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Problems 1-54

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Notes: The *inradius* of a triangle is the radius of the *incircle*, the circle that touches each side of the polygon. The *circumradius* of a triangle is the radius of the *circumcircle*, the circle that passes through its three vertices.

A set of lines of *concurrent* if and only if they have a common point of intersection.

The word *unique* means *exactly one*. A *regular octahedron* is a solid figure with eight faces, each of which is an equilateral triangle. You can think of gluing two square pyramids together along the square bases. The symbol $\lfloor u \rfloor$ denotes the greatest integer that does not exceed u.

An *acute triangle* has all of its angles less than 90°. The *orthocentre* of a triangle is the intersection point of its altitudes. Points are *collinear* iff they lie on a straight line.

For any real number x, $\lfloor x \rfloor$ (the *floor* of x) is equal to the greatest integer that is less than or equal to x.

1. Let M be a set of eleven points consisting of the four vertices along with seven interior points of a square of unit area.

   (a) Prove that there are three of these points that are vertices of a triangle whose area is at most 1/16.

   (b) Give an example of a set M for which no four of the interior points are collinear and each nondegenerate triangle formed by three of them has area at least 1/16.

2. Let a, b, c be the lengths of the sides of a triangle. Suppose that $u = a^2 + b^2 + c^2$ and $v = (a + b + c)^2$. Prove that
$$\frac{1}{3} \leq \frac{u}{v} < \frac{1}{2}$$
   and that the fraction 1/2 on the right cannot be replaced by a smaller number.

3. Suppose that $f(x)$ is a function satisfying
$$|f(m + n) - f(m)| \leq \frac{n}{m}$$
   for all rational numbers n and m. Show that, for all natural numbers k,
$$\sum_{i=1}^{k} |f(2^k) - f(2^i)| \leq \frac{k(k-1)}{2} \ .$$

4. Is it true that any pair of triangles sharing a common angle, inradius and circumradius must be congruent?

5. Each point of the plane is coloured with one of 2000 different colours. Prove that there exists a rectangle all of whose vertices have the same colour.

6. Let n be a positive integer, P be a set of n primes and M a set of at least $n+1$ natural numbers, each of which is divisible by no primes other than those belonging to P. Prove that there is a nonvoid subset of M, the product of whose elements is a square integer.

7. Let
$$S = \frac{1^2}{1\cdot 3} + \frac{2^2}{3\cdot 5} + \frac{3^2}{5\cdot 7} + \cdots + \frac{500^2}{999\cdot 1001} .$$
Find the value of S.

8. The sequences $\{a_n\}$ and $\{b_n\}$ are such that, for every positive integer n,
$$a_n > 0 , \quad b_n > 0 , \quad a_{n+1} = a_n + \frac{1}{b_n} , \quad b_{n+1} = b_n + \frac{1}{a_n} .$$
Prove that $a_{50} + b_{50} > 20$.

9. There are six points in the plane. Any three of them are vertices of a triangle whose sides are of different length. Prove that there exists a triangle whose smallest side is the largest side of another triangle.

10. In a rectangle, whose sides are 20 and 25 units of length, are placed 120 squares of side 1 unit of length. Prove that a circle of diameter 1 unit can be placed in the rectangle, so that it has no common points with the squares.

11. Each of nine lines divides a square into two quadrilaterals, such that the ratio of their area is 2:3. Prove that at least three of these lines are concurrent.

12. Each vertex of a regular 100-sided polygon is marked with a number chosen from among the natural numbers $1, 2, 3, \cdots, 49$. Prove that there are four vertices (which we can denote as A, B, C, D with respective numbers a, b, c, d) such that $ABCD$ is a rectangle, the points A and B are two adjacent vertices of the rectangle and $a + b = c + d$.

13. Suppose that $x_1, x_2, \cdots, x_n$ are nonnegative real numbers for which $x_1 + x_2 + \cdots + x_n < \frac{1}{2}$. Prove that
$$(1 - x_1)(1 - x_2)\cdots(1 - x_n) > \frac{1}{2} ,$$

14. Given a convex quadrilateral, is it always possible to determine a point in its interior such that the four line segments joining the point to the midpoints of the sides divide the quadrilateral into four regions of equal area? If such a point exists, is it unique?

15. Determine all triples (x, y, z) of real numbers for which
$$x(y + 1) = y(z + 1) = z(x + 1) .$$

16. Suppose that $ABCDEZ$ is a regular octahedron whose pairs of opposite vertices are (A, Z), (B, D) and (C, E). The points F, G, H are chosen on the segments AB, AC, AD respectively such that $AF = AG = AH$.

(a) Show that EF and DG must intersect in a point K, and that BG and EH must intersect in a point L.

(b) Let EG meet the plane of AKL in M. Show that $AKML$ is a square.

17. Suppose that r is a real number. Define the sequence x_n recursively by $x_0 = 0$, $x_1 = 1$, $x_{n+2} = rx_{n+1} - x_n$ for $n \geq 0$. For which values of r is it true that

$$x_1 + x_3 + x_5 + \cdots + x_{2m-1} = x_m^2$$

for $m = 1, 2, 3, 4, \cdots$.

18. Let a and b be integers. How many solutions in real pairs (x, y) does the system

$$\lfloor x \rfloor + 2y = a$$

$$\lfloor y \rfloor + 2x = b$$

have?

19. Is it possible to divide the natural numbers $1, 2, \cdots, n$ into two groups, such that the squares of the members in each group have the same sum, if (a) $n = 40000$; (b) $n = 40002$? Explain your answer.

20. Given any six irrational numbers, prove that there are always three of them, say a, b, c, for which $a+b$, $b+c$ and $c+a$ are irrational.

21. The natural numbers x_1, x_2, $\cdots$, x_{100} are such that

$$\frac{1}{\sqrt{x_1}} + \frac{1}{\sqrt{x_2}} + \cdots + \frac{1}{\sqrt{x_{100}}} = 20\ .$$

Prove that at least two of the numbers are equal.

22. Let **R** be a rectangle with dimensions 11×12. Find the least natural number n for which it is possible to cover **R** with n rectangles, each of size 1×6 or 1×7, with no two of these having a common interior point.

23. Given 21 points on the circumference of a circle, prove that at least 100 of the arcs determined by pairs of these points subtend an angle not exceeding $120°$ at the centre.

24. ABC is an acute triangle with orthocentre H. Denote by M and N the midpoints of the respective segments AB and CH, and by P the intersection point of the bisectors of angles CAH and CBH. Prove that the points M, N and P are collinear.

25. Let a, b, c be non-negative numbers such that $a + b + c = 1$. Prove that

$$\frac{ab}{c+1} + \frac{bc}{a+1} + \frac{ca}{b+1} \leq \frac{1}{4}\ .$$

When does equality hold?

26. Each of m cards is labelled by one of the numbers $1, 2, \cdots, m$. Prove that, if the sum of labels of any subset of cards is not a multiple of $m + 1$, then each card is labelled by the same number.

27. Find the least number of the form $|36^m - 5^n|$ where m and n are positive integers.

28. Let A be a finite set of real numbers which contains at least two elements and let $f : A \longrightarrow A$ be a function such that $|f(x) - f(y)| < |x - y|$ for every $x, y \in A$, $x \neq y$. Prove that there is $a \in A$ for which $f(a) = a$. Does the result remain valid if A is not a finite set?

29. Let A be a nonempty set of positive integers such that if $a \in A$, then $4a$ and $\lfloor \sqrt{a} \rfloor$ both belong to A. Prove that A is the set of all positive integers.

30. Find a point M within a regular pentagon for which the sum of its distances to the vertices is minimum.

31. Let x, y, z be positive real numbers for which $x^2 + y^2 + z^2 = 1$. Find the minimum value of

$$S = \frac{xy}{z} + \frac{yz}{x} + \frac{zx}{y} .$$

32. The segments BE and CF are altitudes of the acute triangle ABC, where E and F are points on the segments AC and AB, respectively. ABC is inscribed in the circle $\mathbf{Q}$ with centre O. Denote the orthocentre of ABC by H, and the midpoints of BC and AH be M and K, respectively. Let $\angle CAB = 45°$.

(a) Prove, that the quadrilateral $MEKF$ is a square.

(b) Prove that the midpoint of both diagonals of $MEKF$ is also the midpoint of the segment OH.

(c) Find the length of EF, if the radius of $\mathbf{Q}$ has length 1 unit.

33. Prove the inequality $a^2 + b^2 + c^2 + 2abc < 2$, if the numbers a, b, c are the lengths of the sides of a triangle with perimeter 2.

34. Each of the edges of a cube is 1 unit in length, and is divided by two points into three equal parts. Denote by $\mathbf{K}$ the solid with vertices at these points.

(a) Find the volume of $\mathbf{K}$.

(b) Every pair of vertices of $\mathbf{K}$ is connected by a segment. Some of the segments are coloured. Prove that it is always possible to find two vertices which are endpoints of the same number of coloured segments.

35. There are n points on a circle whose radius is 1 unit. What is the greatest number of segments between two of them, whose length exceeds $\sqrt{3}$?

36. Prove that there are not three rational numbers x, y, z such that

$$x^2 + y^2 + z^2 + 3(x + y + z) + 5 = 0 .$$

37. Let ABC be a triangle with sides a, b, c, inradius r and circumradius R (using the conventional notation). Prove that

$$\frac{r}{2R} \le \frac{abc}{\sqrt{2(a^2+b^2)(b^2+c^2)(c^2+a^2)}} .$$

When does equality hold?

38. Let us say that a set S of nonnegative real numbers if *hunky-dory* if and only if, for all x and y in S, either $x + y$ or $|x - y|$ is in S. For instance, if r is positive and n is a natural number, then $S(n, r) = \{0, r, 2r, \cdots, nr\}$ is hunky-dory. Show that every hunky-dory set with finitely many elements is $\{0\}$, is of the form $S(n, r)$ or has exactly four elements.

39. (a) $ABCDEF$ is a convex hexagon, each of whose diagonals AD, BE and CF pass through a common point. Must each of these diagonals bisect the area?

(b) $ABCDEF$ is a convex hexagon, each of whose diagonals AD, BE and CF bisects the area (so that half the area of the hexagon lies on either side of the diagonal). Must the three diagonals pass through a common point?

40. Determine all solutions in integer pairs (x, y) to the diophantine equation $x^2 = 1 + 4y^3(y + 2)$.

41. Determine the least positive number p for which there exists a positive number q such that

$$\sqrt{1+x} + \sqrt{1-x} \le 2 - \frac{x^p}{q}$$

for $0 \le x \le 1$. For this least value of p, what is the smallest value of q for which the inequality is satisfied for $0 \le x \le 1$?

42. G is a connected graph; that is, it consists of a number of vertices, some pairs of which are joined by edges, and, for any two vertices, one can travel from one to another along a chain of edges. We call two vertices *adjacent* if and only if they are endpoints of the same edge. Suppose there is associated with each vertex v a nonnegative integer $f(v)$ such that all of the following hold:
(1) If v and w are adjacent, then $|f(v) - f(w)| \le 1$.
(2) If $f(v) > 0$, then v is adjacent to at least one vertex w such that $f(w) < f(v)$.
(3) There is exactly one vertex u such that $f(u) = 0$.
Prove that $f(v)$ is the number of edges in the chain with the fewest edges connecting u and v.

43. Two players pay a game: the first player thinks of n integers $x_1, x_2, \cdots, x_n$, each with one digit, and the second player selects some numbers $a_1, a_2, \cdots, a_n$ and asks what is the value of the sum $a_1x_1 + a_2x_2 + \cdots + a_nx_n$. What is the minimum number of questions used by the second player to find the integers $x_1, x_2, \cdots, x_n$?

44. Find the permutation $\{a_1, a_2, \cdots, a_n\}$ of the set $\{1, 2, \cdots, n\}$ for which the sum

$$S = |a_2 - a_1| + |a_3 - a_2| + \cdots + |a_n - a_{n-1}|$$

has maximum value.

45. Prove that there is no polynomial $p(x) = a_nx^n + a_{n-1}x^{n-1} + \cdots + a_0$ with integer coefficients a_i for which $p(m)$ is a prime number for every integer m.

46. Let $a_1 = 2$, $a_{n+1} = \frac{a_n+2}{1-2a_n}$ for $n = 1, 2, \cdots$. Prove that

(a) $a_n \ne 0$ for each positive integer n;

(b) there is no integer $p \ge 1$ for which $a_{n+p} = a_n$ for every integer $n \ge 1$ (*i.e.*, the sequence is not periodic).

47. Let $a_1, a_2, \cdots, a_n$ be positive real numbers such that $a_1a_2 \cdots a_n = 1$. Prove that

$$\sum_{k=1}^{n} \frac{1}{s - a_k} \le 1$$

where $s = 1 + a_1 + a_2 + \cdots + a_n$.

48. Let $A_1A_2 \cdots A_n$ be a regular $n-$gon and d an arbitrary line. The parallels through A_i to d intersect its circumcircle respectively at B_i $(i = 1, 2, \cdots, n$. Prove that the sum

$$S = |A_1B_1|^2 + \cdots + |A_nB_n|^2$$

is independent of d.

49. Find all ordered pairs (x, y) that are solutions of the following system of two equations (where a is a parameter):

$$x - y = 2$$
$$\left(x - \frac{2}{a}\right)\left(y - \frac{2}{a}\right) = a^2 - 1 \ .$$

Find all values of the parameter a for which the solutions of the system are two pairs of nonnegative numbers. Find the minimum value of $x + y$ for these values of a.

50. Let n be a natural number exceeding 1, and let A_n be the set of *all* natural numbers that are not relatively prime with n (*i.e.*, $A_n = \{x \in \mathbf{N} : \ \gcd(x, n) \ne 1\}$. Let us call the number n *magic* if for each two numbers $x, y \in A_n$, their sum $x + y$ is also an element of A_n (*i.e.*, $x + y \in A_n$ for $x, y \in A_n$).

(a) Prove that 67 is a magic number.

(b) Prove that 2001 is **not** a magic number.

(c) Find all magic numbers.

51. In the triangle ABC, $AB = 15$, $BC = 13$ and $AC = 12$. Prove that, for this triangle, the angle bisector from A, the median from B and the altitude from C are concurrent (*i.e.*, meet in a common point).

52. One solution of the equation $2x^3 + ax^2 + bx + 8 = 0$ is $1 + \sqrt{3}$. Given that a and b are rational numbers, determine its other two solutions.

53. Prove that among any 17 natural numbers chosen from the sets $\{1, 2, 3, \cdots, 24, 25\}$, it is always possible to find two whose product is a perfect square.

54. A circle has exactly one common point with each of the sides of a $(2n+1)-$sided polygon. None of the vertices of the polygon is a point of the circle. Prove that at least one of the sides is a tangent of the circle.

Solutions

1. Let M be a set of eleven points consisting of the four vertices along with seven interior points of a square of unit area.

(a) Prove that there are three of these points that are vertices of a triangle whose area is at most $1/16$.

(b) Give an example of a set M for which no four of the interior points are collinear and each nondegenerate triangle formed by three of them has area at least $1/16$.

Solution. (a) We begin by covering the square with non-overlapping triangles whose vertices are found among the eleven points. Begin by drawing one of the diagonals of the square. We then select the remaining seven points in turn. Suppose, as an induction hypothesis, that we have selected $k \geq 0$ points and covered the square with $2(k+1)$ triangles whose vertices are among the four vertices of the square and the k points already selected. Consider the next point. If it is in the interior of an existing triangle, join it to each of the three vertices of the triangle. If it is in the interior of an edge common to two triangles, join it to the remaining vertex of each of the triangles. In each case, we have two more triangles than before, for a total of $2(k+3)$ triangles. When all seven interior points have been selected, we have a total of $2 \times 8 = 16$ triangles. The total area of these sixteen nonoverlapping triangles is 1, so at least one of them must have area not exceeding $1/16$.

(b) Let the square have vertices at $(0,0)$, $(1,0)$, $(0,1)$, $(1,1)$ and let the interior points be at $(\frac{1}{8}, \frac{5}{8})$, $(\frac{2}{8}, \frac{2}{8})$, $(\frac{3}{8}, \frac{7}{8})$, $(\frac{4}{8}, \frac{4}{8})$, $(\frac{5}{8}, \frac{1}{8})$, $(\frac{6}{8}. \frac{6}{8})$ and $(\frac{7}{8}, \frac{3}{8})$. There is a triangulation for which each of the triangles has area exactly equal to $1/16$. (*Exercise:* produce the diagram.) Any other triangle determined by three of the points will have area at least as large as some triangle in the triangulation.

Comment. (a) Most students realized that one had only to look at the triangles involved in some triangulation of the square. There are two issues that need to be addressed: that such a triangulation exists, and the number of triangles obtained. This was neglected by some solvers. The induction argument above looks after both of these issues. Given that there is a triangulation, the number of triangles can be counted in another way. Suppose that there are N triangles. Then the total of all of the angles of the triangles is $2N$ right angles. The angles of the triangles at each corner points of the square total to one right angle, while at the interior seven points of the square total to four right angles, for a total of $4 + 4 \times 7 = 32$ right angles. Since $32 = 2N$, there are 16 triangles.

(b) Most people did not produce an example. The ones who did either transgressed the condition about not having too many collinear points, or else did not ensure that the area of *every* triangle, even those not involved in the triangulation, was at least 1/16.

2. Let a, b, c be the lengths of the sides of a triangle. Suppose that $u = a^2 + b^2 + c^2$ and $v = (a + b + c)^2$. Prove that
$$\frac{1}{3} \leq \frac{u}{v} < \frac{1}{2}$$
and that the fraction 1/2 on the right cannot be replaced by a smaller number.

Solution. The numerator of the difference $\frac{1}{2} - \frac{u}{v}$ is equal to

$$\begin{aligned} v - 2u &= 2(ab + bc + ca) - (a^2 + b^2 + c^2) \\ &= a(b + c - a) + b(c + a - b) + c(a + b - c) \ . \end{aligned}$$

By the triangle inequality, $a < b + c$, $b < c + a$ and $c < a + b$, so that the right side is always positive. Since all variables are positive, the right inequality follows.

The numerator of $\frac{u}{v} - \frac{1}{3}$ is equal to

$$\begin{aligned} 3u - v &= 2(a^2 + b^2 + c^2 - ab - bc - ca) \\ &= (a - b)^2 + (b - c)^2 + (c - a)^2 \ , \end{aligned}$$

The right side, being a sum of squares, is nonnegative and it vanishes if and only if $a = b = c$. The left inequality follows.

To see that u/v can be arbitrarily close to 1/2, let $(a, b, c) = (\epsilon, 1, 1)$ where $0 < \epsilon < 4$. Then

$$\frac{1}{2} - \frac{u}{v} = \frac{\epsilon(4 - \epsilon)}{(2 + \epsilon)^2} \ .$$

This can be made as close to 0 as desired by taking ϵ sufficiently close to 0.

