cities was chosen and a highway was built to connect them. All highways are line segments. Prove that

(a) no two highways intersect;

(b) every city is connected by a highway to no more than 5 other cities;

(c) there is no closed broken line composed of highways only.

Solution. (a) Assume that the highways AC and BD intersect at a point O. From the existence of AC, either C is the closest city to A or A is the closest city to C. A similar statement holds for BD. Wolog, suppose that C is the closest city to A and D the closest city to B. Hence, $AD > AC$, $BC > BD$ so that $AD + BC > AC + BD$. On the other hand, from the triangle inequality,

$$AO + OD > AD \ \& \ BO + OC > BC \ \Rightarrow AC + BD = (AO + OC) + (BO + OD) > AD + BC \ ,$$

which contradicts the earlier inequality. Thus, no two highways intersect.

(b) Consider any three cities A, B, X. If X is connected by a highway to both A and B, then AB must be the longest side in triangle ABX. To prove it, let us suppose, if possible, that one of the other sides, say AX, is longest. Then $AX > AB$ and $AX > XB$, so that AX would not exist as B is closer to A than X is, and B is closer to X than A is.

So AB is the longest side, which implies that $\angle AXB$ is the greatest angle in the triangle. Thus, $\angle AXB > 60°$. Assume that a city X is connect to six other cities A, B, C, D, E, F by highways. Then each of the angles with vertices at X with these cities must exceed $60°$, so the sum of the angles going round X from one highway back to it must exceed $360°$, which yields a contradiction. Therefore, every city is connected by a highway to no more than five other cities.

(c) Suppose that there are n cities $A_1, A_2, \cdots, A_n$ that are connected in this order by a closed broken line of highways. Since all distances between pairs of cities are distinct, there must be a longest distance between a pair of adjacent cities, say A_1 and A_n. Then $A_1A_n > A_1A_2$, so A_n is not the closest city to A_1. Also $A_1A_n > A_{n-1}A_n$, so A_1 is not the closest city to A_n. Therefore, the highway A_1A_n must not exist and we get a contradiction. So there is no broken line composed only of highways.

192. Let ABC be a triangle, D be the midpoint of AB and E a point on the side AC for which $AE = 2EC$. Prove that BE bisects the segment CD.

In the following solutions, F is the intersection point of BE and CD.

Solution 1. Let G be the midpoint of AE. Then $AG = GE = EC$ and $DG \| BE$. In triangle ADC, $DG \| FE$ and $GE = EC$, from which it follows that $DF = FC$, as required.

Solution 2. Let $u = [ADF] = [BDF]$ (where $[\cdots]$ denotes area), $v = [AFE]$, $w = [CFE]$ and $z = [BFC]$. Then $2u + v = 2(w + z)$ and $v = 2w$, whence $2u = 2z$ and $u = z$. It follows from this (two triangles with the same height and equal collinear bases) that F is the midpoint of CD.

Solution 3. [S. Seraj] Let J be the midpoint of BE. Then $DJ = \frac{1}{2}AE = EC$ and $DJ \| AC$. Since $\angle CEF = \angle DJF$ and $\angle ECF = \angle FDJ$, triangles CEF and DJF are congruent. Hence $CF = FD$ as required.

Solution 4. By Menelaus' Theorem, applied to triangle ACD and transversal BFE,

$$\frac{CE}{EA} \cdot \frac{AB}{BD} \cdot \frac{DF}{FC} = -1 \ ,$$

so that $\frac{1}{2} \cdot (-2) \cdot (DF/FC) = -1$ and $DF = FC$, as desired.

Solution 5. [T. Yue] Let K be the midpoint of AC; then $BC = 2DK$ and $BC \| DK$. Suppose that BE produced and DK produced meet at H. Since triangles EBC and EHK are similar and $EC = 2EK$, it follows that $BC = 2KH$ and so $DH = BC$. Thus, $DHCB$ is a parallelogram whose diagonals BH and CD must bisect each other. The result follows.

Solution 6. Place the triangle in the cartesian plane so that $B \sim (0, 0)$, $C \sim (3, 0)$ and $A \sim (6a, 6b)$. Then $D \sim (3a, 3b)$, $E \sim (2(a + 1), 2b)$ and the lines BE and and CD have the respective equations $y = bx/(a + 1)$ and $y = b(x - 3)/(a - 1)$. These lines intersect at the point $((3/2)(a + 1), (3/2)b)$, and the result follows.

