**OLYMON**

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There will be no more regular Olymon problem sets. However, I am willing to keep up a correspondence with Canadian students who wish to prepare for competitions or simply want to solve problems and have them marked on an individual basis. There are many sources of problems on the net, including those in the International Mathematical Talent Search or past Olymon problems on the website of the Canadian Mathematical Society that you can have recourse to. Please contact Ed Barbeau at barbeau@math.utoronto.ca if you are interesting in pursuing this option.

**Solutions.**

**668.** The nonisosceles right triangle $ABC$ has $\angle CAB = 90°$. The inscribed circle with centre $T$ touches the sides $AB$ and $AC$ at $U$ and $V$ respectively. The tangent through $A$ of the circumscribed circle meets $UV$ produced in $S$. Prove that

(a) $ST \parallel BC$;

(b) $|d_1 - d_2| = r$, where $r$ is the radius of the inscribed circle and $d_1$ and $d_2$ are the respective distances from $S$ to $AC$ and $AB$.

(a) *Solution 1.* Wolog, suppose that the situation is as diagrammed. $\angle BAC = \angle AUT = \angle AVT = 90°$, so that $AUVT$ is a rectangle with $AU = AV$ and $UT = VT$. Hence $AUTV$ is a square with diagonals $AT$ and $UV$ which right-bisect each other at $W$. Since $SW$ right-bisects $AT$, by reflection in the line $SW$, we see that $\Delta ASU \equiv \Delta UST$, and so $\angle UTS = \angle UAS$.

Let $M$ be the midpoint of $BC$. Then $M$ is the circumcentre of $\Delta ABC$, so that $MA = MC$ and $\angle MCA = \angle MAC$. Since $AS$ is tangent to the circumcircle of $\Delta ABC$, $AS \perp AM$. Hence

$$\angle UTS = \angle UAS = \angle SAM - \angle BAM = 90° - \angle BAM = \angle MAC = \angle MCA \ .$$

Now $UT \perp AB$ implies that $UT \| AC$. Since $\angle UTS = \angle ACB$, it follows that $ST \| BC$.

*Solution 2.* Wolog, suppose that $S$ is on the opposite side of $AB$ to $C$.

$BT$, being a part of the diameter produced of the inscribed circle, is a line of reflection that takes the circle to itself and takes the tangent $BA$ to $BC$. Hence $\angle UBT = \frac{1}{2}\angle ABC$. Let $\alpha = \angle ABT$. By the tangent-chord theorem applied to the circumscribed circle, $\angle XAC = \angle ABC = 2\alpha$, so that $\angle SAU = 90° - 2\alpha$.

Consider triangles $SAU$ and $STU$. Since $AUTV$ is a square (see the first solution), $AU = UT$ and $\angle AUV = \angle TUV = 45°$ so $\angle SUA = \angle SUT = 135°$. Also $SU$ is common. Hence $\Delta SAU \equiv \Delta STU$, so $\angle STU = \angle SAU = 90° - 2\alpha$. Therefore,

$$\angle STB = \angle UTB - \angle STU = (90° - \alpha) - (90° - 2\alpha) = \alpha = \angle TBC$$

from which it results that $ST \| BC$.

*Solution 3.* As before $\Delta AUS \equiv \Delta TUS$, so $\angle SAU = \angle STU$. Since $UT \| AC$, $\angle STU = \angle SYA$. Also, by the tangent-chord theorem, $\angle SAB = \angle ACB$. Hence $\angle SYA = \angle STU = \angle SAB = \angle ACB$, so $ST \| BC$.

*Solution 4.* In the Cartesian plane, let $A \sim (0,0)$, $B \sim (0,-b)$, $C \sim (c,0)$. The centre of the circumscribed circle is at $M \sim (c/2, -b/2)$. Since the slope of $AM$ is $-b/c$, the equation of the tangent to the circumscribed circle through $A$ is $y = (c/b)x$. Let $r$ be the radius of the inscribed circle. Since $AU = AV$, the equation of the line $UV$ is $y = x - r$. The abscissa of $S$ is the solution of $x - r = (cx)/b$, so $S \sim (\frac{br}{b-c}, \frac{cr}{b-c})$. Since $T \sim (r, -r)$, the slope of $ST$ is $b/c$ and the result follows.

(b) *Solution 1.* $[\cdots]$ denotes area. Wolog, suppose that $d_1 > d_2$, as diagrammed.