Coment. The inequality $v \leq 3u$ can be obtained by applying the Cauchy-Schwarz Inequality (try it!). Robert Barrington Leigh found it convenient to look at the reciprocal v/u. In showing that this could be as close as desired to 1/2, note that, when $(a, b, c) = (\epsilon, 1, 1)$,

$$2 < \frac{v}{u} = 1 + \frac{2(ab + bc + ca)}{a^2 + b^2 + c^2} = 1 + \frac{4\epsilon + 2}{\epsilon + 2} < 1 + \frac{4\epsilon + 2}{2} = 2 + 2\epsilon \ .$$

Some students tried to argue the inequality by looking at extreme situations, without attempting to draw a relationship between the extreme and general situations. Others tried a kind of variation argument, seeming to feel that since we can change the variables to a situation where the inequality holds, it will hold on the way there. While it is often useful to think in terms of extreme cases and variation of the unknowns in an inequality problem in order to get a handle on the situation, this approach generally becomes completely unmanageable in the write-up. It is best to think of the variables as being fixed at certain values and perhaps try to compare the value of a function to a purported extreme value. In many inequalities, including this one, it is often conceptually cleanest to look at the difference between two sides; this difference can usually be manipulated into a form which is clearly positive or negative, in a way in which the reader can easily follow the steps. If you ever depart from this format, you should have a good reason for doing so. Avoid in the write-up starting with what you have to prove and working backwards; this makes the logic harder to follow and could lead to trouble with steps that are not reversible. *Begin with what you know and proceed by logical steps to what has to be obtained.*

3. Suppose that $f(x)$ is a function satisfying

$$|f(m + n) - f(m)| \leq \frac{n}{m}$$

for all positive rational numbers n and m. Show that, for all natural numbers k,

$$\sum_{i=1}^{k} |f(2^k) - f(2^i)| \le \frac{k(k-1)}{2} \ .$$

Solution. Taking the case $m = n = 2^i$, we find that

$$|f(2^{i+1}) - f(2^i)| \le 1$$

for each nonnegative integer i. Hence

$$|f(2^k) - f(2^i)| \le |f(2^k) - f(2^{k-1})| + |f(2^{k-1}) + f(2^{k-2})| + \cdots + |f(2^{i+1} - f(2^i)| \le k - i \ .$$

so that

$$\sum_{i=1}^{k} |f(2^k) - f(2^i)| \le \sum_{i=1}^{k} (k-i) = \sum_{j=0}^{k-1} j = \frac{k(k-1)}{2} \ .$$

Comment. One student picked up that the condition does not make sense when m/n is negative, so I have corrected the statement of the problem. Others (including myself) either rolled with the context or corrected the statement to something reasonable. It does happen that a problem can be misstated on a competition; if you feel that this has happened, you should draw attention to the mistake and make a reasonable nontrivial interpretation of the problem and solve that.

This problem provided a nice illustration of the maxim that a little knowledge is a dangerous thing. Students who had seen some calculus thought that somehow a derivative was involved, and then were lost forever beneath the waves. The context does not suggest a place for calculus. Calculus deals with functions defined on the real numbers (not just the rationals) and possessing derivatives. The most important thing for you to learn about calculus is when not to use it. More seriously, turning to calculus right off the bat inhibits the ability to take the problem on its own terms and come up with a more elementary solution. Resist the urge to categorize problems too quickly. Once the key idea of looking at consecutive powers of 2 is found, the solution then falls right out.

4. Is it true that any pair of triangles sharing a common angle, inradius and circumradius must be congruent?

Solution. Yes. Let ABC be the triangle, with sides a, b, c, circumradius R, inradius r, semiperimeter s, area Δ and u, v, w the respective lengths of the tangents to the incircle from vertices A, B, C. We are given that angle A, and radii r and R are fixed. Then also fixed are $a = 2R \sin A$, $b + c - a = u = r \cot \frac{A}{2}$, $b + c = u + a$, $s = (u + 2a)/2$ and $\Delta = rs$. But $\Delta = \frac{1}{2} bc \sin A$, so we find that bc and $b + c$ are both fixed. Now b and c are the uniquely determined roots of a quadratic equation (whose coefficients are fixed by the sum and product of the roots), and the result follows.

Comment. There is a natural dynamical way of looking at the situation which is hard to capture in a rigorous argument. We can imagine a fixed circumcircle with A fixed as one point on its circumference. Let the angle at A be fixed, but let its arms vary within the circumcircle. Since we know the radius of the incircle enveloped by the arms, this incircle is a fixed distance from A (why?); as we move the arms around, we can convince ourselves that there are two positions (giving congruent triangles) where the incircle will be tangent to the chord of the circumcircle determined by the arms of the angle at A. Nailing all this down is not so easy, and seems to need at the least a continuity argument, which takes us out of the realm of pure geometry into that of analysis.

5. Each point of the plane is coloured with one of 2000 different colours. Prove that there exists a rectangle all of whose vertices have the same colour.

Solution. Let $N = 1000$ and let S consist of points (x, y) with integer coordinates for which $0 \leq x \leq N, 0 \leq y$. Each row $R_y := \{(x, y) \in S : 0 \leq x \leq N\}$ has $N + 1$ points, so that by the Pigeonhole Principle, there is at least one colour used twice. There are $N\binom{N+1}{2}$ possible ways in which a colour can be used in two positions on R_y. Since there are more than this many rows R_y, the same colour must occur in the same two positions in two of these rows. The four points in these two positions on the two rows determine the desired rectangle.

Comment. Many students essentially had this pigeonhole argument, which was set up in a variety of ways.

6. Let n be a positive integer, P be a set of n primes and M a set of at least $n+1$ natural numbers, each of which is divisible by no primes other than those belonging to P. Prove that there is a nonvoid subset of M, the product of whose elements is a square integer.

Solution. Let P consist of the distinct primes $p_1, p_2, \cdots, p_n$. Let S be a nonvoid subset of M, and write the product of the numbers in S in the form $x = y^2 z$, where y^2 is the largest square divisor of x. Then z must have the form

$$p_1^{a_1} p_2^{a_2} \cdots p_n^{a_n}$$

where the vector $(a_1, a_2, \cdots, a_n)$ has all of its entries equal to either 0 or 1. There are 2^n possibilities for this vector. However, there are at least $2^{n+1} - 1 > 2^n$ nonvoid subsets of M and so that many possible products x. Hence there are two such products, $x = y^2 z$ and $u = v^2 w$ which give rise to the same vector. It may transpire that both x and u have factors from M in common; divide through by the product of these factors to obtain products of disjoint subsets P and Q of M. These two products will have the same vector, and so must have the form $r^2 t$ and $s^2 t$, where t is a product of distinct primes from P. The product of the numbers in $P \cup Q$ is $r^2 s^2 t^2$, and this is the desired square.

7. Let

$$S = \frac{1^2}{1 \cdot 3} + \frac{2^2}{3 \cdot 5} + \frac{3^2}{5 \cdot 7} + \cdots + \frac{500^2}{999 \cdot 1001} \ .$$

Find the value of S.

Solution 1. [Miranda Holmes] Each term of S is of the form

$$\frac{i^2}{(2i-1)(2i+1)} = \frac{1}{4}\left(\frac{i}{2i-1} + \frac{i}{2i+1}\right) \ .$$

Thus

$$S = \frac{1}{4}\sum_{i=1}^{500}\left(\frac{i}{2i-1} + \frac{i}{2i+1}\right) \ .$$

$$4S = \sum_{i=1}^{500}\left(\frac{i}{2i-1} + \frac{i}{2i+1}\right) = \frac{1}{1} + \frac{1}{3} + \frac{2}{3} + \frac{2}{5} + \frac{3}{5} + \frac{3}{7} + \cdots + \frac{500}{999} + \frac{500}{1001} \ .$$

Since

$$\frac{i}{2i+1} + \frac{i+1}{2i+1} = 1 \ ,$$

we can combine all such terms to get

$$4S = 1 + 499 + \frac{500}{1001} = \frac{501000}{1001} \ .$$

Thus,

$$S = \frac{125250}{1001} \ .$$

Comment. If you can guess at the formula for the sum of the series to n terms, it takes a straightforward induction argument to establish that

$$\sum_{i=1}^{n} \frac{i^2}{(2i+1)(2i-1)} = \frac{n(n+1)}{2(2n+1)} \ .$$

Solution 2. [Samer Seraj] We have that

$$\begin{aligned}
\sum_{i=1}^{n} \frac{i^2}{(2i-1)(2i+1)} &= \frac{1}{2}\sum_{i=1}^{n}\left[\frac{i^2}{2i-1} - \frac{i^2}{2i+1}\right] \\
&= \frac{1}{2}\left[1 + \sum_{i=1}^{n-1}\left(-\frac{i^2}{2i+1} + \frac{(i+1)^2}{2(i+1)-1}\right) - \frac{n^2}{2n+1}\right] \\
&= \frac{1}{2}\left[1 + \sum_{i=1}^{n-1} 1 - \frac{n^2}{2n+1}\right] = \frac{1}{2}\left[n - \frac{n^2}{2n+1}\right] = \frac{n(n+1)}{2(2n+1} \ ,
\end{aligned}$$

When $n = 500$, we get the answer $125250/1001$.

8. The sequences $\{a_n\}$ and $\{b_n\}$ are such that, for every positive integer n,

$$a_n > 0 \ , \quad b_n > 0 \ , \quad a_{n+1} = a_n + \frac{1}{b_n} \ , \quad b_{n+1} = b_n + \frac{1}{a_n} \ .$$

Prove that $a_{50} + b_{50} > 20$.

Solution 1. Note that $x + (1/x) \geq 2$ for every positive real x. Consider the sequence $c_n = (a_n + b_n)^2$. Since

$$c_2 = (a_2 + b_2)^2 = \left(a_1 + \frac{1}{b_1} + b_1 + \frac{1}{a_1}\right)^2 = \left[\left(a_1 + \frac{1}{a_1}\right) + \left(b_1 + \frac{1}{b_1}\right)\right]^2 \ ,$$

we find that $c_2 \geq (2+2)^2 = 16$.

For each positive integer n,

$$\begin{aligned}
c_{n+1} = (a_{n+1} + b_{n+1})^2 &= \left(a_n + \frac{1}{b_n} + b_n + \frac{1}{a_n}\right)^2 \\
&= [(a_n^2 + b_n^2 + 2a_nb_n) + \left(\frac{1}{a_n^2} + \frac{1}{b_n^2} + 2\cdot\frac{1}{a_n}\cdot\frac{1}{b_n}\right) \\
&\qquad + 2\cdot a_n \cdot \frac{1}{a_n} + 2\cdot b_n\cdot\frac{1}{b_n} + 2\left(\frac{a_n}{b_n} + \frac{b_n}{a_n}\right) \\
&= c_n + \left(\frac{1}{b_n} + \frac{1}{a_n}\right)^2 + 2 + 2 + 2\cdot\left(\frac{a_n}{b_n} + \frac{b_n}{a_n}\right) > c_n + 8 \ ,
\end{aligned}$$

Thus, $c_{n+1} > c_n + 8$. It follows that

$$c_{50} > c_{49} + 8 > c_{48} + 2\cdot 8 > \cdots > c_2 + 48\cdot 8 \geq 16 + 48\cdot 8 = 400 \ .$$

Since $a_{50} + b_{50} > 0$, it follows that $a_{50} + b_{50} > 20$.

Solution 2. [F. Tian] For $n \geq 1$,

$$a_{n+1}b_{n+1} = a_nb_n + 2 + \frac{1}{a_nb_n} \ .$$

Also, $a_2b_2 = a_1b_1 + 2 + 1/(a_1b_1) \geq 4$. By induction, it can be shown that $a_nb_n > 2n$ for all $n \geq 3$. Hence, by the Airthmetic-Geometric Means Inequality,

$$a_{50} + b_{50} \geq 2\sqrt{a_{50}b_{50}} > 2\sqrt{100} = 20 \ .$$

9. There are six points in the plane, no three of them collinear. Any three of them are vertices of a triangle whose sides are of different length. Prove that there exists a triangle whose smallest side is the largest side of another triangle.

Comment. Before giving the solution to the problem, we present a result that you should be aware of; pay also close attention to the proof.

Proposition. There are six points in the plane or in space, no three of them collinear. Each of the segments between two of them is coloured in one of two colours. There exists a triangle whose vertices are three of the given points and whose sides are of the same colour.

Proof. Let A be one of the points. There are five segments joining A to the other points. Since they have one of two colours, by the Pigeonhole Principle, at least three of them must have the same colour. Wolog, suppose that AB, AC, AD are coloured the same. If any of the segments BD, BC, CD has this colour, then we will have a triangle in this colour. Otherwise, BCD must be a triangle all of whose edges have the other colour.

Note 1: The minimum number of points with such a property is 6. If there are five points, it is possible to colour the segments between any two of them so that a triangle with edges of a single colour does not exist. For example, for a regular pentagon, we can colour all the sides with one colour and all the diagonals with the other.

Note 2: A triangle of one-colour always exists when we have 17 points in the plane (no three collinear) and three colours are used for the segments. This can be given a similar proof. From any point, at least six of the segments emanating from it have the same colour. Now look at the six points terminating these segments.

Now we can solve the problem.

Solution. Consider the six points and all triangles whose vertices are any three of them. Colour the (uniquely determined) largest side of each triangle black, and colour the remaining edges red. There must be a triangle all of whose edges are the same colour. This colour cannot be red. (Why?) So there must be a triangle all of whose edges are black; its smallest edge must be the largest edge of some other triangle.

10. In a rectangle, whose sides are 20 and 25 units of length, are placed 120 squares of side 1 unit of length. Prove that a circle of diameter 1 unit can be placed in the rectangle, so that it has no common points with the squares.

Solution. [M. Holmes] If a circle of diameter 1 can be placed, it means that there must be a point in the rectangle such that every point of every square is more than 1/2 units away from it to the centre of the circle. The maximum area A around each square in which the centre of the circle cannot be located is the area of the figure F formed by

(a) the square;

(b) four rectangles of dimensions $1 \times \frac{1}{2}$ external to the sides of the square;

(c) four quarters of circles with radius 1/2 units external to the square with and centres at the vertices of the square.

Hence, $A = 1 + \frac{1}{2} \cdot 1 \cdot 4 + (\frac{1}{2})^2\pi = 3 + \frac{\pi}{4}$. As there are 120 squares, the sum of all such areas within the rectangle does not exceed $120 \cdot (3 + \frac{\pi}{4}) < 455$.

As the circle should be placed inside of the rectangle, its centre cannot be less than 1/2 units away from the rectangle's sides, *i.e.*, it can be only in the rectangle with sides 19 and 24 units of length, whose sides are parallel to the rectangle's sides on the distance 1/2 units from them. The area of this rectangle is $19 \times 24 = 456$. But $456 - 455 > 0$, so at least one point is not covered by any of the 120 figures F described above.. This point can be the centre of a circle of diameter 1 lying within the rectangle and having no point in common with any of the squares.

11. Each of nine lines divides a square into two quadrilaterals, such that the ratio of their area is 2:3. Prove that at least three of these lines are concurrent.

Solution. [M. Holmes] Since the lines divide the square into two quadrilaterals, they cut opposite sides of the square. Let the vertices of the square be A, B, C, D (counterclockwise), and let one of the lines intersect AB at M and CD at N. We can represent these points in an appropriate coordinate plane as $A(0,0)$, $B(1,0)$, $C(1,1)$, $D(0,1)$, $M(m,0)$, $N(n,1)$.

Let $[AMND] : [MBCN] = 2 : 3$. Then $[AMND] = (m+n)/2 = 2/5$, because the area of the whole square is 1. The midpoint of MN is the point $S(\frac{1}{2}(m+n), \frac{1}{2}) = S(\frac{2}{5}, \frac{1}{2})$, which does not depend on the points of intersection of M and N and, hence, is the same for all such lines. So, each line which divides the square into two quadrilaterals in this way, must go through the point S. Because of the symmetry, there are three other possible points in the square $(\frac{3}{5}, \frac{1}{2})$, $(\frac{1}{2}, \frac{2}{5})$, $(\frac{1}{2}, \frac{3}{5})$, and each of the given 9 points must pass through one of them. Applying the Pigeonhole Principle for 9 lines and 4 points, we find that at least three of the lines must pass through the same point, and because of that, they are concurrent.

12. Each vertex of a regular 100-sided polygon is marked with a number chosen from among the natural numbers $1, 2, 3, \cdots, 49$. Prove that there are four vertices (which we can denote as A, B, C, D with respective numbers a, b, c, d) such that $ABCD$ is a rectangle, the points A and B are two adjacent vertices of the rectangle and $a + b = c + d$.

Solution. Since the given polygon is regular, it can be inscribed in a circle. There are exactly 50 diagonals of the polygon which pass through the centre of the circle. As they are diameters, they are of equal length. Consider the positive differences of two vertices which are endpoints of the same diagonal. Since the mark numbers are from among $1, 2, \cdots, 49$, the range of the differences is between 0 and 48. So, we have 49 possible values for 50 differences. Hence, there are at least two diagonals with the same difference. Without loss of generality, denote these diagonals as as AC and BD and suppose that $a \geq c$ and $d \geq b$. Then $a - c = d - b$, so that $a + b = c + d$. The quadrilateral $ABCD$ has two diagonals of equal length and with the same midpoint, so it is a rectangle, which satisfies all of the required conditions.

13. Suppose that $x_1, x_2, \cdots, x_n$ are nonnegative real numbers for which $x_1 + x_2 + \cdots + x_n < \frac{1}{2}$. Prove that

$$(1 - x_1)(1 - x_2) \cdots (1 - x_n) > \frac{1}{2} ,$$

Comments. Some of you tried a "moving variables" argument, or an "extreme case" argument, *i.e.*, if we make x_i as large as possible, we reduce the product, so that if it works for $(x_1, x_2, \cdots, x_n) = (\frac{1}{2}, 0, 0, \cdots, 0)$, then we are in business. Some felt that if it works for the extreme case with each $x_i = 1/2n$, then we are ok. Unless you back this up with solid argument and detailed analysis that relates the general case to the extreme case, then it is worthless. "Moving variable" arguments are always risky and best avoided. Both approaches often muddy the situation rather than clarify it.

Solution 1. If $0 < u, v < 1$, then

$$(1 - u)(1 - v) = 1 - (u + v) + uv > 1 - (u + v) .$$

It can be shown by induction that

$$(1 - x_1)(1 - x_2) \cdots (1 - x_k) > 1 - (x_1 + x_2 + \cdots + x_k)$$

for $2 \le k \le n$, whence

$$(1 - x_1) \cdots (1 - x_n) > 1 - (x_1 + \cdots + x_n) > \frac{1}{2} \ .$$

Comments. You should carry out the necessary induction argument. Note that the problem asks for $>$ rather than $\ge$.

Solution 2. [R. Barrington Leigh] Let

$$t_k = \sum \{x_{i_1} x_{1_2} \cdots x_{i_k} : 1 \le i_1 < i_2 < \cdots < i_k \le n\}$$

for $1 \le k \le n$. Then

$$(1 - x_1)(1 - x_2) \cdots (1 - x_n) = 1 - t_1 + t_2 - t_3 + t_4 - \cdots + (-1)^n t_n \ .$$

For $1 \le k \le n - 1$, we have that

$$\begin{aligned} t_k &= \sum \{x_{i_1} x_{i_2} \cdots x_{i_k} : 1 \le i_1 < i_2 < \cdots < i_k \le n\} \\ &> \sum \{x_{i_1} x_{i_2} \cdots x_{i_k} (x_{i_k+1} + x_{i_k+2} + \cdots + x_n) : 1 \le i_1 < i_2 < \cdots < i_k \le n\} = t_{k+1} \end{aligned}$$

so that

$$(1 - x_1)(1 - x_2) \cdots (1 - x_n) = (1 - t_1) + (t_2 - t_3) + \cdots > (1 - t_1) = 1 - (x_1 + \cdots + x_n) > \frac{1}{2} \ .$$

14. Given a convex quadrilateral, is it always possible to determine a point in its interior such that the four line segments joining the point to the midpoints of the sides divide the quadrilateral into four regions of equal area? If such a point exists, is it unique?

Note. Let $ABCD$ be the quadrilateral and let P, Q, R, S be the respective midpoints of AB, BC, CD, DA. Recall that $PS \| QR$ and $PQ \| SR$ (prove it!).

Solution 1. Yes, and yes. Observe that $[APS] = \frac{1}{4}[ABD]$ and $[CRQ] = \frac{1}{4}[CDB]$, so that $[APS] + [CRQ] = \frac{1}{4}[ABCD]$ (why?). Since

$$[ABCD] = \frac{1}{2} BD \cdot (\text{dist } (A, BD) + \text{dist } (C, BD))$$

$$[APS] = \frac{1}{4} BD \cdot \text{dist } (A, PS)$$

$$[CRQ] = \frac{1}{4} BD \cdot \text{dist } (C, QR)$$

we have that

$$\begin{aligned} \text{dist } (A, PS) + \text{dist } (C, QR) &= \frac{1}{2}[\text{ dist } (A, BD) + \text{ dist } (C, BD) \\ &= \text{dist}(PS, QR) \ . \end{aligned}$$

Hence it is possible to draw a line m parallel to and between PS and QR for which dist $(m, QR) =$ dist (A, PS) and dist $(m, PS) =$ dist (C, QR). For any point M on m, we have that $[MQR] = [APS]$ and $[MSP] = [CRQ]$, so that $[MQCR] = [APMS] = \frac{1}{4}[ABCD]$.