Solution 7. [L. Chen] $[BDE] = [ADE] = \frac{1}{2}[ABE] = [BEC]$. Let M and N be the respective feet of the perpendiculars from D and C to BE. Then $[BDE] = [BEC] \Rightarrow DM = CN$. Since DMF and CNF are similar right triangles with $DM = CN$, they are congruent and so $DF = CF$.

Solution 8. [F. Chung; Y. Jean] As in the previous solution, $[BDE] = [BEC]$. Therefore,

$$DF : FC = [DEF] : [CEF] = [DBF] : [CBF] = ([DEF] + [DBF]) : ([CEF] + [CBF])$$

$$= [BDE] : [BEC] = 1 : 1 \ .$$

Solution 9. [Y. Wei] Let U be a point on BC such that $DU \| AC$. Suppose that DU and BE intersect in V. Then $2EC = AE = 2DV$, so that $DV = EC$. Also $\angle VDF = \angle ECF$ abd $\angle DFV = \angle ECF$, so that triangles DVF and CEF are congruent. Hence $DF = FC$.

Solution 10. Let AF produced meet BC at L. By Ceva's Theorem,

$$\frac{AD}{DB} \cdot \frac{BL}{LC} \cdot \frac{CE}{AE} = 1 \ ,$$

whence $BL = 2LC$ and, so, $LE \| AB$. Since the triangles ABC and ELC are similar with factor 3, $AB = 3EL$. Let EL intersect CD at M. Then the triangles AFB and LFE are similar, so that $FD = 3FM$. But,

$$FD + FM + MC = DC = 3MC \Rightarrow 2FM = MC \Rightarrow FC = FM + MC = 3FM = FD \ ,$$

as desired.

Solution 11. [H. Lee] Let $\mathbf{u} = \overrightarrow{DB}$, $\mathbf{v} = \overrightarrow{EC}$, $\mathbf{a} = \overrightarrow{BF}$, $\lambda\mathbf{a} = \overrightarrow{FE}$, $\mathbf{b} = \overrightarrow{CF}$ and $\mu\mathbf{b} = \overrightarrow{FD}$. Then

$$\mathbf{a} + \mu\mathbf{b} + \mathbf{u} = \mathbf{0}$$

and

$$\mathbf{b} + \lambda\mathbf{a} + \mathbf{v} = \mathbf{0} \ .$$

Hence

$$\mathbf{u} = -\mathbf{a} - \mu\mathbf{b} \quad \text{and} \quad \mathbf{v} = -\mathbf{b} - \lambda\mathbf{a} \ .$$

Therefore, from triangle ABE,

$$\begin{aligned} \mathbf{0} &= (\lambda + 1)\mathbf{a} - 2\mathbf{v} + 2\mathbf{u} \\ &= (\lambda + 1)\mathbf{a} + 2\mathbf{b} + 2\lambda\mathbf{a} - 2\mathbf{a} - 2\mu\mathbf{b} \\ &= (3\lambda - 1)\mathbf{a} + 2(1 - \mu)\mathbf{b} \ . \end{aligned}$$

Since $\{\mathbf{a}, \mathbf{b}\}$ is a linearly independent set, $\lambda = 1/3$ and $\mu = 1$, yielding the desired result.

Solution 12. [M. Zaharia] Place masses 1, 1, 2, respectively, at the vertices A, B, C. We locate the centre of gravity of these masses in two ways. Since the masses at A and B have their centre of gravity at D, we can get an equivalent system by replacing the masses at A and B by a mass 2 at the point D. The centre of gravity of the original set-up is equal to the centre of gravity of masses of 2 placed at each of D and C, namely at the midpoint of CD.

On the other hand, the centre of gravity of the masses at A and C is at E. So the centre of gravity of the original set-up is equal to the centre of gravity of a mass 3 located at E and a mass 1 located at B, namely on the segment BE (at the point F for which $BF = 3FE$). Since both BE and CD contain the centre of gravity of the original set-up, the result follows.