Let $r$ be the inradius of $\Delta ABC$. Then $[AVU] = \frac{1}{2}r^2$, $[AVS] = \frac{1}{2}rd_1$ and $[AUS] = \frac{1}{2}rd_2$. From $[AVU] = [AVS] - [AUS]$, it follows that $r^2 = rd_1 - rd_2$, whence $r = d_1 - d_2$.

*Solution 2.* [F. Crnogorac] Suppose that the situation is as diagrammed. Let $P$ and $Q$ be the respective feet of the perpendiculars from $S$ to $AC$ and $AB$. Since $\angle PVS = 45°$ and $\angle SPV = 90°$, $\Delta PSV$ is isosceles and so $PS = PV = PA + AV = SQ + AV$, *i.e.*, $d_1 = d_2 + r$.

*Solution 3.* Using the coordinates of the fourth solution of (a), we find that

$$d_1 = \left|\frac{cr}{b-c}\right| \qquad \text{and} \qquad d_2 = \left|\frac{br}{b-c}\right|$$

whence $|d_2 - d_1| = r$ as desired.

(b) *Solution.* [M. Boase] Wolog, assume that the configuration is as diagrammed.

Since $\angle SUB = \angle AUV = 45°$, $SU$ is parallel to the external bisector of $\angle A$. This bisector is the locus of points equidistant from $AB$ and $CA$ produced. Wolog, let $PS$ meet this bisector in $W$, as in the diagram. Then $PW = PA$ so that $PS - PA = PS - PW = SW = AU$ and thus $d_1 - d_2 = r$.

**669.** Let $n \geq 3$ be a natural number. Prove that

$$1989 | n^{n^{n^n}} - n^{n^n} \ ,$$

*i.e.*, the number on the right is a multiple of 1989.

*Solution 1.* Let $N = n^{n^{n^n}} - n^{n^n}$. Since $1989 = 3^2 \cdot 13 \cdot 17$,

$$N \equiv 0 \pmod{1989} \Leftrightarrow N \equiv 0 \pmod{9, 13 \ \& \ 17} \ .$$

We require the following facts:
(i) $x^u \equiv 0 \pmod 9$ whenever $u \geq 2$ and $x \equiv 0 \pmod 3$.
(ii) $x^6 \equiv 1 \pmod 9$ whenever $x \not\equiv 0 \pmod 3$.
(iii) $x^u \equiv 0 \pmod{13}$ whenever $x \equiv 0 \pmod{13}$.
(iv) $x^{12} \equiv 1 \pmod{13}$ whenever $x \not\equiv 0 \pmod{13}$, by Fermat's Little Theorem.
(v) $x^u \equiv 0 \pmod{17}$ whenever $x \equiv 0 \pmod{17}$.
(vi) $x^{16} \equiv 1 \pmod{17}$ whenever $x \not\equiv 0 \pmod{17}$, by FLT.
(vii) $x^4 \equiv 1 \pmod{16}$ whenever $x = 2y + 1$ is odd. (For, $(2y+1)^4 = 16y^3(y+2) + 8y(3y+1) + 1 \equiv 1 \pmod{16}$.)

Note that

$$N = n^{n^n}\left[n^{(n^{n^n} - n^n)} - 1\right] = n^{n^n}\left[n^{n^n(n^{n^n - n} - 1)} - 1\right] \ .$$

**Modulo 17**. If $n \equiv 0 \pmod{17}$, then $n^{n^n} \equiv 0$, and so $N \equiv 0 \pmod{17}$.

If $n$ is even, $n \geq 4$, then $n^n \equiv 0 \pmod{16}$, so that

$$n^{n^n(n^{n^n - n} - 1)} \equiv 1^{(n^{n^n - n} - 1)} \equiv 1$$

so $N \equiv 0 \pmod{17}$.

Suppose that $n$ is odd. Then $n^n \equiv n \pmod 4$

$$\Rightarrow n^n - n = 4r \quad \text{for} \quad \text{some} \quad r \in \mathbf{N}$$

$$\Rightarrow n^{n^n - n} = n^{4r} \equiv 1 \pmod{16}$$

$$\Rightarrow n^{n^n - n} - 1 \equiv 0 \pmod{16}$$

$$\Rightarrow n^{n^n(n^{n^n-n}-1)} \equiv 1 \quad (\text{mod } 17$$

$$\Rightarrow N \equiv 0 \pmod{17} \quad .$$

Hence $N \equiv 0 \pmod{17}$ for all $n \geq 3$.