In a similar way, we can draw a line n parallel to and between PQ and SR for which dist $(n, PQ) =$ dist (D, SR) and dist $(n, SR) =$ dist (B, PQ), so that, for any point N on n, $[NPQ] = [DSR]$, $[NRS] = [BRP]$ and $[NPBQ] = [DSNR] = \frac{1}{4}[ABCD]$.

The lines m and n are not parallel, and, so, intersect in a unique point O for which

$$[OQCR] = [APOS] = [OPBQ] = [DSOR] = \frac{1}{4}[ABCD] \ .$$

Note that m and n intersect inside $PQRS$ and so inside $ABCD$.

We show that O is the only point with this property. Let L be another point inside $ABCD$. Then L lies in one of the four partitioning quadrilaterals, say $[OQCR]$, so that $[LQCR] < [OQCR] = \frac{1}{4}[ABCD]$. Hence, L will satisfy the conditions of the problem.

Comments. A careful solution will contain the observation that m and n are not parallel, and will intersect *within* the quadrilateral. The fact that m and n have a unique point of intersection does not, in and of itself, establish that the requisite point cannot be found in some other way. Therefore, uniqueness needs to be explicitly handled, either by showing that the required point *must* lie on m and n or by some other argument.

Solution 2. `Analysis.` Suppose such a point O exists. Then $[OPBQ] = [OCQR] = \frac{1}{4}[ABCD]$, so that

$$[AOB] = 2[POB] = 2([OPBQ] - [OBQ]) = 2([OQCR] - [OQC]) = 2[ROC] = [DOC] \ .$$

Since $[AOB] = \frac{1}{2}AB \cdot \text{dist } (O, AB)$ and $[DOC] = \frac{1}{2}CD \cdot \text{dist } (O, CD)$, we must have that

$$\frac{\text{dist } (O, AB)}{\text{dist } (O, CD)} = \frac{CD}{AB} \ .$$

The locus of points O with this property is a line h through the intersection of AB and CD (or parallel to them if they do not intersect) which lies between them (prove this!). Similarly, O must lie on a line k defined by

$$\frac{\text{dist } (O, BC)}{\text{dist } (O, AD)} = \frac{AD}{BC}$$

through the intersection of BC and AD (if they intersect) that lies between them. These lines are not parallel and intersect within $ABCD$ (why?). If O exists, it must lie on the intersection of h and k.

`Synthesis.` Construct lines h and k to satisfy the foregoing conditions. They must intersect in a unique point O within $ABCD$. Now

$$[AOB] = \frac{1}{2}AB \cdot \text{dist } (O, AB) = \frac{1}{2}CD \cdot \text{dist } (O, CD) = [DOC] \ .$$

Hence $[AOP] = [POB] = [COR] = [DOR]$. Let α be the common value. Similarly, $[BOQ] = [COQ] = [AOS] = [DOS] = \beta$, say. Then each of $[APOS]$, $[BQOP]$, $[CROQ]$ and $[DSOR]$ has the value $\alpha + \beta$ and O is the desired point.

15. Determine all triples (x, y, z) of real numbers for which

$$x(y+1) = y(z+1) = z(x+1) \ .$$

Comments. In solving a system of equations, one begins by assuming a solution and determining what properties it must have. Since this may involve one-way reasoning, such properties may not necessarily yield a solution and extraneous solutions may arise. Thus, when you have solved the equations, for a complete solution, you should check that the solution is valid.

If, in your manipulations, you divide by a certain quantity or find a quantity in the denominator of a fraction, you should explore the possibility that the quantity could vanish, Remember that you cannot divide by zero. When you write up your final solution, it is often a good idea to deal with this possibility ahead of time and get it out of the way.

Solution 1. Suppose one of the variable, say x, is 0. Then $z(x+1) = 0$, so $z = 0$, and $y(z+1) = 0$ so $y = 0$. Suppose one of the variables, say x is -1. The $x(y+1) = 0$, so $y = -1$ and $y(z+1) = 0$, so $z = -1$. Hence, if any variable assumes either of the values -1 and 0, then all are equal. Henceforth, we assume that none of them have either value.

From the equations, we find that

$$\frac{y(z+1)}{y+1} = x = \frac{yz+y-z}{z}$$

whence

$$yz(z+1) = (y+1)yz + (y+1)(y-z)$$

or

$$yz(z-y) = (y+1)(y-z) \ .$$

Hence, either $y = z$ or $yz + y + 1 = 0$.

If $y = z$, then $z + 1 = x + 1$. so that $z = x$ and $x = y = z$. Conversely, the system is satisfied when $x = y = z$.

Suppose that $yz+y+1 = 0$. Then $x(y+1) = z(x+1) = y(z+1) = -1$, and so $xy+x+1 = xz+z+1 = 0$. Suppose $z = t$, Then $x = -(t+1)/t$ and $y = -1/(t+1)$. Conversely, it is straightforward to check that the system is satisfied by

$$(x, y, z) = \left(-\frac{t+1}{t}, -\frac{1}{1+t}, t \right)$$

where $t \neq 0, -1$. Thus, we have obtained exactly the complete set of solutions.

Comment. The solution with x, y, z unequal may look non-symmetrical. Verify that, if $x = s = -(t+1)/t$, then $y = -1/(1+t) = -(s+1)/s$ and $z = t = -1/(1+s)$. Check that x and z can similarly be expressed in terms of $y = r$.

Solution 2. From the equations, we find that

$$x(y-z) = z - x \ , \qquad y(z-x) = x - y \ , \qquad \text{and} \qquad z(x-y) = y - z \ .$$

Multiplying these equations yields

$$xyz(y-z)(z-x)(x-y) = (z-x)(x-y)(y-z) \ .$$

Hence, either two variables (and so all) are equal or $xyz = 1$. The system is satisfied when all variables are equal.

Suppose that $xyz = 1$. Then

$$xy + x + 1 = xy + x + xyz = x(yz + y + 1)$$

and

$$yz + y + 1 = yz + y + xyz = y(zx + z + 1) \ .$$

Since $xy + x = yz + y = zx + z$, either $x = y = z = 1$ or $xy + x + 1 = yz + y + 1 = zx + z + 1 = 0$. We need to check that there are solutions of this type. Select arbitrary $x \neq 0, -1$, then y to satisfy $xy + x + 1 = 0$ and z to satisfy $xyz = 1$. Then

$$zx + z + 1 = zx + z + xyz = z(xy + x + 1) = 0$$

and

$$yz + y + 1 = yz + y + xyz = y(zx + z + 1) = 0$$

so that $x(y+1) = y(x+1) = z(x+1) = -1$ as desired.

Solution 3. As before, we can check that $(x, y, z) = (0, 0, 0)$ is the only solution in which any variable vanishes. Henceforth, suppose that $xyz \neq 0$. Let $z = vx$. Then $y + 1 = v(x + 1)$, whence $y = vx + v - 1$. Therefore

$$x(vx + v) = x(y + 1) = y(z + 1) = (vx + v - 1)(vx + 1)$$

so that

$$v(v-1)x^2 + v(v-1)x + (v-1) = 0 \ .$$

Either $v = 1$ and we are led to $x = y = z$, which works, or

$$vx^2 + vx + 1 = 0 \ .$$

Hence

$$(x, y, z) = \left(\frac{-v \pm \sqrt{v^2 - 4v}}{2v}, \frac{(v-2) \pm \sqrt{v^2 - 4v}}{2}, \frac{-v \pm \sqrt{v^2 - 4v}}{2} \right) \ .$$

where $v < 0$ or $v \geq 4$. It can be checked that these values satisfy the equations. (`Exercise:` Check that this colution is consistent with the other solutions.)

Solution 4. As in the foregoing solutions, we can check that $(x, y, z) = (0, 0, 0), (-1, -1, -1)$ are the only solutions in which any variable assumes either of the values 0 or -1. Henceforth, suppose x, y, z all differ from 0 and -1.

Let $x(y + 1) = y(z + 1) = z(x + 1) = k$. Then $z = k/(x + 1)$ and $y = (k/x) - 1 = (k - x)/x$. Therefore,

$$\begin{aligned} k = y(z+1) &= \left(\frac{k - x}{x} \right) \left(\frac{k + x + 1}{x + 1} \right) \\ &\Longrightarrow k(x^2 + x) = k^2 - x^2 + k - x = k(k + 1) - (x^2 + x) \\ &\Longrightarrow (k + 1)(x^2 + x - k) = 0 \\ &\Longrightarrow k = -1 \quad \text{or} \quad x^2 + x = k \ . \end{aligned}$$

Suppose that $x^2 + x = k$. Then $x(x + 1) = z(x + 1)$. Since $x \neq -1$, $x = z$, and so $y + 1 = x + 1$. Thus, $x = y = z$. Suppose that $k = -1$. Then $z = -1/(x + 1)$ and $y = -(x + 1)/x$. It can be checked that this works.

Comment. It is not hard to check independently that the only nonzero solution in which x, y, z have the same sign is in fact given by $x = y = z$. For in this case, the ratio of any pair is positive, and the system can be rewritten

$$\begin{aligned} \frac{1}{x} - \frac{1}{y} &= 1 - \frac{z}{x} \\ \frac{1}{z} - \frac{1}{x} &= 1 - \frac{y}{z} \\ \frac{1}{y} - \frac{1}{z} &= 1 - \frac{x}{y} \end{aligned}$$

whence $3 = (x/y) + (y/z) + (z/x)$. By the Arithmetic-Geometric Means Inequality (applicable since the three summands are positive), $(x/y) + (y/z) + (z/x) \geq 3$ with equality if and only if $x/y = y/z = z/x$ or $x = y = z$.

16. Suppose that $ABCDEZ$ is a regular octahedron whose pairs of opposite vertices are (A, Z), (B, D) and (C, E). The points F, G, H are chosen on the segments AB, AC, AD respectively such that $AF = AG = AH$.

(a) Show that EF and DG must intersect in a point K, and that BG and EH must intersect in a point L.

(b) Let EG meet the plane of AKL in M. Show that $AKML$ is a square.

Comment. Many students had complicated arguments involving similar triangles. You should try to envisage the situation in terms of transformations, as this gives you a better sense of what is going on. Of course, if a synthetic argument cannot be found, you always have recourse to the "refuge of the destitute", coordinate geometry and the hope that the computations will not be too horrendous. In this case, they are not bad at all.

Solution 1. (a) Since $AF : AB = AG : GC$, it follows that $FG \| BC \| ED$, while $FG < BC = ED$. Hence, $FGDE$ is a coplanar isosceles trapezoid and so EF and DG must intersect in a point K. A $90°$ rotation about the axis AZ takes $B \to C$, $F \to G$, $C \to D$, $G \to H$, $D \to E$, $E \to B$. Hence $EF \to BG$ and $DG \to EH$, so that BG and EH must intersect in a point L, which is the image of K under the rotation.

(b) KE and AB intersect in F, so that the two lines are coplanar. Also $KF : KE = FG : ED = FG : BC = AF : FB$ so that $\Delta KAF \sim \Delta EFB$ and $AK \| EB$. Hence K lies in a plane through A parallel to $BCDE$. Because the $90°$ rotation about the axis AZ (which is perpendicular to the planes $BCDE$ and AKL takes $K \to L$, $AK = AL$ and $\angle KAL = 90°$.

Consider a dilation with centre E and factor $|AB|/|FB|$. Let I be on AE with $AI = AF$. The the dilation takes $F \to K$, $H \to L$, $I \to A$ and the plane of $FGHI$ to the parallel plane AKL. The image of G under this dilation is the intersection of EG and the plane of AKL, namely M. Thus the square $FGHI$ goes to $KMLA$ which must also be a square.

Solution 2. The dilation-reflection perpendicular to a plane through FG perpendicular to BE and CD with factor $|AF|/|FB|$ takes $B \to E$, $C \to D$, and fixes F and G. The lines BF and CG with intersection A gets carried to lines EF and DG which intersect in a point K for which AK is perpendicular to the plane and so parallel to BE and CD, and the distance from K to the plane is $|AF|/|FB|$ times the distance from A to the plane.

Similarly, considering a dilation-reflection to a plane through GH perpendicular to BC and DE with the same factor produces the point L with AL perpendicular to this plane and so parallel to BC and ED. Thus $\angle KAL = 90°$.

The reflection in the plane $AECZ$ fixes A, C, E, G and interchanges the points in each of the pairs (B, D) and (F, H). Hence the line pairs (EF, DG) and (EH, BG) are interchanged as is the pair K and L. Thus $AK = AL$ and KL is perpendicular to $AECZ$. The triangle AKL is in a plane through A parallel to $BCDE$. The proof that $AKML$ is a square can be completed as in Solution 1.

Solution 3. Assign solid coordinates: $A \sim (0, 0, 1)$, $B \sim (0, -1, 0)$, $C \sim (1, 0, 0)$, $D \sim (0, 1, 0)$, $E \sim (-1, 0, 0)$. For some t with $0 < t < 1$, $F \sim (0, -(1-t), t)$, $G \sim (1-t, 0, t)$, $H \sim (0, 1-t, t)$. The line EF consists of points $(-u, -(1-u)(1-t), (1-u)t)$, with real u and the line DG consists of points $((1-v)(1-t), v, (1-v)t)$ with real v. They meet in $K \sim ((1-t)/t, (t-1)/t, 1)$. Similarly, $L \sim ((1-t)/t, (1-t)/t, 1)$, so A, K, L lie on the plane $z = 1$. The line EG consists of points $(-w + (1-t)(1-w), 0, t(1-w))$ with real w, and this intersects the plane $z = 1$ in the point $(2(1-t)/t, 0, 1)$. The desired result can now be verified.

17. Suppose that r is a real number. Define the sequence x_n recursively by $x_0 = 0$, $x_1 = 1$, $x_{n+2} = rx_{n+1} - x_n$ for $n \geq 0$. For which values of r is it true that

$$x_1 + x_3 + x_5 + \cdots + x_{2m-1} = x_m^2$$

for $m = 1, 2, 3, 4, \cdots$.

Answer. The property holds for all values of r.

Solution 1. Define $x_{-1} = -1$; the recurrence still holds with this extension of the index. Note that, for $n \geq 0$,

$$\begin{aligned}(x_{n+1}^2 - x_n^2) - (x_n x_{n+2} - x_{n-1}x_{n+1}) &= x_{n+1}(x_{n+1} + x_{n-1}) - x_n(x_n + x_{n+2}) \\ &= x_{n+1}(rx_n) - x_n(rx_{n+1}) = 0\end{aligned}$$

so that $x_{n+1}^2 - x_n^2 = x_n x_{n+2} - x_{n-1}x_{n+1}$.

We prove by induction that for each nonnegative integer m,

$$x_{2m} = x_m(x_{m+1} - x_{m-1})$$

$$x_{2m+1} = x_{m+1}^2 - x_m^2 = x_m x_{m+2} - x_{m-1}x_{m+1} \ .$$

These equations check out for $m = 0$. Suppose they hold for $m = k$. Then

$$\begin{aligned}x_{2k+2} = rx_{2k+1} - x_{2k} &= r(x_{k+1}^2 - x_k^2) - x_k(x_{k+1} - x_{k-1}) \\ &= x_{k+1}(rx_{k+1} - x_k) - x_k(rx_k - x_{k-1}) \\ &= x_{k+1}x_{k+2} - x_k x_{k+1} = x_{k+1}(x_{k+2} - x_k)\end{aligned}$$

$$\begin{aligned}x_{2k+3} &= rx_{2k+2} - x_{2k+1} = r(x_{k+1}x_{k+2} - x_{k+1}x_k) - (x_{k+1}^2 - x_k^2) \\ &= x_{k+1}(rx_{k+2} - x_{k+1}) - x_k(rx_{k+1} - x_k) \\ &= x_{k+1}x_{k+3} - x_k x_{k+2} = x_{k+2}^2 - x_{k+1}^2\end{aligned}$$

so the result holds for $m = k + 1$.

Hence

$$x_1 + x_3 + \cdots + x_{2m-1} = (x_1^2 - x_0^2) + (x_2^2 - x_1^2) + \cdots + (x_m^2 - x_{m-1}^2) = x_m^2$$

as desired.

Solution 2. [R. Mong] We prove by induction that, for $m \geq 1$,

$$x_1 + x_3 + \cdots + x_{2m-1} = x_m^2$$

$$x_2 + x_4 + \cdots + x_{2m} = x_m x_{m+1} \ .$$

These hold for $m = 1$. Assume they hold for $1 \leq k \leq m$. Then

$$\begin{aligned}x_1 + x_3 + \cdots + x_{2k+1} &= 1 + (rx_2 - x_1) + (rx_4 - x_3) + \cdots + (rx_{2k} - x_{2k-1}) \\ &= 1 + r[x_2 + x_4 + \cdots + x_{2k}] - [1 + r(x_2 + x_4 + \cdots + x_{2k-2}) - (x_1 + x_3 + \cdots + x_{2k-3})] \\ &= r(x_k x_{k+1} - x_{k-1}x_k) + x_{k-1}^2 = rx_k x_{k+1} - x_{k-1}(rx_k - x_{k-1}) \\ &= rx_k x_{k+1} - x_{k-1}x_{k+1} = x_{k+1}(rx_k - x_{k-1}) = x_{k+1}^2\end{aligned}$$

$$\begin{aligned}x_2 + x_4 + \cdots + x_{2k+2} &= (rx_1 - x_0) + (rx_3 - x_2) + \cdots + (rx_{2k+1} - x_{2k}) \\ &= rx_{k+1}^2 - x_k x_{k+1} = x_{k+1}(rx_{k+1} - x_k) = x_{k+1}x_{k+2} \ .\end{aligned}$$

Hence the result holds for all m and the result follows.

Solution 3. The recursion $x_{n+2} = rx_{n+1} - x_n$ has characteristic polynomial $t^2 - rt + 1$ with discriminant $r^2 - 4$. Its roots are distinct as long as $r \neq \pm 2$, so we deal with $r = \pm 2$ separately.

The solution for $r = 2$ is $x_n = n$ and for $r = -2$ is $x_n = (-1)^{n-1}n$, and it is straightforward to establish the desired relation in these cases. When $r \neq \pm 2$, the characteristic polynomial has distinct roots σ and $1/\sigma$, where $\sigma + 1/\sigma = r$, *i.e.* $\sigma^2 = r\sigma - 1$. Note that $\sigma \neq \pm 1$ since $r \neq \pm 2$. The solution of the recursion is

$$x_n = \frac{\sigma}{\sigma^2 - 1}(\sigma^n - \sigma^{-n}) \ .$$

Then

$$\begin{aligned}
&x_1 + x_3 + \cdots + x_{2m-1} \\
&\quad = \frac{\sigma}{\sigma^2 - 1}(\sigma + \sigma^3 + \cdots + \sigma^{2m-1}) - \frac{\sigma}{\sigma^2 - 1}(\sigma^{-1} + \sigma^{-3} + \cdots + \sigma^{-(2m-1)}) \\
&\quad = \frac{\sigma^2}{\sigma^2 - 1}\left[\frac{\sigma^{2m} - 1}{\sigma^2 - 1}\right] - \frac{1}{\sigma^2 - 1}\left[\frac{\sigma^{-2m} - 1}{\sigma^{-2} - 1}\right] \\
&\quad = \frac{\sigma^2}{(\sigma^2 - 1)^2}[\sigma^{2m} - 1 - 1 + \sigma^{-2m}] = \frac{\sigma^2}{(\sigma^2 - 1)^2}(\sigma^m - \sigma^{-m})^2 = x_m^2 \ .
\end{aligned}$$

Comment. Solvers who used the approach of Solution 3 failed to consider the case in which the characteristic polynomial had a double root.