Solution 13. Place the triangle in the complex plane with C at 0, B at $12z$ and A at 12. Then D is located at $6(z+1)$ and E at 4. Let P be the midpoint $3(z+1)$ of CD. Then, BP and PE are collinear since

$$12z - 3(z + 1) = 3(3z - 1) = 3[3(z + 1) - 4] \ ,$$

i.e., the vector $\overrightarrow{BP}$ is a real multiple of $\overrightarrow{PE}$. The result follows.

193. Determine the volume of an isosceles tetrahedron for which the pairs of opposite edges have lengths a, b, c. Check your answer independently for a regular tetrahedron.

Solution 1. The edges of the tetrahedron can be realized as the diagonals of the six faces of a rectangular parallelepiped with edges of length u, v, w in such a way that $a^2 = v^2 + w^2$, $b^2 = u^2 + w^2$ and $c^2 = u^2 + v^2$. The tetrahedran can be obtained from the parallelepiped by trimming away four triangular pyramids each with three mutually perpendicular faces (surrounding a corner of the parallelepiped) and three pairwise orthogonal edges of lengths u, v, w. Hence the volume of the tetrahedron is equal to

$$uvw - 4((1/6)uvw) = (1/3)uvw \ .$$

From the foregoing equations, $2u^2 = b^2 + c^2 - a^2$, $2v^2 = c^2 + a^2 - b^2$ and $2w^2 = a^2 + b^2 - c^2$. (By laying out the tetrahedron flat, we see that the triangle of sides a, b, c is acute and the right sides of these equations are indeed positive.) It follows that the volume of the tetrahedron is

$$\frac{\sqrt{2}}{12}\sqrt{(b^2 + c^2 - a^2)(c^2 + a^2 - b^2)(a^2 + b^2 - c^2)} \ .$$

In the case of a regular tetrahedron of side 1, the height is equal to $\sqrt{2/3}$ and the area of a side is equal to $\sqrt{3}/4$, and the formula checks out.

Solution 2. [D. Yu] Let the base of the tetrahedron be triangle ABC, eith $a = |BC| = |AD|$, $b = |AC| = |BD|$, $c = |AB| = |CD|$; let P be the foot of the perpendicular from D to the plane of ABC and let $h = |DP|$. Then $|AP| = \sqrt{a^2 - h^2}$, $|BP| = \sqrt{b^2 - h^2}$, $|CP| = \sqrt{c^2 - h^2}$.

Suppose that $\alpha = \angle BCP$ and $\beta = \angle ACP$. Then using the Law of Cosines on triangles BCP, ACP and ABC, we obtain that

$$\cos\alpha = \frac{a^2 + c^2 - b^2}{2a\sqrt{c^2 - h^2}}$$

$$\cos\beta = \frac{b^2 + c^2 - a^2}{2b\sqrt{c^2 - h^2}}$$

and

$$\cos(\alpha + \beta) = \frac{a^2 + b^2 - c^2}{2ab} \ ,$$

whence

$$\frac{a^2 + b^2 - c^2}{2ab} =$$

$$\frac{(a^2 + c^2 - b^2)(b^2 + c^2 - a^2) - \sqrt{4a^2(c^2 - h^2) - (a^2 + c^2 - b^2)^2}\sqrt{4b^2(c^2 - h^2) - (b^2 + c^2 - a^2)^2}}{4ab(c^2 - h^2)} \ .$$

Shifting terms and squaring leads to

$$[2(a^2+b^2-c^2)(c^2-h^2)-(a^2+c^2-b^2)(b^2+c^2-a^2)]^2 = [4a^2(c^2-h^2)-(a^2+c^2-b^2)^2][4b^2(c^2-h^2)-(b^2+c^2-a^2)^2] \ .$$

With $u = b^2 + c^2 - a^2$, $v = c^2 + a^2 - b^2$, $w = a^2 + b^2 - c^2$, $z = c^2 - h^2$, this can be rendered

$$\begin{aligned} 0 &= [2wz - uv]^2 - [4a^2 z - v^2][4b^2 z - u^2] \\ &= z[4(w^2 - 4a^2b^2)z - 4(uvw - a^2u^2 - b^2v^2)] \end{aligned}$$

so that

$$c^2 - h^2 = z = \frac{a^2u^2 + b^2v^2 - uvw}{4a^2b^2 - w^2}$$

and

$$h^2 = \frac{4a^2b^2c^2 + uvw - a^2u^2 - b^2v^2 - c^2w^2}{4a^2b^2 - w^2} .$$

Now

$$\begin{aligned} 4a^2b^2 - w^2 &= -a^4 - b^4 - c^4 + 2a^2b^2 + 2a^2c^2 + 2b^2c^2 \\ &= (a+b+c)(a+b-c)(b+c-a)(c+a-b) \\ &= 16S^2 , \end{aligned}$$

where S is the area of triangle ABC.