**Modulo 13**. If $n \equiv 0 \pmod{13}$, then $n^{n^n} \equiv 0$ and $N \equiv 0 \pmod{13}$.

Suppose that $n$ is even. Then $n^n \equiv 0 \pmod 4$, so that $n^{n^n} - n^n \equiv 0 \pmod 4$. Suppose that $n$ is odd. Then $n^{n^n - n} - 1 \equiv 0 \pmod{16}$ and so $n^{n^n} - n^n \equiv 0 \pmod 4$.

If $n \equiv 0 \pmod 3$, then $n^n \equiv 0$ so $n^n(n^{n^n-n} - 1) \equiv 0 \pmod 3$. If $n \equiv 1 \pmod 3$, then $n^{n^n-n} \equiv 1$ so $n^n(n^{n^n-n} - 1) \equiv 0 \pmod 3$. If $n \equiv 2 \pmod 3$, then, as $n^n - n$ is always even, $n^{n^n-n} \equiv 1$ so $n^n(n^{n^n-n} - 1) \equiv 0 \pmod 3$. Hence, for all $n$, $n^{n^n} - n^n \equiv 0 \pmod 3$.

It follows that $n^{n^n} - n^n \equiv 0 \pmod{12}$ for all values of $n$. Hence, when $n$ is not a multiple of 13, $n^{(n^{n^n} - n)} \equiv 1$ so $N \equiv 0 \pmod{13}$.

**Modulo 9**. If $n \equiv 0 \pmod 3$, then $n^{n^n} \equiv 0 \pmod 9$, so $N \equiv 0 \pmod 9$. Let $n \not\equiv 0 \pmod 9$. Since $n^{n^n} - n^n$ is divisible by 12, it is divisible by 6, and so $n^{(n^{n^n} - n^n)} \equiv 1$ and $N \equiv 0 \pmod 9$. Hence $N \equiv 0 \pmod 9$ for all $n$.

The required result follows.

**670.** Consider the sequence of positive integers $\{1, 12, 123, 1234, 12345, \cdots\}$ where the next term is constructed by lengthening the previous term at the right-hand end by appending the next positive integer. Note that this next integer occupies only one place, with "carrying"occurring as in addition. Thus, the ninth and tenth terms of the sequence are 123456789 and 1234567900 respectively. Determine which terms of the sequence are divisible by 7.

*Solution 1.* For positive integer $n$, let $x_n$ be the $n$th term of the sequence, and let $x_0 = 0$. Then, for $n \geq 0$, $x_{n+1} = 10x_n + (n+1)$ so that $x_{n+1} \equiv 3x_n + (n+1) \pmod 7$. Suppose that $m$ is a nonnegative integer and that $x_{7m} = a$. Then

$$\begin{array}{llll} x_{7m+1} \equiv 3a + 1 & x_{7m+2} \equiv 2a + 5 & x_{7m+3} \equiv 6a + 4 & x_{7m+4} \equiv 4a + 2 \\ x_{7m+5} \equiv 5a + 4 & x_{7m+6} \equiv a + 4 & x_{7m+7} \equiv 3a + 5 & \end{array}$$

In particular, we find that, modulo 7, $\{x_{7m}\}$ is periodic with the values $\{0, 5, 6, 2, 4, 3\}$ repeated, so that $0 \equiv x_0 \equiv x_{42} \equiv x_{84} \equiv \cdots$. Hence, modulo 7, $x_{7m+1} \equiv 0$ iff $a \equiv 2$, $x_{7m+2} \equiv 0$ iff $a \equiv 1$, $x_{7m+3} \equiv 0$ iff $a \equiv 4$, $x_{7m+4} \equiv 0$ iff $a \equiv 3$, $x_{7m+5} \equiv 0$ iff $a \equiv 2$ and $x_{7m+6} \equiv 0$ iff $a \equiv 3$. Putting this all together, we find that $x_n \equiv 0 \pmod 7$ if and only if $n \equiv 0, 22, 26, 31, 39, 41 \pmod{42}$.

*Solution 2.* [C. Deng] Recall the formula

$$r^{n-1} + 2r^{n-2} + \cdots + (n-1)r + n = \frac{r^{n+1} - r - (r-1)n}{(r-1)^2} \ .$$

[Derive this.] Noting that

$$a_n = 1 \cdot 10^{n-1} + 2 \cdots 10^{n-2} + \cdots + (n-1) \cdot 10 + n \ ,$$

we find that

$$81a_n = 10^{n+1} - 10 - 9n$$

for each positive integer $n$. Therefore

$$81(a_{n+42} - a_n) = 10^{n+1}((10^6)^7 - 1) - 9(42)$$

for each positive integer $n$. Since $10^6 \equiv 1$ (modulo 7), it follows that $a_{n+42} \equiv a_n$ (modulo 7), so that the sequence has period 42 (modulo 7). Thus, the value of $n$ for which $a_n$ is divisible by 7 are the solutions of the congruence $3^{n+1} \equiv 2n + 3$ (modulo 7). These are $n \equiv 22, 26, 31, 39, 41, 42$ (modulo 7).