18. Let a and b be integers. How many solutions in real pairs (x, y) does the system

$$\lfloor x \rfloor + 2y = a$$

$$\lfloor y \rfloor + 2x = b$$

have?

Comments. Several of the solvers got all mixed up with the status of the variables in the problem, and, for example, found infinitely many solutions to the equations. a and b are fixed in advance; they are parameters, and your final answer will be conditioned by the the characteristics of various pairs (a, b). Rather than rush blindly into the problem, it is a good strategy to gain some understanding of the situation by looking at particular cases. For example, if one takes $(a, b) = (0, 0)$, it is not too hard to see that $(x, y) = (0, 0)$ is the only solution. Similarly, if $(a, b) = (1, 1)$, one arrives at the sole solution $(x, y) = (\frac{1}{2}, \frac{1}{2})$. However, if $(a, b) = (1, 0)$, we find two distinct solutions, $(x, y) = (-\frac{1}{2}, 1), (0, \frac{1}{2})$. This not only clarifies the situation, but gives you a couple of examples against which you can check your final answer. Get in the habit of using examples to help understanding. It is an excellent exercise to plot, on the same axes, the graphs of the two equations for various values of a and b.

Solution 1. Since $2x$ and $2y$ are the difference of two integers, x and y must have the form $m + \frac{u}{2}$ and $n + \frac{v}{2}$ respectively, where m and n are integers and u and v each take one of the values 0 and 1. Plugging these in yields

$$m + 2n + v = a$$

$$n + 2m + u = b \ .$$

Solving for m and n gives

$$3m = 2b - a + v - 2u$$

$$3n = 2a - b + u - 2v \ .$$

For a viable solution, u and v must be such that the right side of each equation is a multiple of 3 (and this is where the characteristics of the parameters a and b enter in). We consider three cases:

(i) Let $a + b \equiv 0 \pmod 3$. Then, for a solution, we must have

$$0 \equiv 2b - a + v - 2u = 3b - (a + b) + v - 2u \equiv v - 2u$$

and

$$0 \equiv 2a - b + u - 2v \equiv u - 2v$$

modulo 3. Since $\{u, v\} \subseteq \{0, 1\}$, we must have $u = v = 0$, and we obtain the solution

$$(x, y) = \left(\frac{2b - a}{3}, \frac{2a - b}{3}\right) .$$

This checks out.

(ii) Let $a+b \equiv 1 \pmod 3$. Then, for a solution, we must have

$$0 \equiv 2b - a + v - 2u \equiv -1 + v - 2u$$

and

$$0 \equiv 2a - b + u - 2v \equiv -1 + u - 2v$$

modulo 3. The only solutions of $u - 2v \equiv v - 2u \equiv 0 \pmod 3$ with u and v equal to 0 or 1 are given by $(u,v) = (1,0), (0,1)$, so, either

$$(x,y) = (m + \frac{1}{2}, n) \qquad \text{with} \qquad 3m = 2b - a - 2, 3n = 2a - b + 1$$

or

$$(x,y) = (m, n + \frac{1}{2}) \qquad \text{with} \qquad 3m = 2b - a + 1, 3n = 2a - b - 2 \ .$$

Thus, we get two solutions

$$(x,y) = \left(\frac{4b-2a-1}{6}, \frac{2a-b+1}{3}\right), \left(\frac{2b-a+1}{3}, \frac{4a-2b-1}{6}\right) \ .$$

These check out.

(iii) Let $a+b \equiv 2 \pmod 3$. Then

$$0 \equiv 2b - a + v - 2u \equiv -2 + v - 2u$$

and

$$0 \equiv 2a - b + u - 2v \equiv -2 + u - 2v$$

modulo 3. For a solution, we must have $v - 2u \equiv u - 2v \equiv 2 \pmod 3$, so that $(u,v) = (1,1)$ and $(x,y) = (m + \frac{1}{2}, n + \frac{1}{2})$ with $3m = 2b - a - 1$ and $3n = 2a - b - 1$. Hence there is a unique solution

$$(x,y) = \left(\frac{4a-2a+1}{6}, \frac{4a-2b+1}{6}\right) \ .$$

This checks out.

Hence, when $a+b \equiv 0$ or $a+b \equiv 2 \pmod 3$, there is one solution to the system, while when $a+b \equiv 1 \pmod 3$, there are two solutions.

Solution 2. Observe that x and y must be half integers. Since

$$\lfloor x \rfloor + \lfloor y \rfloor + 2x + 2y = a + b$$

and $x - \frac{1}{2} \le \lfloor x \rfloor \le x$, $y - \frac{1}{2} \le \lfloor y \rfloor \le y$, we must have that

$$a + b \le 3(x+y) \le a + b + 1 \ .$$

Hence, $x+y$, being a half integer, must have exactly one of the values

$$\frac{a+b}{3}, \frac{a+b+\frac{1}{2}}{3}, \frac{a+b+1}{3}$$

depending on the divisibility of the numerator of the fractions by 3. Consider cases:

(i) Let $a+b \equiv 0 \pmod 3$. Then $x+y$ is the integer $(a+b)/3$. If x and y are not themselves integers, then

$$\lfloor x \rfloor + \lfloor y \rfloor = (x - \frac{1}{2}) + (y - \frac{1}{2}) = x + y - 1$$

so that

$$3(x+y) - 1 \equiv a + b \equiv 0$$

modulo 3, a contradiction. Hence, x and y are both integers and

$$(x, y) = \left(\frac{2b-a}{3}, \frac{2a-b}{3} \right) .$$

This checks out.

(ii) Let $a+b \equiv 1 \pmod 3$. Then $2(a+b)+1 \equiv 0 \pmod 3$, and so

$$x + y = \frac{a+b+\frac{1}{2}}{3} = \frac{2(a+b)+1}{6} ,$$

a half-integer. Hence there are integers m and n for which

$$(x, y) = (m + \frac{1}{2}, n) \qquad \text{or} \qquad (m+1, n - \frac{1}{2})$$

where $m + n = (a+b-1)/3$. We find that

$$(x, y) = \left(\frac{4b-2a-1}{6}, \frac{2a-b+1}{3} \right), \left(\frac{2b-a+1}{3}, \frac{4a-2b-1}{6} \right) .$$

These check out.

(iii) $a+b \equiv 2 \pmod 3$. Then $x+y = (a+b+1)/3$, an integer. In this case, we can check that x and y cannot be integers, so that $x = \lfloor x \rfloor + 1/2$ and $y = \lfloor y \rfloor + 1/2$, and so

$$(x, y) = \left(\frac{4a-2a+1}{6}, \frac{4a-2b+1}{6} \right) .$$

This checks out.

19. Is it possible to divide the natural numbers $1, 2, \cdots, n$ into two groups, such that the squares of the members in each group have the same sum, if (a) $n = 40000$; (b) $n = 40002$? Explain your answer.

Solution. [M. Holmes] (a) Yes, it is possible. Partition the numbers into two sets A and B such that
- if $i \equiv 1, 4, 6, 7 \pmod 8$, put i in set A;
- if $i \equiv 2, 3, 5, 8 \pmod 8$, put i in set B.

Since 40000 is a multiple of 8, there are 5000 strings of eight consecutive natural numbers. For each of them, it is straightforward to see that

$$\begin{aligned}(8k+1)^2 + (8k+4)^2 + (8k+6)^2 + (8k+7)^2 &= 4 \times (8k)^2 + 16 \times (1+4+6+7) + (1+16+36+49) \\ &= 4 \times (8k)^2 + 16 \times (2+3+5+8) + (4+9+25+64) \\ &= (8k+2)^2 + (8k+3)^2 + (8k+5)^2 + (8k+8)^2\end{aligned}$$

for $0 \le k \le 4999$. So, if the numbers are put into the sets as suggested, the squares of the numbers in each group will have the same sum.

(b) No, it is impossible. Suppose it were possible to partition the numbers from 1 to 40002 inclusive into two sets A and B as required. There is a well-known formula for the sum of the squares of the first n natural numbers,

$$1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

which we recommend that you prove by induction. When $n = 40002$, this sum is odd, and so we cannot express it as the sum of two equal numbers, the sums of the squares in A and in B. Hence, the desired partition is not possible.

Comment. One does not need the formula for the sum of squares to establish that the sum is odd; just note that the sum has 20001 odd summands.

20. Given any six irrational numbers, prove that there are always three of them, say a, b, c, for which $a+b$, $b+c$ and $c+a$ are irrational.

Solution. [O. Bormashenko] Recall the result given in the solution to Problem 9 in *Olymon* 1:5 (August, 2000): *For any six points in space, let the full graph of all fifteen edges between two of them be coloured with two colours. There exists a triangle of three of its vertices, each edge of which has the same colour.* Let each of the six irrational numbers be assigned a point in space, and colour an edge joining two points representing a pair (u, v) red if $u + v$ is rational and green if $u + v$ is irrational. Then there must be a red triangle or a green triangle. Suppose, if possible, there is a red triangle. Then three of the numbers, say a, b, c, have $a+b$, $b+c$, $c+a$ all rational. But then $2a = (a+b)+(c+a)-(b+c)$ would be rational, contrary to hypothesis. So there is no red triangle, and so there must be a green triangle. The triple corresponding to the vertices of this triangle must satisfy the requirement of the problem.

21. The natural numbers $x_1, x_2, \cdots, x_{100}$ are such that

$$\frac{1}{\sqrt{x_1}} + \frac{1}{\sqrt{x_2}} + \cdots + \frac{1}{\sqrt{x_{100}}} = 20 \ .$$

Prove that at least two of the numbers are equal.

Solution. [R. Barrington Leigh] We construct a proof by contradiction. Assume that the natural numbers are distinct, and, wolog, in increasing order. Thus, $1 \le x_1$, $2 \le x_2$, $3 \le x_3$, $\cdots$, $100 \le x_{100}$, so that

$$\frac{1}{\sqrt{x_1}} + \frac{2}{\sqrt{x_2}} + \cdots + \frac{1}{\sqrt{x_{100}}} \le \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{100}} \ .$$

On the other hand, for each natural number n,

$$2\sqrt{n} > \sqrt{n} + \sqrt{n-1} = \frac{1}{\sqrt{n} - \sqrt{n-1}}$$

whence

$$\frac{1}{\sqrt{n}} < 2(\sqrt{n} - \sqrt{n-1}) \ .$$

It follows that

$$\begin{aligned} 20 &= \frac{1}{\sqrt{x_1}} + \frac{1}{\sqrt{x_2}} + \cdots + \frac{1}{\sqrt{x_{100}}} \\ &\le \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{100}} \\ &< 2[(\sqrt{1} - \sqrt{0}) + (\sqrt{2} - \sqrt{1}) + \cdots + (\sqrt{100} - \sqrt{99})] = 20 \end{aligned}$$

or $20 < 20$, which is palpably false. Therefore, the assumption that the numbers x_1, x_2, $\cdots$, x_{100} is false, and the desired result holds.

22. Let **R** be a rectangle with dimensions 11×12. Find the least natural number n for which it is possible to cover **R** with n rectangles, each of size 1×6 or 1×7, with no two of these having a common interior point.

Comment. Clearly, we can use 22 rectangles of size 1×6, so the optimum number is no greater than this. We show in fact that the optimum value of n is 20. First, we establish that at least 20 rectangles are needed, and then display a case in which 20 suffice.

Solution. Coordinatize the squares (i, j) with $1 \le i \le 11$, $1 \le j \le 12$. Mark the twenty squares $(1+t, 1+t)$ $(0 \le t \le 10)$, $(1+s, 8+s)$ $(0 \le s \le 4)$, $(8+r, 1+r)$ $(0 \le r \le 3)$. It is impossible to cover more than one of the marked squares with a small rectangle, whether 1×6 or 1×7, so we need at least as many rectangles as marked squares, *i.e.*, at least 20. On the other hand, we can achieve the covering with 12 rectangles of size 1×7 and 8 of size 1×6. Cover the top seven rows with 12 rectangles of size 7×1 and the bottom four rows with rectangles of size 1×6.

23. Given 21 points on the circumference of a circle, prove that at least 100 of the arcs determined by pairs of these points subtend an angle not exceeding $120°$ at the centre.

Comment. Before providing the solution of this problem, it is necessary to recall some basic concepts from graph theory and to prove one theorem. A *graph* is a topological combination of two sets, points and line segments between some pairs of the points. The points are referred to as the *vertices* or *nodes* of the graph and the line segments as *edges* or *arcs*. Let G be a graph. If A is a vertex of G, then the *degree*, $d(A)$, of A is the number of edges emanating from A. The number of edges is denoted by $l(G)$. If there are three vertices, A, B, C, of G such that any two of them are connected by an edge, the set of vertices A, B, C and edges AB, BC, CA is a *triangle* in G, and the number of triangles in G is denoted by $t(G)$. If there are four vertices with any pair connected by an edge, then the set of the four vertices and their six edges is a *tetrahedron* in G, and the number of all tetrahedra is denoted by $T(G)$. The angle of an arc is the angle subtended by the arc at the centre of the circle.

Turan's Theorem. Let G be a graph with n vertices. If $t(G) = 0$, then $l(G) \le \lfloor n^2/4 \rfloor$.

(This theorem is named after the Eastern European mathematician Pal Turan and is very useful in a variety of problems, for which it is possible to model the given objects and their relationships with a graph. In other words, it says that in a graph with n vertices with no triangles, there are no more than $\lfloor n^2/4 \rfloor$ edges. It is easy to check that the result holds for $n = 3$ and $n = 4$, and you should do this.)

Proof of Turan's theorem. In the graph G, let A be the vertex of greatest degree (*i.e.*, the number of edges from any other vertex does not exceed the number from A). Suppose that $d(A) = k$ (a natural number), that B_1, B_2, $\cdots$, B_k are the vertices connected to A by an edge, and that G' is the graph that can be obtained from G by removing A and all edges emanating from A. Then, $l(G) = d(A) + l(G')$. Obviously, there is no edge B_iB_j because, otherwise, the vertices A, B_i, B_j will form a triangle. So, in G' are counted only all edges emanating from the $n-k-1$ vertices of G other than A, B_1, B_2, $\cdots$, B_k. Since $d(A) = k$ is the maximum number of edges emanating from any vertex, $l(G') \le (n-k-1)k$. Therefore

$$l(G) \le k + (n-k-1)k = k(n-k) \ .$$

Applying the arithmetic-geometric means inequality, we get $k(n-k) \le [k + (n-k)]^2/4 = n^2/4$, so that $l(G) \le n^2/4$. Since $l(G)$ is a natural number, the desired result follows. ♠

Two examples show that this value can be reached, so that it is the largest possible value of $l(G)$.

Example 1. Let $n = 2p$ be an even number. Partition the vertices into two sets with p vertices in each. Draw all edges connecting a vertex from one set to the other. There are example $p^2 = n^2/4$ edges, with no triangles formed. Any additional edge will form a triangle with some other two.

Example 2. Let $n = 2p+1$ be an odd number. Partition the vertices into two sets with p and $p+1$ vertices. As in Example 1, connect ay vertex from one set with one in the other to obtain a total of $p(p+1) = \lfloor n^2/4 \rfloor$ edges, with no triangle in the graph.

(There is a similar theorem about tetrahedra in the graph, to wit: *for a graph with n vertices with $T(G) = 0$, then $l(G) \le \lfloor n^2/3 \rfloor$*. This can be proved using similar ideas to those for the triangle case. Here is an example of a graph with $T(G) = 0$ and $l(G) = \lfloor n^2/3 \rfloor$. Divide the vertices of G into three "almost equal" sets (the difference between the numbers of vertices in any two of the sets is at most 1). Connect two vertices with an edge if and only if they are from two different sets.)

Now we apply Turan's Theorem to solve Problem 23.

Solution 1. Count all arcs not exceeding 180° and ending in any two of the given 21 points. There are $210 = \binom{21}{2}$ arcs in all. Connect with a segment any two points determining an arc exceeding 120°; consider all such segments as edges in a graph G whose vertices are the 21 given points. There are no triangles in G (otherwise, each of its angles would exceed 60° giving an angle sum in excess of 180°. According to Turan's theorem, there are no more than $\lfloor 21^2/4 \rfloor = 110$ such segments, so there must be at least $210 - 110 = 100$ arcs that do not exceed 120°.

Solution 2. [M. Tancer, O. Bormashenko] There are 210 arcs not exceeding 180° for the 21 points on the circumference of the circle, one for each pair. Given three points on the circle, at least one of the arcs between two of them must not exceed 120°. If all of the 210 arcs do not exceed 120°, then the problem is solved.

Suppose that there is an arc AB exceeding 120°. For any third point C, among the three arcs AB, AC, BC, at least one must not exceed 120°; since it is not AB, it must be one of the other two. Similarly, for each of the nineteen points other than A and B, there must be at least one arc determined by that point and one of A and B not exceeding 120°. Thus, there are at least 19 arcs one of whose endpoints is either A or B exceeding 120°. Since there are $2 \times 19 + 1 = 39$ arcs with at least one endpoint A or B, the maximum number of arcs among them exceeeding 120° is 20. If there are no further arcs exceeding 120°, the problem is solved.

Otherwise, let DE be a second arc exceeding 120°, with D and E distinct from A and B. There are at least 19 arcs with at least one endpoint D or E not exceeding 120°. Since all arcs emanating form A and B have been counted, there are at least 17 new such arcs, and at most 18 arcs exceeding 120°. Continuing this procedure, we find that at most $20 + 18 + 16 + \cdots + 2 = 110$ arcs exceeding 120°. So there must be at least 100 arcs that do not exceed 120°.

Comment. Here is an example in which the number of arcs not exceeding 120° is exactly 100. Let the centre of the circle be O, and let AB and CD be two diameters with $\angle AOC = 60°$. On the circumference of the circle, put 10 points on the smaller arc AC and 11 on the smaller arc BD, all distinct from A, B, C, D. Consider all arcs between a point from the first set and a point from the second set; there are $10 \times 11 = 110$ such arcs, all exceeding 120°. The remaining arcs do not exceed 120°.

24. ABC is an acute triangle with orthocentre H. Denote by M and N the midpoints of the respective segments AB and CH, and by P the intersection point of the bisectors of angles CAH and CBH. Prove that the points M, N and P are collinear.

Solution. Let h_a and h_b be the altitudes from A and B respectively, and let them $BC \cap h_a = \{E\}$ and $AC \cap h_b = \{D\}$. Let l_1 and l_2 be the respective angle bisectors of angles CAH and CBH. Since $\angle CDH = \angle CEH = 90°$, the quadrilaterial $CDHE$ is inscribed in a circle k_1 with centre N and diameter CH. Hence $ND = NE$, as radii of the circle k_1. Similarly, since $\angle ADB = \angle AEB = 90°$, the quadrilateral $ADEB$ is inscribed in a circle k_2 with centre M and diameter AB. Hence $MD = ME$ as radii of the circle k_2. It follows that MN is the right bisector of the segment DE; denote it by S_{DE}. So, to prove M, N and P collinear, it suffices to prove that $P \in S_{DE}$, or $PD = PE$.

Consider the circle $ADEB$. Let G be the centre of the arc DE. Since $\angle DAG = \angle GAE$, l_1 must pass through G; since $\angle EBG = \angle GED$, l_2 must pass through G. But then the point of intersection of l_1 and l_2 is P, so that P must coincide with G and therefore be the midpoint of the arc DE. Now it is easy to see that the triangles DPM and EPM are congruent ($DM = PM = EM$ as radii and $\angle DMP = \angle PME$). Hence, $PD = PE$ and so M, N and P are collinear.

25. Let a, b, c be non-negative numbers such that $a + b + c = 1$. Prove that

$$\frac{ab}{c+1} + \frac{bc}{a+1} + \frac{ca}{b+1} \leq \frac{1}{4} \quad .$$

When does equality hold?

Solution. It is straightforward to show that the inequality holds when one of the numbers is equal to zero. Equality holds if and only if the other two numbers are each equal to $1/2$. Henceforth, assume that all values are positive.