Now consider the numerator of h^2. Its value when $w = a^2 + b^2 - c^2$ is set equal to 0 is $4a^2b^2c^2 - a^2u^2 - b^2v^2 = 4a^2b^2c^2 - a^2(2b^2) - b^2(2a^2) = 0$, so that w divides the numerator. So also do u and v. Hence the numerator of degree 6 in a, b, c must be a multiple of uvw, also of degree 6 in a, b, c. Hence the numerator is a multiple of uvw. Comparing the coefficients of a^6 (say) gives that the numerator must be $2uvw$. Hence

$$h^2 = \frac{2uvw}{16S^2} = \frac{uvw}{8S^2} .$$

The volume V of the tetrahedron satisfies

$$V^2 = \left(\frac{Sh}{3}\right)^2 = \frac{S^2h^2}{9} = \frac{uvw}{72} ,$$

whence

$$V = \frac{\sqrt{uvw}}{6\sqrt{2}} = \frac{2uvw}{12} .$$

The checking for the tetrahedron proceeds as before.

194. Let ABC be a triangle with incentre I. Let M be the midpoint of BC, U be the intersection of AI produced with BC, D be the foot of the perpendicular from I to BC and P be the foot of the perpendicular from A to BC. Prove that

$$|PD||DM| = |DU||PM| .$$

Solution 1. Suppose that the lengths of the sides of the triangle are a, b and c, using the conventional notation. Then the distance from B of the following points on the side BC are given by $(B, 0)$, (C, a), $(M, a/2)$, $(U, ca/(b+c))$, $(D, (a+c-b)/2)$ and $(P, c\cos B) = (P, (a^2+c^2-b^2)/(2a))$. One can then verify the desired relation by calculation.

Solution 2. [L. Chen] Let the side lengths of the triangle be a, b, c, as conventional, and, wolog, suppose that $c < b$. Let $u = |BP|$ and $v = |PC|$. Then, equating two expressions for the area of the triangle, with $r = |ID|$ as the inradius, we find that $|AP| = 2rs/a$. From similar triangle, we have that

$$\frac{|PU|}{|DU|} = \frac{|AP|}{|ID|} = \frac{2s}{a} = 1 + \frac{b+c}{a} ,$$

whence

$$\frac{|PD|}{|DU|} = \frac{b+c}{a} .$$

Now $|PM| = (a/2) - u = (v-u)/2$ and $|DM| = (b-c)/2$. Hence

$$\frac{|PM|}{|DM|} = \frac{v-u}{b-c} .$$

By Pythagoras' Theorem, $c^2 - u^2 = b^2 - v^2$, whence

$$\frac{v-u}{b-c} = \frac{b+c}{v+u} = \frac{b+c}{a}$$

and the result follows.

195. Let $ABCD$ be a convex quadrilateral and let the midpoints of AC and BD be P and Q respectively, Prove that

$$|AB|^2 + |BC|^2 + |CD|^2 + |DA|^2 = |AC|^2 + |BD|^2 + 4|PQ|^2 \ .$$

Solution 1. Let X denote the vector from an origin to a point X. Then, vectorially, it can be verified that

$$\begin{aligned}(A-B)\cdot(A-B) + (B-C)\cdot(B-C) + (C-D)\cdot(C-D) + (D-A)\cdot(D-A)&\\ -(A-C)\cdot(A-C) - (B-D)\cdot(B-D)&\\ = -2A\cdot B - 2B\cdot C - 2C\cdot D - 2D\cdot A + 2A\cdot C + 2B\cdot D + A^2 + B^2 + C^2 + D^2&\\ = 4\Big(\frac{A+C}{2} - \frac{B+D}{2}\Big)\cdot\Big(\frac{A+C}{2} - \frac{B+D}{2}\Big)\ ,&\end{aligned}$$

which yields the desired result.