Since $a+b+c=1$, at least one of the numbers is less than $4/9$. Assume that $c<4/9$. Let E denote the left side of the inequality. Then

$$E = abc\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}-\frac{1}{a+1}-\frac{1}{b+1}-\frac{1}{c+1}\right) .$$

Since

$$\frac{1}{a+1}+\frac{1}{b+1}+\frac{1}{c+1} = \frac{1}{4}[(a+1)+(b+1)+(c+1)]\left[\frac{1}{a+1}+\frac{1}{b+1}+\frac{1}{c+1}\right] \geq \frac{9}{4} ,$$

(by the Cauchy-Schwarz Inequality for example), we have that

$$E \leq abc\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}-\frac{9}{4}\right) = ab+bc+ca-\frac{9}{4}abc .$$

On the other hand, $(1-c)^2=(a+b)^2 \geq 4ab$, whence $ab \leq \frac{1}{4}(1-c)^2$. Therefore

$$\begin{aligned}
E-\frac{1}{4} &\leq ab+c(a+b)-\frac{9}{4}abc-\frac{1}{4} \\
&= ab\left(1-\frac{9}{4}c\right)+c(1-c)-\frac{1}{4} \\
&\leq \frac{1}{4}(1-c)^2\left(1-\frac{9}{4}c\right)+c(1-c)-\frac{1}{4} \\
&= -\frac{1}{16}c(3c-1)^2 \leq 0 .
\end{aligned}$$

Equality occurs everywhere if and only if $a=b=c=1/3$.

26. Each of m cards is labelled by one of the numbers $1,2,\cdots,m$. Prove that, if the sum of labels of any subset of cards is not a multiple of $m+1$, then each card is labelled by the same number.

Solution. Let a_k be the label of the kth card, and let $s_n=\sum_{k=1}^{n} a_k$ for $n=1,2,\cdots,m$. Since the sum of the labels of any subset of cards is not a multiple of $m+1$, we get different remainders when we divide the s_n by $m+1$. These remainders must be $1,2,\cdots,m$ in some order. Hence there is an index $i\in\{1,2,\cdots,m\}$ for which $a_2 \equiv s_i \pmod{m+1}$. If i were to exceed 1, then we would have a contradiction, since then s_i-a_2 would be a multiple of $m+1$. Therefore, $a_2\equiv s_1=a_1$, so that $a_2 \equiv a_1 \pmod{m+1}$, whence $a_2=a_1$. By cyclic rotation of the a_k, we can argue that all of the a_k are equal.

27. Find the least number of the form $|36^m-5^n|$ where m and n are positive integers.

Solution. Since the last digit of 36^m is 6 and the last digit of 5^n is 5, then the last digit of 36^m-5^n is 1 when $36^m>5^n$ and the last digit of 5^n-36^m is 9 when $5^n>36^m$. If $36^m-5^n=1$, then

$$5^n=36^m-1=(6^m+1)(6^m-1)$$

whence 6^m+1 must be a power of 5, an impossibility. 36^m-5^n can be neither -1 nor 9. If $5^n-36^m=9$, then $5^n=9(4\cdot 36^{m-1}+1)$, which is impossible. For $m=1$ and $n=2$, we have that $36^m-5^n=36-25=11$, and this is the least number of the given form.

28. Let A be a finite set of real numbers which contains at least two elements and let $f:A\longrightarrow A$ be a function such that $|f(x)-f(y)|<|x-y|$ for every $x,y\in A$, $x\neq y$. Prove that there is $a\in A$ for which $f(a)=a$. Does the result remain valid if A is not a finite set?

Solution 1. Let $a \in A$, $a_1 = f(a)$, and, for $n \geq 2$, $a_n = f(a_{n-1})$. Consider the sequence $\{x_n\}$ with

$$x_n = |a_{n+1} - a_n|$$

where $n = 1, 2, \cdots$. Since A is a finite set and each a_n belongs to A, there are only a finite number of distinct x_n. Let $x_k = \min_{n \geq 1}\{x_n\}$; we prove by contradiction that $x_k = 0$.

Suppose if possible that $x_k > 0$. Then

$$x_k = |a_{k+1} - a_k| > |f(a_{k+1}) - f(a_k)| = |a_{k+2} - a_{k+1}| = x_{k+1} \ .$$

But this does not agree with the selection of x_k. Hence, $x_k = 0$, and this is equivalent to $a_{k+1} = a_k$ or $f(a_k) = a_k$. The desired result follows.

Solution 2. We first prove that $f(A) \neq A$. Suppose, if possible, that $f(A) = A$. Let M be the largest and m be the smallest number in A. Since $f(A) = A$, there are elements a_1 and a_2 in A for which $M = f(a_1)$ and $m = f(a_2)$. Hence

$$M - m = |f(a_1) - f(a_2)| < |a_1 - a_2| \leq |M - m| = M - m$$

which is a contradiction. Therefore, $f(A) \subset A$.

Note that $A \supseteq f(A) \supseteq f^2(A) \supseteq \cdots \supseteq f^n(A) \supseteq \cdots$. In fact, we can extend the foregoing argument to show *strict* inclusion as long as the sets in question have more than one element:

$$A \supset f(A) \supset f^2(A) \supset \cdots \supset f^n(A) \supset \cdots \ .$$

(The superscripts indicate multiple composites of f.) Since A is a finite set, there must be a positive integer m for which $f^m(A) = \{a\}$, so that $f^{m+1}(A) = f^m(A)$. Thus, $f(a) = a$.

Example. If A is not finite, the result may fail. Indeed, we can take $A = (0, \frac{1}{2})$ (the open interval of real numbers strictly between 0 and $\frac{1}{2}$) or $A = \{2^{-2n} : n = 1, 2, \cdots\}$ and $f(x) = x^2$.

29. Let A be a nonempty set of positive integers such that if $a \in A$, then $4a$ and $\lfloor \sqrt{a} \rfloor$ both belong to A. Prove that A is the set of all positive integers.

Solution. (i) Let us first prove that $1 \in A$. Let $a \in A$. Then we have

$$\lfloor a^{1/2} \rfloor \in A \ , \quad \lfloor \lfloor a^{1/2} \rfloor^{1/2} \rfloor \in A \ , \quad \cdots \quad , \lfloor \cdots \lfloor a^{1/2} \rfloor^{1/2} \rfloor^{1/2} \cdots \rfloor \in A \ , \quad \cdots \ .$$

Also, the following inequalities are true

$$1 \leq \lfloor a^{1/2} \rfloor \leq a^{1/2} \ , \quad 1 \leq \lfloor \lfloor a^{1/2} \rfloor^{1/2} \rfloor \leq a^{1/2^2} \ , \quad \cdots \ , \quad 1 \leq \lfloor \cdots \lfloor a^{1/2} \rfloor^{1/2} \rfloor^{1/2} \cdots \rfloor \leq a^{1/2^n} \ ,$$

where there are n brackets in the general inequality. There is a sufficiently large positive integer k for which $a^{1/2^k} \leq 1.5$, and for this k, we have, with k brackets,

$$1 \leq \lfloor \cdots \lfloor a^{1/2} \rfloor^{1/2} \rfloor^{1/2} \cdots \rfloor \leq a^{1/2^k} \leq 1.5 \ ,$$

and thus

$$\lfloor \cdots \lfloor \lfloor a^{1/2} \rfloor^{1/2} \rfloor^{1/2} \cdots \rfloor = 1 \ ,$$

(ii) We next prove that $2^n \in A$ for $n = 1, 2, \cdots$. Indeed, since $1 \in A$, we obtain that, for each positive integer n, $2^{2n} \in A$ so that $2^n = \lfloor \sqrt{2^{2n}} \rfloor \in A$.

(iii) We finally prove that an arbitrary positive integer m is in A. It suffices to show that there is a positive integer k for which $m^{2^k} \in A$. For each positive integer k, there is a positive integer p_k such that $2^{p_k} \leq m^{2^k} < 2^{p_k+1}$ (we can take $p_k = \lfloor \log_2 m^{2^k} \rfloor$). For k sufficiently large, we have the inequality

$$\left(1 + \frac{1}{m}\right)^{2^k} \geq 1 + \frac{1}{m} \cdot 2^k > 4 \ .$$

This combined with the foregoing inequality produces

$$2^{p_k} \leq m^{2^k} < 2^{p_k+1} < 2^{p_k+2} < (m+1)^{2^k} \ . \qquad (*)$$

Since $2^{2(p_k+1)+1} \in A$, we have that

$$\lfloor \sqrt{2^{2(p_k+1)+1}} \rfloor = \lfloor 2^{p_k+1}\sqrt{2} \rfloor \in A \ .$$

Hence, with $k+1$ brackets,

$$\lfloor \cdots \lfloor \lfloor 2^{(p_k+1)}\sqrt{2} \rfloor^{1/2} \rfloor^{1/2} \cdots \rfloor \in A \ .$$

On the other hand, using $(*)$, we get

$$m^{2^k} < 2^{p_k+1} \leq \lfloor 2^{(p_k+1)}\sqrt{2} \rfloor < (m+1)^{2^k}$$

and, then, with $k+1$ brackets,

$$m \leq \lfloor \cdots \lfloor \lfloor 2^{(p_k+1)}\sqrt{2} \rfloor^{1/2} \rfloor^{1/2} \cdots \rfloor < m+1 \ .$$

Thus,

$$m = \lfloor \cdots \lfloor \lfloor 2^{(p_k+1)}\sqrt{2} \rfloor^{1/2} \rfloor^{1/2} \cdots \rfloor \in A \ .$$

30. Find a point M within a regular pentagon for which the sum of its distances to the vertices is minimum.

Solution. We solve this problem for the regular $n-$gon $A_1A_2\cdots A_n$. Choose a system of coordinates centred at O (the circumcentre) such that

$$A_k \sim \left(r\cos\frac{2k\pi}{n}, r\sin\frac{2k\pi}{n}\right) \ , \quad r = \|\overrightarrow{OA_k}\|$$

for $k = 1, 2, \cdots, n$. Then

$$\sum_{k=1}^{n} \overrightarrow{OA_k} = \sum_{k=1}^{n} \left(r\cos\frac{2k\pi}{n}, r\sin\frac{2k\pi}{n}\right) = \left(r\sum_{k=1}^{n}\cos\frac{2k\pi}{n}, r\sum_{k=1}^{n}\sin\frac{2k\pi}{n}\right) = (0,0) \ .$$

Indeed, letting $\zeta = \cos\frac{2\pi}{n} + i\sin\frac{2\pi}{n}$ and using DeMoivre"s Theorem, we have that $\zeta^n = 1$ and

$$\begin{aligned}\sum_{k=1}^{n}\cos\frac{2k\pi}{n} + i\sum_{k=1}^{n}\sin\frac{2k\pi}{n} &= \sum_{k=1}^{n}\left(\cos\frac{2\pi}{n} + i\sin\frac{2\pi}{n}\right)^k \\ &= \sum_{k=1}^{n}\zeta^k = \zeta\left(\frac{1-\zeta^n}{1-\zeta}\right) = 0 \ ,\end{aligned}$$

whence $\sum_{k=1}^n \cos(2\pi k/n) = \sum_{k=1}^n \sin(2\pi k/n) = 0$. On the other hand,

$$\begin{aligned}
\sum_{k=1}^n \|\overrightarrow{MA_k}\| &= \frac{1}{r}\sum_{k=1}^n \|\overrightarrow{OA_k} - \overrightarrow{OM}\|\|\overrightarrow{OA_k}\| \\
&\geq \frac{1}{r}\sum_{k=1}^n (\overrightarrow{OA_k} - \overrightarrow{OM}) \cdot \overrightarrow{OA_k} \\
&= \frac{1}{r}\Big(\sum_{k=1}^n \|\overrightarrow{OA_k}\|^2 - \|\overrightarrow{OM}\| \sum_{k=1}^n \overrightarrow{OA_k}\Big) \\
&= \sum_{k=1}^n r = \sum_{k=1}^n \|\overrightarrow{OA_k}\| \ .
\end{aligned}$$

Equality occurs if and only if $M = O$, so that O is the desired point.

31. Let x, y, z be positive real numbers for which $x^2 + y^2 + z^2 = 1$. Find the minimum value of

$$S = \frac{xy}{z} + \frac{yz}{x} + \frac{zx}{y} \ .$$

Solution 1. [S. Niu] Let $a = yz/x$, $b = zx/y$ and $c = xy/z$. Then a, b, c are positive, and the problem becomes to minimize $S = a + b + c$ subject to $ab + bc + ca = 1$. Since

$$2(a^2 + b^2 + c^2 - ab - bc - ca) = (a-b)^2 + (b-c)^2 + (c-a)^2 \geq 0,$$

we have that $a^2 + b^2 + c^2 \geq ab + bc + ca$. Thus,

$$\begin{aligned}
1 &= ab + bc + ca \leq a^2 + b^2 + c^2 \\
&= (a+b+c)^2 - 2(ab+bc+ca) = (a+b+c)^2 - 2 = S^2 - 2
\end{aligned}$$

so $S \geq \sqrt{3}$ with equality if and only if $a = b = c$, or $x = y = z = 1/\sqrt{3}$. The desired result follows.

Solution 2. We have that

$$\begin{aligned}
S^2 &= \frac{x^2y^2}{z^2} + \frac{y^2z^2}{x^2} + \frac{z^2x^2}{y^2} + 2(x^2 + y^2 + z^2) \\
&= \frac{1}{2}\left[\left(\frac{x^2y^2}{z^2} + \frac{z^2x^2}{y^2}\right) + \left(\frac{x^2y^2}{z^2} + \frac{y^2z^2}{x^2}\right) + \left(\frac{y^2z^2}{x^2} + \frac{z^2x^2}{y^2}\right)\right] + 2 \\
&\geq x^2 + y^2 + z^2 + 2 = 3
\end{aligned}$$

by the Arithmetic-Geometric Means Inequality. Equality holds if and only if $xy/z = yz/x = zx/y$, which is equivalent to $x = y = z$. Hence $S \geq \sqrt{3}$ if and only if $x = y = z = 1/\sqrt{3}$.

32. The segments BE and CF are altitudes of the acute triangle ABC, where E and F are points on the segments AC and AB, respectively. ABC is inscribed in the circle $\mathbf{Q}$ with centre O. Denote the orthocentre of ABC by H, and the midpoints of BC and AH be M and K, respectively. Let $\angle CAB = 45°$.

(a) Prove, that the quadrilateral $MEKF$ is a square.

(b) Prove that the midpoint of both diagonals of $MEKF$ is also the midpoint of the segment OH.

(c) Find the length of EF, if the radius of $\mathbf{Q}$ has length 1 unit.

Solution 1. (a) Since AH is the hypotenuse of right triangles AFH and AHE, $KF = KH = KA = KE$. Since BC is the hypotenuse of each of the right triangles BCF and BCE, we have that $MB = MF = ME = MC$. Since $\angle BAC = 45°$, triangles ABE, HFB and ACF are isosceles right triangles, so $\angle ACF = \angle ABE = \angle FBH = \angle FHB = 45°$ and $FA = FC$, $FH = FB$.

Consider a $90°$ rotation with centre F that takes $H \to B$. Then $FA \to FC$, $FH \to FB$, so $\Delta FHA \to \Delta FBC$ and $K \to M$. Hence $FK = FM$ and $\angle KFM = 90°$.

But $FK = KE$ and $FM = ME$, so $MEKF$ is an equilateral quadrilateral with one right angle, and hence is a square.

(b) Consider a $180°$ rotation (half-turn) about the centre of the square. It takes $K \leftrightarrow M$, $F \leftrightarrow E$ and $H \leftrightarrow H'$. By part (a), $\Delta FHA \equiv \Delta FBC$ and $AH \perp BC$. Since $KH \| MH'$ (by the half-turn), $MH' \perp BC$. Since $AH = BC$, $BM = \frac{1}{2}BC = \frac{1}{2}AH = KH = MH'$, so that BMH' is a right isosceles triangle and $\angle CH'M = \angle BH'M = 45°$. Thus, $\angle BH'C = 90°$. Since $\angle BAC = 45°$, H' must be the centre of the circle through ABC. Hence $H' = O$. Since O is the image of H by a half-turn about the centre of the square, this centre is the midpoint of OH as well as of the diagonals.

(c) $|EF| = \sqrt{2}|FM| = \sqrt{2}|BM| = |OB| = 1$.

Solution 2. [M. Holmes] (a) Consider a Cartesian plane with origin $F(0,0)$ and $x-$axis along the line AB. Let the vertices of the triangle be $A(-1,0)$, $C(0,1)$, $B(b,0)$. Since the triangle is acute, $0 < b < 1$. The point E is at the intersection of the line AC ($y = x + 1$) and a line through B with slope -1, so that $E = (\frac{1}{2}(b-1), \frac{1}{2}(b+1)$. H is the intersection point of the lines BE and CF, so H is at $(0,b)$; K is the midpoint of AH, so K is at $(-\frac{1}{2}, \frac{b}{2})$; M is the midpoint of BC, so M is at $(\frac{b}{2}, \frac{1}{2})$. It can be checked that the midpoints of EF and KM are both at $(\frac{1}{4}(b-1), \frac{1}{4}(b+1))$. The slope of EF is $(b+1)/(b-1)$ and that of KM is the negative reciprocal of this, so that $EF \perp KM$. It is straightforward to check that the lengths of EF and KM are equal, and we deduce that $EKFM$ is a square.

(b) O is the intersection point of the right bisectors of AB, AC and BC. The line $x + y = 0$ is the right bisector of AC and the abscissae of points on the right bisector of BC are all $\frac{1}{2}(b-1)$. Hence O is at $(\frac{1}{2}(b-1), \frac{1}{2}(1-b))$. It can be checked that the midpoint of OH agress with the joint midpoint of EF and KM.

(c) This can be checked by using the coordinates of points already identified.

Comment. One of the most interesting theorems in triangle geometry states that for each triangle there exists a circle that passes through the following nine special points: *the three midpoints of the sides*; *the three intersections of sides and altitudes (pedal points); and the three midpoints of the segments connecting the vertices to the orthocentre*. This circle is called the *nine-point circle*. If H is the orthocentre and O is the circumcentre, then the centre of the nine-point circle is the midpoint of OH. Note that in this problem, the points M, E, F, K belong to the nine-point circle.

33. Prove the inequality $a^2 + b^2 + c^2 + 2abc < 2$, if the numbers a, b, c are the lengths of the sides of a triangle with perimeter 2.

Solution 1. Let $u = b + c - a$, $v = c + a - b$ and $w = a + b - c$, so that $2a = v + w$, $2b = u + w$ and $2c = u + v$. Then u, v, w are all positive and $u + v + w = 2$. The difference of the right and left sides multiplied by 4 is equal to

$$\begin{aligned}
4[2-(a^2+b^2+c^2+2abc)]& \\
&= 8-(v+w)^2-(u+w)^2-(u+v)^2-(v+w)(u+w)(u+v)] \\
&= 8-(u+v+w)^2-(u^2+v^2+w^2)-(2-u)(2-v)(2-w) \\
&= 8-4-(u^2+v^2+w^2)-8+4(u+v+w)-2(vw+uw+uv)+uvw \\
&= 4-(u+v+w)^2-8+4\times 2+uvw \\
&= 4-4-8+8+uvw = uvw > 0
\end{aligned}$$

as desired.

Solution 2. [L. Hong] The perimeter of the triangle is $a+b+c=2$. We have that

$$\begin{aligned}
a^2+b^2+c^2+2abc &= (a+b+c)^2+2(a-1)(b-1)(c-1)+2-2(a+b+c) \\
&= 4+2(a-1)(b-1)(c-1)+2-4 = 2(a-1)(b-1)(c-1)+2 .
\end{aligned}$$

Since $a<b+c$, $b<c+a$ and $c<a+b$, it follows that $a<1$, $b<1$, $c<1$, from which the result follows.

34. Each of the edges of a cube is 1 unit in length, and is divided by two points into three equal parts. Denote by **K** the solid with vertices at these points.

(a) Find the volume of **K**.

(b) Every pair of vertices of **K** is connected by a segment. Some of the segments are coloured. Prove that it is always possible to find two vertices which are endpoints of the same number of coloured segments.

Solution. (a) The solid figure is obtained by slicing off from each corner a small tetrahedron, three of whose faces are pairwise mutually perpendicular at one vertex; the edges emanating from that vertex all have length $1/3$, and so the volume of each tetrahedron removed is $1/3(1/2\cdot 1/3\cdot 1/3)(1/3)=1/162$. Since there are eight such tetrahedra removed, the volume of the resulting solid is $1-4/81=77/81$. (The numbers of vertices, edges and faces of the solid are respectively 24, 36 and 14.)

(b) The polyhedron has $3\times 8=24$ vertices. Each edge from a given vertex is joined to 23 vertices. The possible number of coloured segments emanating from a vertex is one of the twenty-four numbers, 0, 1, 2, $\cdots$, 23. But it is not possible for one vertex to be joined to all 23 others and another vertex to be joined to no other vertex. So there are in effect only 23 options for the number of coloured segments emanating from each of the 24 vertices. By the Pigeonhole Principle, there must be two vertices with the same number of coloured segments emanating from it.

35. There are n points on a circle whose radius is 1 unit. What is the greatest number of segments between two of them, whose length exceeds $\sqrt{3}$?

Solution. [O.Bormashenko] The side of the equilateral triangle inscribed in a circle of unit radius is $\sqrt{3}$. So the segment with length $\sqrt{3}$ is a chord subtending an angle of 120° at the centre. Therefore, there is no triangle with three vertices on the circle each of whose sides are longer than $\sqrt{3}$. Consider the graph whose vertices are all n given points and whose arcs all have segments longer than $\sqrt{3}$. This graph contains no triangles.

Recall Turan's theorem (see the solution of problem 23 in *Olymon 1:4*: *Let G be a graph with n vertices. Denote by $l(G)$ the number of its edges and $t(G)$ the number of triangles contained in the graph. If $t(G)=0$, then $l(G)\leq\lfloor n^2/4\rfloor$*. From this theorem, it follows that the number of segments with chords exceeding $\sqrt{3}$ is at most $\lfloor n^2/4\rfloor$.

To show that this maximum number can be obtained, first construct points A,B,C,D on the circle, so that the disjoint arcs AB and CD subtend angles of 120° at the centre. If $n=2k+1$ is odd, place k points on the arc BC and $k+1$ points on the arc DA. Any segment containing a point in BC to a point in DA must subtend an angle exceeding $120°$, so its length exceeds $\sqrt{3}$. There are exactly $k(k+1) = \lfloor (2k+1)^2/4 \rfloor$ such segments. If $n = 2k$ is even, place k points in each of the arcs BC and CA, so that there are exactly $k^2 = \lfloor (2k)^2/4 \rfloor$ such segments. In either case, the maximum number of segments whose length exceeds $\sqrt{3}$ is $\lfloor n^2/4 \rfloor$.

36. Prove that there are not three rational numbers x, y, z such that

$$x^2 + y^2 + z^2 + 3(x + y + z) + 5 = 0 \ .$$

Solution. Suppose the $x = u/m$, $y = v/m$ and $z = w/m$, where m is the least common multiple of the denominators of x, y and z. Then, multiplying the given equation by m^2 yields

$$u^2 + v^2 + w^2 + 3(um + vm + wm) + 5m^2 = 0 \ .$$

A further multiplication by 4 and a rearrangement of terms yields

$$(2u + 3m)^2 + (2v + 3m)^2 + (2w + 3m)^2 = 7m^2 \ .$$

This is of the form

$$p^2 + q^2 + r^2 = 7n^2 \qquad (*)$$

for integers p, q, r, n. Suppose that n is even. Then, considering the equation modulo 4, we deduce that p, q, r must also be even. We can divide the equation by 4 to obtain another equation of the form (*) with smaller numbers. We can continue to do this as long as the resulting n turns out to be even. Eventually, we arrive at an equation for which n is odd, so that the right side is congruent to 7 modulo 8. But there is not combination of integers p, q and r for which $p^2 + q^2 + r^2 \equiv 7 \pmod 8$, so that the equation is impossible.

37. Let ABC be a triangle with sides a, b, c, inradius r and circumradius R (using the conventional notation). Prove that

$$\frac{r}{2R} \le \frac{abc}{\sqrt{2(a^2+b^2)(b^2+c^2)(c^2+a^2)}} \ .$$

When does equality hold?

Solution. Suppose, first, that all angle of the triangle are acute.

$$\begin{aligned} a^2 - (b^2 + c^2)(1 - \cos A) &= b^2 + c^2 - 2bc \cos A - (b^2 + c^2) + (b^2 + c^2)\cos A \\ &\ge (b - c)^2 \cos A \ge 0 \end{aligned}$$

$$\Longrightarrow a^2 \ge (b^2 + c^2)(1 - \cos A) = 2(b^2 + c^2)\sin^2(A/2) \ .$$

With similar inequalities for b and c, we find that

$$a^2b^2c^2 \ge 8(a^2 + b^2)(b^2 + c^2)(c^2 + a^2)\sin^2(A/2)\sin^2(B/2)\sin^2(C/2) \ .$$

Since $r = 4R\sin(A/2)\sin(B/2)\sin(C/2)$, the desired result follows. Equality holds if and only if the triangle is equilateral.

Suppose that the angle A is obtuse. Then $\cos A < 0$,

$$a^2 = b^2 + c^2 - 2bc\cos A > b^2 + c^2$$

and

$$2[ABC] = bc\sin A < bc \ .$$

Since $b+c-a$, $a^2-b^2-c^2$ and b^2-bc+c^2 are all positive, we have that

$$a^2(a+b+c)^2 - 2(a^2+b^2)(c^2+a^2) \\ = 2(b+c-a)a^3 + (a^2-b^2-c^2)(a^2+2bc) + 2bc(b^2-bc+c^2) > 0 \ ,$$

whence

$$2(a^2+b^2)(c^2+a^2) < a^2(a+b+c)^2 \ .$$

Hence

$$(b^2+c^2)\cdot 16[ABC]^4 \cdot 2(a^2+b^2)(c^2+a^2) < a^2\cdot b^4c^4\cdot a^2(a+b+c)^2 \ .$$

Since $[ABC] = abc/4R = (a+b+c)r/2$, this becomes

$$(b^2+c^2)(c^2+a^2)(a^2+b^2)\cdot\left(\frac{(abc)^2(a+b+c)^2r^2}{4R^2}\right) < (abc)^4\cdot(a+b+c)^2 \ ,$$

from which the result follows.

Comment. The identity in the solution can be obtained as follows. Let $2s = a+b+c$. Then

$$\frac{r}{s-a} = \tan\frac{A}{2} = \frac{\sin\frac{A}{2}}{\cos\frac{A}{2}}$$

while

$$a = 2R\sin A = 4R\sin\frac{A}{2}\cos\frac{A}{2} \ .$$

Hence

$$\frac{ar}{s-a} = 4R\sin^2\frac{A}{2} \ .$$

Using similar identities for the other sides, we find that

$$\frac{abcr^3}{(s-a)(s-b)(s-c)} = 64R^3\sin^2\frac{A}{2}\sin^2\frac{B}{2}\sin^2\frac{C}{2} \ . \qquad (*)$$

Note that the area Δ of the triangle is given by

$$\Delta = rs = \frac{abc}{4R} = \sqrt{s(s-a)(s-b)(s-c)} \ ,$$

so that the left side of $(*)$ becomes $4R\Delta r^2(rs)\Delta^{-2} = 4Rr^2$. Substituting this in, dividing by $4R$ and taking the square root yields

$$r = 4R\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2} \ .$$

38. Let us say that a set S of nonnegative real numbers if *hunky-dory* if and only if, for all x and y in S, either $x+y$ or $|x-y|$ is in S. For instance, if r is positive and n is a natural number, then $S(n,r) = \{0, r, 2r, \cdots, nr\}$ is hunky-dory. Show that every hunky-dory set is $\{0\}$, is of the form $S(n,r)$ or has exactly four elements.

Solution 1. $\{0\}$ and sets of the form $\{0, r\}$ are clearly hunky-dory. Let S be a nontrivial hunky-dory set with largest postive element z. Then $2z \notin S$, so $0 = z - z \in S$. Thus, every hunky-dory set contains 0. Suppose that S has at least three elements, with least positive element a.

Suppose, if possible, that S contains an element that is not a positive integer multiple of a. Let b be the least nonmultiple of a. Then $0 < b-a < b$. Since $b-a$ cannot be a multiple of a (why?), we must have $b-a \notin S$ and $b+a \in S$. Since z is the largest element of S, $z-a$ and $z-b$ belong to S. However,

$(z-a)-(z-b)=b-a$ does not belong to S, so $2z-(a+b)=(z-a)+(z-b)\in S$. Therefore, $2z-(a+b)\le z$, whence $z\le a+b$, so that $z=a+b$. Thus, S contains $\{0,a,b,a+b\}$, with $a+b$ the largest element. This subset is already hunky-dory. But suppose, if possible, S contains more elements. Let c be the smallest such element. Then $0<(a+b)-c\in S\Rightarrow a\le(a+b)-c\Rightarrow c<b\Rightarrow c=ma$ for some positive integer $m\ge 2$. Since $b+ma>b+a$, $b-ma$ must belong to S, and so be a multiple of a. This yields a contradiction. Hence, S must be equal to $\{0,a,b,a+b\}$.

The only remaining case is that S consists solely of nonnegative multiples of some element a. Let na be the largest such multiple. If $n=2$, then $S=S(2,a)$. Suppose that $n>2$. Then $(n-1)a\in S$, so S contains $\{0,a,(n-1)a,na\}$, which is hunky-dory.

Suppose S contains a further multiple ma with $2\le m\le n-2$. Since $a\in S$ and $na+a>na$, $(n-1)a\in S$, so that $n-(m+1)a=(n-1)a-ma\in S\Rightarrow(m+1)a=n-[n-(m+1)a]\in S$. By induction, it can be shown that $ka\in S$ for $m\le k\le n$. In particular, $(n-2)a\in S$ so that $2a=na-(n-2)a\in S$. But then $3a,4a,\cdots,na$ are in S and so $S=S(n,a)$. The desired result follows,

Solution 2. [S. Niu] Let $S=\{a_0,a_1,\cdots,a_n\}$, with $a_0<a_1<\cdots<a_n$. The elements a_n-a_n, a_n-a_{n-1}, $\cdots$, a_n-a_0 are $n+1$ distinct elements of S listed in increasing order, and so $a_0=0$, and for each i with $0\le i\le n$, we must have that $a_n-a_i=a_{n-i}$. Let $i\le\frac{n}{2}$. Then $i\le n-i$ and so $a_i\le a_{n-i}$; thus, $a_i\le(a_n)/2\le a_{n-i}$. Thus, if $j>k\ge n/2$, $a_j-a_k\in S$.

Since $0<a_{n-1}-a_{n-2}<a_n-a_{n-2}=a_2$, it follows that $a_{n-1}-a_{n-2}=a_1$. Also, $0<a_{n-1}-a_{n-2}=a_1<a_{n-1}-a_{n-3}<a_n-a_{n-3}=a_3$, so that $a_{n-1}-a_{n-3}=a_2$. Continuing on in this way, we find that, for $i\ge n/2$,

$$0<a_{n-1}-a_{n-2}<a_{n-1}-a_{n-3}<\cdots<a_{n-1}-a_{n-i}<a_n-a_{n-i}=a_i,$$

whence $a_{n-1}-a_{n-j}=a_{j-1}$ for $1\le j\le(n/2)$.

Now $0<a_{n-2}-a_{n-3}<a_{n-1}-a_{n-3}=a_2$ so $a_{n-2}-a_{n-3}=a_1$. We can proceed in this fashion to obtain that, for $j\ge n/2$, $a_{j+1}-a_j=a_1$. Hence, for $i\le(n/2)-1$, $a_{i+1}-a_i=(a_n-a_{n-i-1})-(a_n-a_{n-i})=a_{n-i}-a_{n-i-1}=a_1$.

Let $n=2m$. Then $a_i=ia_1$ and $a_{n-i}=a_n-ia_1$ for $1\le i\le m$, so that $a_m=ma_1$ and $a_n=a_m+ma_1=na_1$. It follows that $a_k=ka_1$ for $1\le k\le n$ and $S=S(n,a_1)$.

Let $n=2m+1$. If $m=0$, then $S=\{0,a_1\}=S(1,a_1)$. If $m=1$, then $S=\{0,a_1,a_3-a_1,a_3\}=\{0,a_1,a_2,a_1+a_2\}$ is a 4-element hunky-dory set. Let $m\ge 2$. Then, for $1\le i\le m$, $a_i=ia_1$ and $a_{n-i}=a_n-ia_i$. Now $a_{m+1}=a_n-a_m>a_{n-1}-a_m>\cdots\ge a_{m+2}-a_m>a_{m+1}-a_m\ge a_1$. Since $\{a_n-a_m=a_{m+(m+1)}-a_m,a_{n-1}-a_m,\cdots,a_{m+2}-a_m,a_{m+1}-a_m\}$ contains $m+1$ elements, we must have $a_{m+j}-a_m=a_j$ for $1\le j\le n-m=m+1$. Therefore, $a_i=ia_i$ for $1\le i\le n$. (Why does this last statement fail to follow when $m=1$?)

39. (a) $ABCDEF$ is a convex hexagon, each of whose diagonals AD, BE and CF pass through a common point. Must each of these diagonals bisect the area?

(b) $ABCDEF$ is a convex hexagon, each of whose diagonals AD, BE and CF bisects the area (so that half the area of the hexagon lies on either side of the diagonal). Must the three diagonals pass through a common point?

Solution 1. (a) No, they need not bisect the area. Let the vertices of the hexagon have coordinates $(-1,0)$, $(-1,-1)$, $(1,-1)$, $(1,0)$, $(-t,t)$, $(t,-t)$ with $t>0$ but $t\ne 1$. The diagonals with equations $y=0$, $y=x$ and $y=-x$ intersect in the origin but do not bisect the area of the hexagon.

(b) Let the hexagon be $ABCDEF$ and suppose that the intersection of the diagonals AD and BE is on the same side of CF as the side AB. Thus, AB, CD and EF border on triangles whose third vertices form a triangle at the centre of the hexagon (we will show this triangle to be degenerate). Let a, b, c, d, e, f be the lengths of the rays from the respective vertices A, B, C, D, E, F to the vertices of the central triangle, whose sides are x, y, z so that the lengths of AD, BE and CF are respectively $a+x+d$, $b+y+e$, $c+z+f$. All lower-case variables represent nonnegative real numbers.

Let the areas of the bordering on FA, AB, BC, CD, DE, EF be respectively α, β, γ, δ, ϵ, ϕ, and let the area of the central triangle be λ. Then, since each diagonal bisects the area of the hexagon, we have that

$$\alpha+\beta+\gamma+\lambda=\delta+\epsilon+\phi$$

$$\epsilon+\phi+\alpha+\lambda=\beta+\gamma+\delta$$

$$\gamma+\delta+\epsilon+\lambda=\phi+\alpha+\beta\ .$$

From the first two equations, we find that $\delta=\alpha+\lambda$. Similarly, $\phi=\gamma+\lambda$ and $\beta=\epsilon+\lambda$.

Using the fact that the area of a trangle is half the product of adjacent sides and the sine of the angle between them, and the equality of opposite angles, we find that

$$1=\frac{\alpha+\lambda}{\delta}=\frac{(a+x)(f+z)}{cd}$$

$$1=\frac{\gamma+\lambda}{\phi}=\frac{(b+y)(c+z)}{ef}$$

$$1=\frac{\epsilon+\lambda}{\beta}=\frac{(d+x)(e+y)}{ab}\ .$$

Multiplying these three equations together yields that

$$abcdef=(a+x)(b+y)(c+z)(d+x)(e+y)c+z)\ ,$$

whence $x=y=z=0$. Thus, the central triangle degenerates and the three diagonals intersect in a common point.

Solution 2. (a) No. Let $ABCDEF$ be a regular hexagon. The diagonals AD, BE, CF intersect and each diagonal *does* bisect the area. Let X be any point other than F on the diagonal CF for which $ABCDEX$ is still a convex hexagon. The diagonals of this hexagon are the same as those of the regular hexagon, and so have a common point of intersection. However, the diagonals AD and BE no longer intersect the area of the hexagon.

(b) [X. Li] Let $ABCDEF$ be a given convex hexagon, each of whose diagonals bisect its area. Suppose that the diagonals AD and CF intersect at G. As in Solution 1, we can determine that the areas of triangles AGF and DGC are equal, whence $AG\cdot GF=CG\cdot GD$, or $AG/GD=CG/GF$. Therefore, $\Delta AGC\sim\Delta DGF$ (SAS). It follows that $AC/DF=AG/GD=CG/GF$, $\angle CAG=\angle FDG$, and so $AC\|DF$. In a similar way, we find that $BF\|CE$ and $AE\|BD$, so that $\Delta ACE\sim\Delta DFB$ and $AC/DF=CE/FB=EA/BD$.

Suppose diagonals AD and BE intersect at H. Then, as above, we find that $AG/GD=AC/DF=EA/BD=EH/HB=AH/HD$, so that $H=G$. Hence, the three diagonals have the point G in common.

40. Determine all solutions in integer pairs (x,y) to the diophantine equation $x^2=1+4y^3(y+2)$.

Solution 1. Clearly, $(x,y)=(\pm1,0),(\pm1,-2)$ are solutions. When $y=-1$, the right side is negative and there is no solution. Suppose that $y\geq1$; then

$$(2y^2+2y)^2=4y^4+8y^3+4y^2>4y^4+8y^3+1$$

and

$$(2y^2+2y-1)^2=4y^4+8y^3-4y+1<4y^4+8y^3+1$$

so that the right side is between two consecutive squares, and hence itself cannot be square.

Suppose that $y \leq -3$. We first observe that for a given product p of two positive integers, the sum of these positive integers has a minium value of $2\sqrt{p}$ (why?) and a maximum value of $1+p$. This follows from the fact that, for integers u with $1 \leq u \leq p$,

$$(1+p) - (u + p/u) = (u-1)[(p/u) - 1] \geq 0 \ .$$

We have that

$$\begin{aligned}[(2y^2+2y-1)+x][(2y^2+2y-1)-x] &= (2y^2+2y-1)^2 - x^2 \\ &= (4y^4+8y^3-4y+1) - (4y^4+8y^3+1) \\ &= -4y \ .\end{aligned}$$

Since $2y^2+2y-1 = y^2+(y+1)^2-2$ is positive, at least one of the factors on the left is positive. Since the product is positive, both factors are positive. By our observation on the sum of the factors, we find that

$$4y^2+4y-2 \leq 1-4y \ ,$$

which is equivalent to

$$4(y-1)^2 \leq 7 \ .$$

However, this does not hold when $y \leq -3$. Therefore, the only solutions are the four that we identified at the outset.

Solution 2. Since x must be odd, we can let $x = 2z+1$ for some integer z, so that the equation becomes $z(z+1) = y^4+2y^3$. We can deal with the cases that $y = 0, -1, -2$ directly to obtain the solutions $(x, y, z) = (1, 0, 0), (-1, 0, -1), (1, -2, 0), (-1, -2, -1)$. Henceforth, suppose that $y \geq 1$ or $y \leq -3$, so that y^4+2y^3 is positive. Let $\phi(t) = t(t+1)$. Then $\phi(t)$ is increasing for $t \geq 0$ and $\phi(-t) = \phi(t-1)$ for every integer t; thus, we need check only that y^4+2y^3 does not coincide with a value taken by $\phi(t)$ for nonnegative values of t.

Now

$$\phi(y^2+y) = y^4+2y^3+2y^2+y > y^4+2y^3 \quad ;$$

$$\phi(y^2+y-1) = y^4+2y^3-y \neq y^4+2y^3 \quad ;$$

$$\begin{aligned}\phi(y^2+y-2) &= y^4+2y^3-2y^2-3y+2 \\ &= y^4+2y^3-(2y+1)(y+1)+3 < y^4+2y^3 \quad .\end{aligned}$$

It follows that $\phi(t)$ can never assume the value y^4+2y^3 for any positive t, and hence for any t. Thus, the solutions already listed comprise the complete solution set.