Solution 2. [T. Yin] We use the result that for any parallelogram $KLMN$, $2|KL|^2 + 2|LM|^2 = |KM|^2 + |LN|^2$. This is straightforward to verify using the Law of Cosines, for example. Let W, X, Y, Z be the respective midpoints of the sides AB, BC, CD, DA. Using the fact that all of $WXYZ$, $PXQZ$ and $PWQY$ are parallelograms, we have that

$$\begin{aligned}|AB|^2 + |BC|^2 + |CD|^2 + |DA|^2 &= 4[|PX|^2 + |PW|^2 + |PZ|^2 + |PY|^2]\\ &= 2[|PQ|^2 + |XZ|^2 + |PQ|^2 + |WY|^2]\\ &= 4|PQ|^2 + 2[|XZ|^2 + |WY|^2]\\ &= 4|PQ|^2 + 4[|WZ|^2 + |WX|^2]\\ &= 4|PQ|^2 + |BD|^2 + |AC|^2\ .\end{aligned}$$

196. Determine five values of p for which the polynomial $x^2 + 2002x - 1002p$ has integer roots.

Answer. Here are some values of $(p;u,v)$ with u and v the corresponding roots: $(0;0,-2002)$, $(4;2,-2004)$, $(784;336,-2338)$, $(1780;668,-2670)$, $(3004;2002,-3004)$, $(3012;1004,-3006)$, $(4460;1338,-3340)$, $(8012;2004,-4006)$, $(8024;2006,-4008)$, $(-556;-334,-1668)$, $(-1000;-1000,-1002)$.

Solution 1. If x satisfies the equation $x^2+2002x-1002p = 0$, then we must have $p = x(x+2002)/(1002)$. If we choose integers x for which $x(x+2002)$ is a multiple of 1002, then this value of p will be an integer that yields a quadratic with two integer roots, namely x and $-2002-x$. One way to do this is to select either $x \equiv 0$ or $x \equiv 2 \pmod{1002}$. Observing that $1002 = 2 \times 3 \times 167$, we can also try to make $x \equiv 0$ (mod 167) and $x \equiv 2 \pmod 6$. For example, $x = 668$ works. We can also try $x \equiv 2 \pmod{167}$ and $x \equiv 0 \pmod 6$; in this case, $x = 336$ works.

Solution 2. The discriminant of the quadratic is 4 times $1001^2 + 1002p$. Suppose that p is selected to make this equal to a square q^2. Then we have that

$$1002p = q^2 - 1001^2 = (q-1001)(q+1001)\ .$$

We select q so that either $q - 1001$ or $q + 1001$ is divisible by 1002. For example $q = 2003, 1, 3005, 4007$ all work. We can also make one factor divisible by 667 and the other by 6.

197. Determine all integers x and y that satisfy the equation $x^3 + 9xy + 127 = y^3$.

Solution 1. Let $x = y + z$. Then the equation becomes $(3z + 9)y^2 + (3z^2 + 9z)y + (z^3 + 127) = 0$, a quadratic in y whose discriminant is equal to

$$\begin{aligned}(3z+9)^2z^2 &- 4(3z+9)(z^3+127)\\ &= (3z+9)[z^2(3z+9) - 4(z^3+127)]\\ &= -(3z+9)(z^3 - 9z^2 + 508)\ .\end{aligned}$$

Note that $z^3 - 9z^2 + 508 = z^2(z - 9) + 508$ is nonnegative if and only if $z \geq -5$ (z being an integer) and that $3z + 9$ is nonnegative if and only if $z \geq -3$. Hence the discriminant is nonnegative if and only if $z = -3, -4, -5$. From the quadratic equation, we have that $z^3 + 127 \equiv 0 \pmod 3$. The only possibility is $z = -4$ and this leads to the equation $0 = -3y^2 + 12y + 63 = -3(y - 7)(y + 3)$ and the solutions $(x, y) = (3, 7), (-7, -3)$.