41. Determine the least positive number p for which there exists a positive number q such that

$$\sqrt{1+x} + \sqrt{1-x} \leq 2 - \frac{x^p}{q}$$

for $0 \leq x \leq 1$. For this least value of p, what is the smallest value of q for which the inequality is satisfied for $0 \leq x \leq 1$?

Comments. Recall the binomial expansion

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \cdots + \frac{n(n-1)\cdots(n-r+1)}{r!}x^r + \cdots \ .$$

When n is not a nonnegative integer, this is an infinite series that converges when $0 \leq |x| < 1$ to $(1+x)^n$. The partial sums constitute a close approximation. When $n = \frac{1}{2}$, we have that

$$(1 \pm x)^{\frac{1}{2}} = 1 \pm \frac{1}{2}x - \frac{1}{8}x^2 \pm \frac{1}{16}x^3 - \frac{5}{128}x^4 \pm \cdots$$

so that

$$(1+x)^{\frac{1}{2}} + (1-x)^{\frac{1}{2}} \sim 2 - \frac{x^2}{4} - \frac{x^4}{8} \leq 2 - \frac{x^4}{4} \ .$$

This suggests that we are looking for $(p, q) = (2, 4)$. However, the approximation approach is not sufficiently rigorous, and we need to find an argument in finite terms that will work.

Solution 1. Observe that, for $0 \leq x \leq 1$,

$$\sqrt{1 \pm x} \leq 1 \pm \frac{1}{2}x - \frac{1}{8}x^2 \pm \frac{1}{16}x^3$$

$$\Leftrightarrow 1 \pm x \leq 1 \pm x + \frac{5}{64}x^4 \mp \frac{1}{64}x^5 + \frac{1}{256}x^6$$

$$\Leftrightarrow 0 \leq 5 \mp x + 4x^2 \ .$$

The last inequality clearly holds, so the first must as well. Hence

$$\sqrt{1+x} + \sqrt{1-x} \leq 2(1 - \frac{1}{8}x^2) = 2 - \frac{1}{4}x^2$$

so the pair $(p, q) = (2, 4)$ works for all $x \in [0, 1]$.

Suppose, for some constants p and c with $0 < p < 2$ and $c > 0$,

$$\sqrt{1+x} + \sqrt{1-x} \leq 2 - 2cx^p$$

for $0 \leq x \leq 1$. For this range of x, this is equivalent to

$$2 + 2\sqrt{1-x^2} \leq 4 - 8cx^p + 4c^2x^{2p}$$

$$\Leftrightarrow \sqrt{1-x^2} \leq 1 - 4cx^p + 2c^2x^{2p}$$

$$\Leftrightarrow 1 - x^2 \leq 1 - 8cx^p + 20c^2x^{2p} - 16c^3x^{3p} + 4c^4x^{4p}$$

$$8c \leq x^{2-p} + 4c^2x^p(5 - 4cx^{2p} + c^3x^{3p}) \ .$$

However, for x sufficiently small, the right side can be made less than $8c$, yielding a contradiction. Hence, when $0 < p < 2$, there is no value that yields the desired inequality.

Now we look at the situation when $p = 2$ and $q > 0$. For $0 \leq x \leq 1$,

$$\sqrt{1+x} + \sqrt{1-x} \leq 2 - \frac{x^2}{q}$$

$$\Leftrightarrow 2(1 + \sqrt{1-x^2}) \leq 4 - \frac{4x^2}{q} + \frac{x^4}{q^2}$$

$$\Leftrightarrow \sqrt{1-x^2} \leq 1 - \frac{2x^2}{q} + \frac{x^4}{2q^2}$$

$$\Leftrightarrow 1 - x^2 \leq 1 - \frac{4x^2}{q} + \frac{5x^4}{q^2} - \frac{2x^6}{q^3} + \frac{x^8}{4q^4}$$

$$\Leftrightarrow 0 \leq x^2\left[\left(1 - \frac{4}{q}\right) + \frac{x^2}{q^2}\left(5 - \frac{2x^2}{q} + \frac{x^4}{4q^2}\right)\right] \ .$$

If $q < 4$, then the quantity in square brackets is negative for small values of x. Hence, for the inequality to hold for all x in the interval $[0, 1]$, we must have $q \geq 4$. Hence, p must be at least 2, and for $p = 2$, q must be at least 4.

Solution 2. [R. Furmaniak] The given inequality is equivalent to

$$\begin{aligned} q &\geq \frac{x^p}{2-\sqrt{1+x}-\sqrt{1-x}} \\ &= \frac{x^p(2+\sqrt{1+x}+\sqrt{1-x})}{4-(2+2\sqrt{1-x^2})} \\ &= \frac{x^p(2+\sqrt{1+x}+\sqrt{1-x})}{2(1-\sqrt{1-x^2})} \\ &= \frac{x^p(2+\sqrt{1+x}+\sqrt{1-x})(1+\sqrt{1-x^2})}{2x^2} . \end{aligned}$$

If $p < 2$, then the right side becomes arbitrarily large as x gets close to zero, so the inequality becomes unsustainable for any real q. Hence, for the inequality to be viable, we require $p \geq 2$. When $p = 2$, we can cancel x^2 and see by taking $x = 0$ that $q \geq 4$. It remains to verify the inequality when $(p, q) = (2, 4)$. We have the following chain of logically equivalent statements, where $y = \sqrt{1-x^2}$ (note that $0 \leq x \leq 1$):

$$\sqrt{1+x}+\sqrt{1-x} \leq 2 - \frac{x^2}{4}$$

$$\Leftrightarrow 2+2\sqrt{1-x^2} \leq 4 - x^2 + \frac{x^4}{16}$$

$$\Leftrightarrow 32\sqrt{1-x^2} \leq x^4 - 16x^2 + 32$$

$$\Leftrightarrow 32y \leq 1 - 2y^2 + y^4 - 16 + 16y^2 + 32$$

$$\Leftrightarrow$$

$$0 \leq y^4 + 14y^2 - 32y + 17 = (y-1)^2(y^2+2y+17) = (y-1)^2[(y+1)^2+16] .$$

Since the last inequality is clearly true, the first holds and the result follows.

42. G is a connected graph; that is, it consists of a number of vertices, some pairs of which are joined by edges, and, for any two vertices, one can travel from one to another along a chain of edges. We call two vertices *adjacent* if and only if they are endpoints of the same edge. Suppose there is associated with each vertex v a nonnegative integer $f(v)$ such that all of the following hold:
(1) If v and w are adjacent, then $|f(v) - f(w)| \leq 1$.
(2) If $f(v) > 0$, then v is adjacent to at least one vertex w such that $f(w) < f(v)$.
(3) There is exactly one vertex u such that $f(u) = 0$.
Prove that $f(v)$ is the number of edges in the chain with the fewest edges connecting u and v.

Solution. We prove by induction that $f(x) = n$ if and only if the shortest chain from u to x has n members. This is true for $n = 0$ (and for $n = 1$). Suppose that this holds for $0 \leq n \leq k$.

Let $f(x) = k + 1$. There exists a vertex y adjacent to x for which $h = f(y) < k + 1$. By the induction hupothesis, y can be connected to u by a chain of h edges, so x can be connected to u by a chain of $h + 1$ edges. Hence, $h + 1 \geq k + 1$. From these two inequalities, we must have $h = k$, so x can be connected to u by a chain of $k + 1$ edges. There cannot be a shorter chain, as, by the induction hypothesis, this would mean that $f(x)$ would have to be less than $k + 1$.

Let the shortest chain connecting x to u have $k + 1$ edges. Following along this chain, we can find an element z adjacent to x connected to u by k edges. This must be one of the shortest chains between u and z, so that $f(z) = k$. By hypothesis (1), $f(x)$ must take one of the values $k - 1$ and $k + 1$. The first is not admissible, since there is no chain with $k - 1$ edges connecting u and x. Hence $f(x) = k + 1$.

43. Two players play a game: the first player thinks of n integers $x_1, x_2, \cdots, x_n$, each with one digit, and the second player selects some numbers $a_1, a_2, \cdots, a_n$ and asks what is the value of the sum

$a_1x_1 + a_2x_2 + \cdots + a_nx_n$. What is the minimum number of questions used by the second player to find the integers x_1, x_2, $\cdots$, x_n?

Solution. We are going to prove that the second player needs only one question to find the integers x_1, x_2, $\cdots$, x_n. Indeed, let him choose $a_1 = 100$, $a_2 = 100^2$, $\cdots$, $a_n = 100^n$ and ask for the value of the sum

$$S_n = 100x_1 + 100^2x_2 + \cdots + 100^nx_n \quad .$$

Note that

$$\begin{aligned}
&\left|\frac{100x_1 + 100^2x_2 + \cdots + 100^{n-1}x_{n-1}}{100^n}\right| \\
&\le \frac{100|x_1| + 100^2|x_2| + \cdots + 100^{n-1}|x_{n-1}|}{100^n} \\
&\le \frac{9(100 + 100^2 + \cdots + 100^{n-1})}{100^n} \\
&< \frac{10^{2n-1}}{10^{2n}} = \frac{1}{10} \; .
\end{aligned}$$

Hence

$$\left|\frac{S_n}{100^n} - x_n\right| = \frac{100x_1 + 100^2x_2 + \cdots + 100^{n-1}x_{n-1}}{100^n} < \frac{1}{10} \; ,$$

and x_n can be obtained. Now, we can find the sum

$$S_{n-1} = S_n - 100^nx_n = 100x_1 + 100^2x_2 + \cdots + 100^{n-1}x_{n-1}$$

and similarly obtain x_{n-1}. The procedure continues until all the numbers are found.

44. Find the permutation $\{a_1, a_2, \cdots, a_n\}$ of the set $\{1, 2, \cdots, n\}$ for which the sum

$$S = |a_2 - a_1| + |a_3 - a_2| + \cdots + |a_n - a_{n-1}|$$

has maximum value.

Solution. Let $a = (a_1, a_2, \cdots, a_n)$ be a permutation of $\{1, 2, \cdots, n\}$ and define

$$f(a) = \sum_{k=1}^{n-1} |a_{k+1} - a_k| + |a_n - a_1| \; .$$

With $a_{n+1} = a_1$ and $\epsilon_0 = \epsilon_n$, we find that

$$f(a) = \sum_{k=1}^{n} \epsilon_k(a_{k+1} - a_k) = \sum_{k=1}^{n} (\epsilon_{k-1} - \epsilon_k)a_k$$

where ϵ_1, ϵ_2, $\cdots$, ϵ_n are all equal to 1 or -1. Thus

$$f(a) = \sum_{k=1}^{n} \beta_k k \; .$$

where each β_k is one of -2, 0, 2, and $\sum_{k=1}^{n} \beta_k = 0$ (there are the same number of positive and negative numbers among the β_k).

Therefore

$$f(a) = 2(x_1 + x_2 + \cdots + x_m) - 2(y_1 + y_2 + \cdots + y_m)$$

where $x_1, \cdots, x_m, y_1, \cdots, y_m \in \{1, 2, \cdots, n\}$ and are distinct from each other. Hence $f(a)$ is maximum when $\{x_1, x_2, \cdots, x_m\} = \{n, n-1, \cdots, n-m+1\}$, $\{y_1, y_2, \cdots, y_m\} = \{1, 2, \cdots, m\}$ with $m = \lfloor n/2 \rfloor$, *i.e.*, m is as large as possible. The maximum value is

$$2\left\lfloor \frac{n}{2} \right\rfloor \left(n - \left\lfloor \frac{n}{2} \right\rfloor\right) .$$

This value is attained by taking

$$a = (s+1, 1, s+2, 2, \cdots, 2s, s) \qquad \text{when} \quad n = 2s$$

and

$$a = (s+2, 1, s+3, 2, \cdots, 2s+1, s, s+1) \qquad \text{when} \quad n = 2s+1 .$$

Since $|a_n - a_1| = 1$ for these permutations, the maximum value of the given expression is

$$2\left\lfloor \frac{n}{2} \right\rfloor \left(n - \left\lfloor \frac{n}{2} \right\rfloor\right) - 1 .$$

This is equal to $2s^2 - 1$ when $n = 2s$, and to $2s(s+1) - 1$ when $n = 2s+1$.

45. Prove that there is no nonconstant polynomial $p(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_0$ with integer coefficients a_i for which $p(m)$ is a prime number for every integer m.

Solution. Let a be an integer, for which $p(a) \neq -1, 0, 1$. (If there is no such a, then p cannot take all prime values.) Suppose that b is a prime divisor of $p(a)$. Now, for any integer k,

$$p(a+kb) - p(a) = a_n[(a+kb)^n - a^n] + a_{n-1}[(a+kb)^{n-1} - a_{n-1}] + \cdots + a_1[(a+kb) - a] .$$

It can be seen that b is a divisor of $p(a+kb) - p(a)$ and hence of $p(a+kb)$ for every integer k. Both of the equations $p(x) = b$ and $p(x) = -b$ have at most finitely many roots. So some of the values of $p(a+kb)$ must be composite, and the result follows.

Comment. It should have been stated in the problem that the polynomial was nonconstant, or had positive degree.

46. Let $a_1 = 2$, $a_{n+1} = \frac{a_n+2}{1-2a_n}$ for $n = 1, 2, \cdots$. Prove that

(a) $a_n \neq 0$ for each positive integer n;

(b) there is no integer $p \geq 1$ for which $a_{n+p} = a_n$ for every integer $n \geq 1$ (*i.e.*, the sequence is not periodic).

Solution. (a) We prove that $a_n = \tan n\alpha$ where $\alpha = \arctan 2$ by mathematical induction. This is true for $n = 1$. Assume that it holds for $n = k$. Then

$$a_{k+1} = \frac{2 + a_n}{1 - 2a_n} = \frac{\tan\alpha + \tan n\alpha}{1 - \tan\alpha \tan n\alpha} = \tan(n+1)\alpha ,$$

as desired.

Suppose that $a_n = 0$ with $n = 2m+1$. Then $a_{2m} = -2$. However,

$$a_{2m} = \tan 2m\alpha = \frac{2\tan m\alpha}{1 - \tan^2 m\alpha} = \frac{2a_m}{1 - a_m^2} ,$$

whence

$$\frac{2a_m}{1 - a_m^2} = -2 \Leftrightarrow a_m = \frac{1 \pm \sqrt{5}}{2} ,$$

which is not possible, since a_m has to be rational.

Suppose that $a_n = 0$ with $n = 2^k(2m+1)$ for some positive integer k. Then

$$0 = a_n = \tan 2 \cdot 2^{k-1}(2m+1)\alpha = \frac{2\tan 2^{k-1}(2m+1)}{1-\tan^2 2^{k-1}(2m+1)} = \frac{2a_{n/2}}{1-a_{n/2}^2} ,$$

so that $a_{n/2} = 0$. Continuing step by step backward, we finally come to $a_{2m+1} = 0$, which has already been shown as impossible.

(b) Assume, if possible, that the sequence is periodic, *i.e.*, there is a positive integer p such that $a_{n+p} = a_n$ for every positive integer n. Thus

$$\tan(n+p)\alpha - \tan n\alpha = \frac{\sin p\alpha}{\cos(n+p)\alpha \cos n\alpha} = 0 .$$

Therefore $\sin p\alpha = 0$ and so $a_p = \tan p\alpha = 0$, which, as we have shown, is impossible. The desired result follows.

47. Let $a_1, a_2, \cdots, a_n$ be positive real numbers such that $a_1 a_2 \cdots a_n = 1$. Prove that

$$\sum_{k=1}^{n} \frac{1}{s-a_k} \le 1$$

where $s = 1 + a_1 + a_2 + \cdots + a_n$.

Solution. First, we recall that Chebyshev's Inequalities:
(1) if the vectors $(a_1, a_2, \cdots, a_n)$ and $(b_1, b_2, \cdots, b_n)$ are similarly sorted (that is, both rising or both falling), then

$$\frac{a_1b_1 + \cdots + a_nb_n}{n} \ge \frac{a_1 + \cdots + a_n}{n} \cdot \frac{b_1 + \cdots + b_n}{n} ;$$

(2) if the vectors $(a_1, a_2, \cdots, a_n)$ and $(b_1, b_2, \cdots, b_n)$ are oppositely sorted (that is, one rising and the other falling), then

$$\frac{a_1b_1 + \cdots + a_nb_n}{n} \le \frac{a_1 + \cdots + a_n}{n} \cdot \frac{b_1 + \cdots + b_n}{n} .$$

If $x_1, x_2, \cdots, x_n$ are positive real numbers with $x_1 \le x_1 \le \cdots \le x_n$, then $x_1^n \le x_2^n \le \cdots \le x_n^n$. From Chebyshev's Inequality (1), we have, for each $k = 1, 2, \cdots, n$, that

$$\sum_{i=1, i\ne k}^{n} x_i^n = \sum_{i=1, i\ne k}^{n} x_i^{n-1} x_i \ge \frac{1}{n-1} \Big(\sum_{i=1, i\ne k}^{n} x_i \Big) \Big(\sum_{i=1, i\ne k}^{n} x_i^{n-1} \Big) .$$

The Arithmetic-Geometric Means Inequality yields

$$\sum_{i=1, i\ne k}^{n} x_i^{n-1} \ge (n-1) x_1 \cdots x_{k-1} x_{k+1} \cdots x_n ,$$

for $k = 1, \cdots, n$. Therefore,

$$\sum_{i=1, i\ne k}^{n} x_i^n \ge (x_1 \cdots x_{k-1} x_{k+1} \cdots x_n) \sum_{i=1, i\ne k}^{n} x_i .$$

for each k. This inequality can also be written

$$x_1^n + \cdots + x_{k-1}^n + x_{k+1}^n + \cdots x_n^n + x_1 x_2 \cdots x_n$$

$$\geq x_1 \cdots x_{k-1}x_{k+1}\cdots x_n(x_1+x_2+\cdots+x_n) \ ,$$

or

$$\frac{1}{x_1x_2\cdots x_n + x_1^n+\cdots+x_{k-1}^n+x_{k+1}^n+\cdots+x_n^n} \leq \frac{1}{x_1+x_2+\cdots+x_n}\cdot\frac{1}{x_1\cdots x_{k-1}x_{k+1}\cdots x_n} \ .$$

Adding up these inequalities, for $1 \leq k \leq n$, we get

$$\sum_{k=1}^{n} \frac{1}{x_1x_2\cdots x_n + x_1^n+\cdots+x_{k-1}^n+x_{k+1}^n+\ldots x_n^n} \leq \frac{1}{x_1x_2\cdots x_n} \ .$$

Now, let the x_k^n be equal to the a_k in increasing order to obtain the desired result.

48. Let $A_1A_2\cdots A_n$ be a regular $n-$gon and d an arbitrary line. The parallels through A_i to d intersect its circumcircle respectively at B_i $(i = 1, 2, \cdots, n$. Prove that the sum

$$S = |A_1B_1|^2 + \cdots + |A_nB_n|^2$$

is independent of d.

Solution. Select a system of coordinates so that O is the centre of the circumcircle and the $x-$axis (or real axis) is orthogonal to d. Without loss of generality, we may assume that the radius of the circumcircle is of length 1. Let a_k the the affix (complex number representative) of A_k $(1 \leq k \leq n)$. Then the a_k are solutions of the equation $z^n = \lambda$, where λ is a complex number with $|\lambda| = 1$. Since A_k and B_k are symmetrical with respect to the real axis, the affix of B_k is $\overline{a_k}$, the complex conjugate of a_k, for $1 \leq k \leq n$, Thus

$$A_kB_k^2 = |a_k - \overline{a_k}|^2 = (a_k - \overline{a_k})(\overline{a_k} - a_k) = 2a_k\overline{a_k} - a_k^2 - \overline{a_k}^2 = 2 - a_k - \overline{a_k}^2 \ .$$

Summing these inequalities yields that

$$\sum_{k=1}^{n} A_kB_k^2 = 2n - \sum_{k=1}^{n} a_k^2 - \sum_{k=1}^{n} \overline{a_k}^2 \ .$$

Since $\{a_k : 1 \leq k \leq n\}$ is a complete set of solutions of the equation $z^n = \lambda$, their sum and the sum of their pairwise products vanishes. Hence

$$0 = \sum_{k=1}^{n} a_k^2 = \sum_{k=1}^{n} \overline{a_k}^2 \ .$$

Hence $\sum_{k=1}^{n} A_kB_k^2 = 2n$.

49. Find all ordered pairs (x, y) that are solutions of the following system of two equations (where a is a parameter):

$$x - y = 2$$

$$\left(x - \frac{2}{a}\right)\left(y - \frac{2}{a}\right) = a^2 - 1 \ .$$

Find all values of the parameter a for which the solutions of the system are two pairs of nonnegative numbers. Find the minimum value of $x + y$ for these values of a.

Solution. We need to assume that $a \neq 0$. Substitute $y = x - 2$ into the second equation to obtain

$$x^2 - 2x - \frac{4}{a}(x-1) + \frac{4}{a^2} = a^2 - 1 \ .$$

This can be manipulated to

$$\left[x - \left(1 + \frac{2}{a}\right)\right]^2 = a^2 \ ,$$

from which we find that

$$(x,y) = (1 + (2/a) + a, -1 + (2/a) + a), (1 + (2/a) - a, -1 + (2/a) - a) \ .$$

For x and y to be nonnegative for both solutions, we require all of the four inequalities:

$$a(a^2 + a + 2) \geq 0 \ , \quad a(a^2 - a + 2) \geq 0 \ ,$$

$$a(a-2)r(a+1) = a(a^2 - a - 2) \leq 0 \ , \quad a(a+2)(a-1) = a(a^2 + a - 2) \leq 0 \ .$$

The last two equations entail that $a \leq -2$ or $0 < a \leq 1$. When $a \leq -2$, the first two inequalities fail to hold, while if $0 < a \leq 1$, all four equations hold. Thus, the solutions are two pairs of nonnegative numbers when $0 < a \leq 1$.

When both x and y are nonnegative, then $x + y \geq x - y = 2$. When $a = 1$, we obtain the solutions $(x, y) = (4, 2)$ and $(2, 0)$, and in the latter case, $x + y = 2$. Hence the minimum value of $x + y$ is 2.

Comment. The minimum of $x + y$ can be also obtained with more effort directly from the solutions in terms of a. Now $x + y = (4/a) \pm 2a$. Since

$$\left(\frac{4}{a} + 2a\right) - 6 = \frac{2(2-a)(1-a)}{a} \geq 0$$

for $0 < a \leq 1$, $(4/a) + 2a$ assumes its minimum value of 6 when $a = 1$. Since

$$\left(\frac{4}{a} - 2a\right) - 2 = \frac{2(2+a)(1-a)}{a} \geq 0$$

for $0 < a \leq 1$, $(4/a) - 2a$ assumes its minimum value of 2 when $a = 1$. Hence the minimum possible value of $x + y$ is 2.

50. Let n be a natural number exceeding 1, and let A_n be the set of *all* natural numbers that are not relatively prime with n (*i.e.*, $A_n = \{x \in \mathbf{N} : \ \gcd\ (x, n) \neq 1\}$. Let us call the number n *magic* if for each two numbers $x, y \in A_n$, their sum $x + y$ is also an element of A_n (*i.e.*, $x + y \in A_n$ for $x, y \in A_n$).

    (a) Prove that 67 is a magic number.

    (b) Prove that 2001 is **not** a magic number.

    (c) Find all magic numbers.

Solution. [O. Bormashenko] (a) 67 is prime, so that all numbers that are not relatively prime with 67 and are elements of A_{67} are the multiples of 67. The sum of two such numbers is also a multiple of 67, and hence belongs to A_{67}. Therefore 67 is a magic number.

(b) $2001 = 3 \times 23 \times 29$. Now 3 and 23 belong to A_{2001}, but $26 = 3 + 23$ does not, because gcd (26, 2001) is equal to 1. Thus, 2001 is not a magic number.

(c) First, let us prove that all prime powers are magic numbers. Suppose that $n = p^k$ for some prime p and positive integer k. Then the numbers not relatively prime to n are precisely those that are divisible by p (as this is the only prime that can divide any divisor of n). The sum of any two such numbers is also divisible by p, so that A_n is closed under addition and n is magic.

Now suppose that n is not a power of a prime. Then $n = ab$, where a and b are two relatively prime numbers exceeding 1. Clearly, a and b belong to A_n. However, $\gcd(a, a+b) = \gcd(a, b) = 1$, and $\gcd(b, a+b)$

$= \gcd(b,a) = 1$, so that $a+b$ is relatively prime to both a and b, and hence to $n = ab$. Thus, $a+b \notin A_n$, so that n is not magic.

We conclude that the set of prime powers equals the set of magic numbers.

51. In the triangle ABC, $AB = 15$, $BC = 13$ and $AC = 12$. Prove that, for this triangle, the angle bisector from A, the median from B and the altitude from C are concurrent (*i.e.*, meet in a common point).

Solution. [K. Ho] Denote by E the intersection of AC with the median from B (so that E is the midpoint of AC), by F the intersection of AB with the altitude from C (so that $CF \perp AB$), by D the intersection of BC and the bisector from A, and by x the length of AF. Then $BF = AB - AF = 15 - x$. By the angle-bisector theorem, $CD : DB = AC : AB = 12 : 15 = 4 : 5$. By the pythagorean theorem for the triangles AFC and BFC, $AC^2 - AF^2 = CF^2 = BC^2 - BF^2 \Leftrightarrow 12^2 - x^2 = 13^2 - (15-x)^2 = -56 + 30x - x^2$. Thus, $x = 20/3$ and $15 - x = 25/3$, so that $AF : FB = 4 : 5$. Since $(AF/FB)(BD/DC)/(CE/EA) = (4/5)(5/4)1 = 1$, Ceva's theorem tells us that AD, BE and CF are concurrent.

52. One solution of the equation $2x^3 + ax^2 + bx + 8 = 0$ is $1 + \sqrt{3}$. Given that a and b are rational numbers, determine its other two solutions.

Solution 1. [R. Barrington Leigh] Since $1+\sqrt{3}$ satisfies the given equation,

$$0 = 2(1+\sqrt{3})^3 + a(1+\sqrt{3})^2 + b(1+\sqrt{3}) + 8 = (28 + 4a + b) + (12 + 2a + b)\sqrt{3}\ .$$

Since a and b are rational, this is possible only when $28 + 4a + b = 12 + 2a + b = 0$, or $(a,b) = (-8, 4)$. The equation is thus $0 = 2x^3 - 8x^2 + 4x + 8$. By inspection, we find that 2 is a root, and so the three roots turn out to be 2, $1+\sqrt{3}$ and $1-\sqrt{3}$.

Comment. Once 2 and $1+\sqrt{3}$ are known to be roots, we can get the third root by noting that the sum of the roots is $-(-8)/2 = 4$.

Solution 2. A more structural way of getting the result is to note that the mapping that takes a surd $u + v\sqrt{3}$ (with u and v rational) to its surd conjugate $u - v\sqrt{3}$ preserves addition, subtraction and multiplication (*i.e.,* the surd conjugate of the sum (resp. product) or two surds is equal to the sum (resp. product) of the surd conjugates. The surd conjugate of a rational is the rational itself. Tranforming all the elements of the equation to their surd conjugates, gives the same equation with x replaced by its surd conjugate. Thus, the surd conjugate $1-\sqrt{3}$ of $1+\sqrt{3}$ also satisfies the equation. The quadratic with these as roots is $x^2 - 2x - 2$, and this quadratic must be a factor of the given cubic. Since $2x^3 + ax^2 + bx + 8 = (x^2 - 2x - 2)(2 + [a+4]x) + (2a + b + 12)x + (16 + 2a)$, we must have $2a + b + 12 = 16 + 2a = 0$, whence $(a,b) = (-8,4)$, and

$$2x^3 + ax^2 + bx + 8 = 2(x^2 - 2x - 2)(x - 2)$$

and the third root is 2.

53. Prove that among any 17 natural numbers chosen from the sets $\{1, 2, 3, \cdots, 24, 25\}$, it is always possible to find two whose product is a perfect square.

Solution. [K. Ho] Consider the following 16 sets: $\{1,4,9,16,25\}$, $\{2,8,18\}$, $\{3,12\}$, $\{5,20\}$, $\{6,24\}$, $\{7\}$, $\{10\}$, $\{11\}$, $\{13\}$, $\{14\}$, $\{15\}$, $\{17\}$, $\{19\}$, $\{21\}$, $\{22\}$, $\{23\}$. In any of the sets with more than one element, the product of any two elements is a perfect square. Any choice of 17 numbers from among these sets must, by the Pigeonhole Principle, yield at least two from the same set. The product of these two must be a square. The desired result follows.

54. A circle has exactly one common point with each of the sides of a $(2n+1)-$sided polygon. None of the vertices of the polygon is a point of the circle. Prove that at least one of the sides is a tangent of the circle.

Solution. [J.Y. Jin] Assume that none of the sides is tangent to the circle. Let $A_1, A_2, \cdots, A_{2n-1}$ be the consecutive vertices of a $(2n+1)-$sided polygon. Each side of the polygon has exactly one common point with the circle, none of the vertices lies on the circle and, by assumption, none of the sides is tangent to the circle. Therefore, for each side, one of the endpoints lies inside the circle and the second endpoint lies outside the circle. Colour the vertices inside the circle blue and the ones outside the circle red. Apparently, the colours of the consecutive vertices alternate (*i.e.*, blue, red, blue, red, etc.). Since there are $2n+1$ vertices, A_1 and A_{2n+1} must have the same colour. However, these two vertices are the endpoints of a side of the polygon and so they cannot have the same colour. Thus, the assumption leads to a contradiction, and so at least one of the sides is tangent to the circle.

GEOMETRY ON THE 2000 USAMO

There were two geometry problems on the 2000 USAMO, both of which were reasonably well handled by the Canadian candidates. However, each of them illustrate the importance of not jumping to conclusions and ending up with a score that does not reflect your understanding of the situation.

2. Let S be the set of all triangles ABC for which

$$5\left(\frac{1}{AP} + \frac{1}{BQ} + \frac{1}{CR}\right) - \frac{3}{\min\{AP, BQ, CR\}} = \frac{6}{r} ,$$

where r is the inradius and P, Q, R are the points of tangency of the incircle with sides AB, BC, CA, respectively. Prove that all triangles in S are isosceles and similar to one another.

Comments. Since the statement either holds or does not hold for both of a pair of similar triangles (*i.e.*, it does not depend on scale), we can eliminate a possible variable by assuming that $r = 1$. If the angles of the triangle at A, B and C are respectively, 2α, 2β, 2γ, we find that the lengths of AP, BQ and CR are, respectively, $\cot\alpha$, $\cot\beta$ and $\cot\gamma$. Assuming that $\cot\alpha$ is the smallest of these, we find that the condition is

$$0 = 2\tan\alpha + 5(\tan\beta + \tan\gamma) - 6 .$$

In our quest to reduce the number of variables we have to deal with, we note that $\alpha + \beta + \gamma = 90°$, so we try to combine the expression involving β and γ into something depending on their sum. We need the observation that $\tan x$ is convex for $0 \leq x < 90°$, so that $\tan((\alpha+\beta)/2) \leq (1/2)(\tan\alpha + \tan\beta)$.

So let us dispose of this first. From the tangent of a difference, we have the identity

$$\tan\rho - \tan\sigma = \tan(\rho - \sigma)[1 + \tan\rho\tan\sigma] .$$

Applying this, when $0 \leq \alpha, \beta < 90°$ yields

$$\begin{aligned}\left[\tan\alpha - \tan\left(\frac{\alpha+\beta}{2}\right)\right] - \left[\tan\left(\frac{\alpha+\beta}{2}\right) - \tan\beta\right] \\ = \tan\frac{\alpha-\beta}{2}\left[\left(1 + \tan\alpha\tan\frac{\alpha+\beta}{2}\right) - \left(1 + \tan\beta\tan\frac{\alpha+\beta}{2}\right)\right] \\ = \tan\frac{\alpha-\beta}{2} \cdot \tan\frac{\alpha+\beta}{2} \cdot (\tan\alpha - \tan\beta) \geq 0 ,\end{aligned}$$

whence

$$2\tan\frac{\alpha+\beta}{2} \leq \tan\alpha + \tan\beta .$$

Using the convexity of $\tan x$ and making the substitution $t = \tan(\alpha/2)$, we find that

$$\begin{aligned} 0 &\geq 2\tan\alpha + 10\tan\left(\frac{\beta+\gamma}{2}\right) - 6 \\ &= 2\tan\alpha + 10\tan\left(45^\circ - \frac{\alpha}{2}\right) - 6 \\ &= \frac{4t}{1-t^2} + \frac{10(1-t)}{1+t} - 6 \\ &= \frac{4(2t-1)^2}{1-t^2} \ . \end{aligned}$$

Since $\alpha/2 < 45^\circ$, $t < 1$ so that we must have equality, $\beta = \gamma$ and $t = 1/2$. This yields $\tan\frac{1}{2}\alpha = 1/2$ and so $\tan\alpha = 4/3$.

The triangle is isosceles, and the ratio of the semi-base to height is $\tan\alpha : 1 = 4 : 3$, so that ratio of the sides of the triangle must be $8 : 5 : 5$.

Several students avoided the appeal to convexity. Noting that $\tan\alpha = \cot(\beta+\gamma)$ and multiplying the equation by $\tan\beta + \tan\gamma$, we get

$$\begin{aligned} 0 &= 2(1 - \tan\beta\tan\gamma) + 5(\tan\beta + \tan\gamma)^2 - 6(\tan\beta + \tan\gamma) \\ &= 5\tan^2\beta + 5\tan^2\gamma + 8\tan\beta\tan\gamma - 6\tan\beta - 6\tan\gamma + 2 \ . \end{aligned}$$

Considering this as a quadratic equation in $\tan^2\beta$, we find that its discriminant is

$$(8\tan\gamma - 6)^2 - 20(5\tan^2\gamma - 6\tan\gamma + 2) = -4(3\tan\gamma - 1)^2 \ .$$

We are presuming that there is a suitable triangle, so that the quadratic for $\tan\beta$ has a real solution; this happens if and only if $\tan\gamma = 1/3$. By a parallel argument, we must also have $\tan\beta = 1/3$. This gives the same triangle as before.

In hindsight, we can verify that the condition holds with these values of $\tan\beta$ and $\tan\gamma$ by rewriting it as

$$0 = 5(\tan\beta - \tan\gamma)^2 + 2(3\tan\beta - 1)(3\tan\gamma - 1) \ .$$

Several students identified the circumradius in terms of the lengths x, y and z of the respective segments AP, BQ, CR. This can be done by comnparing two expressions for the area of the triangle, the first involving the product of the inradius and the semiperimeter $x + y + z$, and the second using Heron's formula:

$$r(x + y + z) = \sqrt{(x + y + z)xyz} \ .$$

Using this and assuming that x is the minimum of x, y, z, transforms the given condition to

$$0 = \frac{2}{x} + \frac{5}{y} + \frac{5}{z} = 6\sqrt{\frac{x+y+z}{xyz}} \ .$$

We can clear the denominators of this equation and get a fairly complicated expression in x, y, z to analyze. But Yin Ren of Windsor decided to deal with the reciprocals of these variables: $u = 1/x$, $v = 1/y$ and $w = 1/z$. This makes the condition

$$2u + 5v + 5w = 6\sqrt{uv + vw + wu} \ .$$

Squaring and collectin terms gives

$$0 = 4u^2 + 25v^2 + 25w^2 - 16uv - 16uw + 14vw = 4u^2 + 25(v^2 + w^2) - 16u(v + w) + 14vw \ ,$$

Yin the made the assumption that $v = w$ and found the isosceles triangle that delivers the condition. But he still had the non-isosceles possibility to consider, and left his proof incomplete. Actually, he made more progress than he thought, and could follow up on his inspiration by looking at a perturbation from the isoceles case. Let $v = r + e$ and $w = r - e$ where r is the average of v and w and e is the perturbation from the average. Then $v^2 + w^2 = 2(r^2 + e^2)$, $v + w = 2r$ and $vw = r^2 - e^2$ (notice how products of re drop out). Then the condition becomes

$$0 = 4(u - 4r)^2 + 36e^2 \ ,$$

which evidently holds if and only if $u = 4r$ and $e = 0$. Thus, $y = 4x$ and the ratio of the sides of the triangle must be $x + y : x + z : y + z = 5 : 5 : 8$.

To nail the thing down completely, one should make the observation that the condition does in fact hold when the sides are in this ratio. Logically, without this, all one has demontrated is that presence of the condition means that the sides cannot have any other ratio.

5. Let $A_1A_2A_3$ be a triangle and let ω_1 be a circle in its plane passing through A_1 and A_2. Suppose there exist circles ω_2, ω_3, $\cdots$, ω_7 such that, for $k = 2, 3, \cdots, 7$, ω_k is externally tangent to ω_{k-1} and passes through A_k and A_{k+1}, where $A_{n+3} = A_n$ for all $n \geq 1$. Prove that $\omega_7 = \omega_1$.

Comments. The presence of the circles¡F4¿ makes the problem conceptually quite complex, and the students who succeeded (with two notable exceptions) got the circles out of the picture early on by making the key observation that the line joining the centres of two tangent circles is perpendicular to their common tangent line. This enabled a succession of angles to be related and so the problem reduced to a rather simple angle-chasing exercise.

Note that ω_k is externally tangent to ω_{k-1} at the point A_k. Let the centre of circle ω_k be O_k and let $\beta_k = \angle O_kA_kA_{k+1} = \angle O + kA_{k+1}A_k$, $\alpha_k = \angle A_{k-1}A_kA_{k+1}$. Then $\beta_2 = 180° - \beta_1 - \alpha_2$, $\beta_3 = \beta_1 + \alpha_2 - \alpha_3$, $\beta_5 = \beta_3 + \alpha_1 - \alpha_2 = \beta_1 + \alpha_1 - \alpha_3$ and $\beta_7 = \beta_5 + \alpha_3 - \alpha_1$. Thus $\angle O_7A_2A_1 = \angle O_1A_2A_1$ and it follows that ω_1 and ω_7 have the same centre and radius.

Some students were not careful with the argument and concluded that ω_1 and ω_4 coincided. You may wish to verify that this is not necessarily so. Can it ever occur?

The alternative approach, adopted by David Arthur and David Pritchard, was to use inversion. Invert the whole configuration in a circle with centre A_1 and use primes to indicate the images of entities under inversion. Note that A_1 gets transported "to infinity", and each circle through A_1 to lines passing through the intersection points (if any) of the circle and the circle of inversion. Inversion preserves tangency.

ω_1' is a straight line through A_2'; ω_2' is a circle through A_2' and A_3' that is tangent to ω_1' at A_2': ω_3' is a straight line tangent to ω_2' at A_3'; ω_4' is a straight line through A_2' parallel to ω_3'; ω_5' is a circle through A_2' and A_3' tangent to ω_4' at A_2'; ω_6' is a line through A_3' tangent to ω_5'; ω_7' is a line through A_2' parallel to ω_6'. So it remains to see that ω_5' and ω_1' are tangent. One way to do this is to observe that the half-turn (180° rotation) about the midpoint of $A_2'A_3'$ interchanges A_2' and A_3' and interchanges ω_3' and ω_4'. Thus, ω_2', a circle through A_2' and A_3' tangent to ω_3' at A_2', goes to a circle through the same points, but tangent to ω_4' at A_4'. Thus, ω_2' and ω_5' get switched, as do ω_1' and ω_6'. Hence, ω_1', ω_6' and ω_7 are parallel, and so ω_1' and ω_7' must coincide. The desired result follows.