Solution 2. The equation can be rewritten

$$(x - y)[(x - y)^2 + 3xy] + 9xy = -127$$

or

$$u^3 + 3v(u + 3) = -127$$

where $u = x - y$ and $v = xy$. Hence

$$3v = -\frac{u^3 + 127}{u + 3} = -\left[(u^2 - 3u + 9) + \frac{100}{u + 3}\right]\ .$$

Therefore, $u^3 + 127 \equiv 0 \pmod 3$, so that $u \equiv 2 \pmod 3$, and $u + 3$ divides 100. The candidates are

$$u = -103, -28, -13, -7, -4, -1, 2, 17, 47\ .$$

Checking these out leads to the posible solutions.

198. Let p be a prime number and let $f(x)$ be a polynomial of degree d with integer coefficients such that $f(0) = 0$ and $f(1) = 1$ and that, for every positive integer n, $f(n) \equiv 0$ or $f(n) \equiv 1$, modulo p. Prove that $d \geq p - 1$. Give an example of such a polynomial.

Solution. Since the polynomial is nonconstant, $d \geq 1$, so that the result holds for $p = 2$. Henceforth, assume that p is an odd prime. Let $0 \leq k \leq p - 2$. Consider the polynomial

$$p_k(x) = \frac{x(x-1)(x-2)\cdots(x-k+1)(x-k-1)\cdots(x-p+2)}{k!(p-k-2)!(-1)^{p-k}}\ .$$

We have that $p_k(k) = 1$ and $p_k(x) = 0$ when $x = 0, 1, 2, \cdots, k - 1, k + 1, \cdots, p - 2$. Let

$$g(x) = \sum_{k=0}^{p-2} f(k)p_k(x)\ .$$

Then the degree of $g(x)$ does not exceed $p - 2$ and $g(x) = f(x)$ for $x = 0, 1, 2, \cdots, p - 2$; in fact, $g(x)$ is the unique polynomial of degree less than $p - 1$ that agrees with f at these $p - 1$ points (why?).

Now

$$g(p-1) = \sum_{k=0}^{p-2} (-1)^{p-k} \frac{(p-1)!}{k!(p-k-1)!} f(k) = \sum_{k=0}^{p-2} (-1)^{p-k} \binom{p-1}{k} f(k) \ .$$

Since $\binom{p-1}{k} = \binom{p}{k} - \binom{p-1}{k-1}$ and $\binom{p}{k} \equiv 0 \pmod p$ for $1 \leq k \leq p-1$, and induction argument yields that $\binom{p-1}{k} \equiv (-1)^k$ for $1 \leq k \leq p-1$, so that

$$g(p-1) \equiv (-1)^p \sum_{k=0}^{p-2} f(k)$$

(mod p). Since $f(0) = 0$ and $f(1) = 1$, it follows that $\sum_{k=0}^{p-2} f(k)$ is congruent to some number between 1 and $p-2$ inclusive, so that $g(p-1) \not\equiv 0$ and $g(p-1) \not\equiv 1 \pmod p$. Hence $f(p-1) \neq g(p-1)$, so that f and g are distinct polynomials. Thus, the degree of f exceeds $p-2$ as desired.

By Fermat's Little Theorem, the polynomial x^{p-1} satisfies the condition.

Solution 2. [M. Guay-Paquet] Let

$$h(x) = f(x) + f(2x) + \cdots + f((p-1)x) \ .$$

Then $h(1) \not\equiv 0 \pmod p$ and $h(0) = 0$. The degree of h is equal to d, the degree of f.

Let $x \not\equiv 0 \pmod p$. Then $(x, 2x, 3x, \cdots, (p-1)x)$ is a permutation of $(1, 2, 3, \cdots, p-1)$, so that $h(x) \equiv h(1) \pmod p$.

Suppose that $g(x) = h(x) - h(1)$. The degree of g is equal to d, $g(0) \equiv -h(1) \not\equiv 0 \pmod p$ and $g(x) \equiv 0$ whenever $x \not\equiv 0 \pmod p$. Therefore, $g(x)$ differs from a polynomial of the form $k(x-1)(x-2)\cdots(x-\overline{p-1})$ by a polynomial whose coefficients are multiples of p. Since $k \not\equiv 0 \pmod p$ (check out the value at 0), the coefficient of x^{k-1} must be nonzero, and so $d \geq p-1$, as desired.