**671.** Each point in the plane is coloured with one of three distinct colours. Prove that there are two points that are unit distant apart with the same colour.

*Solution 1.* Suppose that the points in the plane are coloured with three colours. Select any point $P$.

We form two rhombi $PQSR$ and $PUWV$, one the rotated image of the other for which all of the following segments have unit length: $PQ$, $PR$, $SQ$, $SR$, $QR$, $PU$, $PV$, $WU$, $WV$, $UV$, $SW$. If $P, Q, R$ are all coloured differently, then either the result holds or $S$ must have the same colour as $P$. If $P, U, V$ are all coloured differently, then either the result holds or $W$ must have the same colour as $P$. Hence, either one of the triangles $PQR$ and $PUV$ has two vertices the same colour, or else $S$ and $W$ must be coloured the same.

*Solution 2.* Suppose, if possible, the planar points can be coloured without two points unit distance apart being coloured the same. Then if $A$ and $B$ are distant $\sqrt{3}$ apart, then there are distinct points $C$ and $D$ such that $ACD$ and $BCD$ are equilateral triangles ($ACBD$ is a rhombus). Since $A$ and $B$ must be coloured differently from the two colours of $C$ and $D$, $A$ and $B$ must have the same colour. Hence, if $O$ is any point in the plane, every point on the circle of radius $\sqrt{3}$ consists of points coloured the same as $O$. But there are two points on this circle unit distant apart, and we get a contradiction of our initial assumption.

*Solution 3.* Suppose we can colour the points of the plane with three colours, red, blue and yellow so that the result fails. We show that three collinear points at unit distance are coloured with three different colours. Let $P, Q, R$ be three such points, and let $P, R$ be opposite sides of a unit hexagon $ABPCDR$ whose centre is $Q$.

If, say, $Q$ is red, $B$ and $A$ must be coloured differently, as are $A$ and $R$, $R$ and $D$, $D$ and $C$, $C$ and $P$, $P$ and $B$. Thus, $B$, $R$, $C$, are one colour, say, blue, and $A$, $D$, $P$ the other, say yellow. The preliminary result follows.

Now consider any isosceles triangle $UVW$ with $|UV| = |UW| = 3$ and $|VW| = 2$. It follows from the preliminary result that $U$ and $V$ must have the same colour, as do $U$ and $W$. But $V$ and $W$ cannot have the same colour and we reach a contradiction.

*Solution 4.* [D. Arthur] Suppose that the result is false. Let $A$, $B$ be two points with $|AB| = 3$. Within the segment $AB$ select $P, Q$ with $|AP| = |PQ| = |QB| = 1$, and suppose that $R$ and $S$ are points on the same side of $AB$ with $\Delta RAP$ and $\Delta SPQ$ equilateral. Then $|RS| = 1$. Suppose if possible that $A$ and $Q$ have the same colour. Then $P$ must have a second colour and $R$ and $S$ the third, leading to a contradiction. Hence $A$ must be coloured differently from both $P$ and $Q$. Similarly $B$ must be coloured differently from both $P$ and $Q$. Since $P$ and $Q$ are coloured differently, $A$ and $B$ must have the same colour.

Now consider a trapezoid $ABCD$ with $|CB| = |AB| = |AD| = 3$ and $|CD| = 1$. By the foregoing observation, $C$, $A$, $B$, $D$ must have the same colour. But this yields a contradiction. The result follows.

**672.** The Fibonacci sequence $\{F_n\}$ is defined by $F_1 = F_2 = 1$ and $F_{n+2} = F_{n+1} + F_n$ for $n = 0, \pm 1, \pm 2, \pm 3, \cdots$. The real number $\tau$ is the positive solution of the quadratic equation $x^2 = x + 1$.

(a) Prove that, for each positive integer $n$, $F_{-n} = (-1)^{n+1} F_n$.

(b) Prove that, for each integer $n$, $\tau^n = F_n \tau + F_{n-1}$.

(c) Let $G_n$ be any one of the functions $F_{n+1}F_n$, $F_{n+1}F_{n-1}$ and $F_n^2$. In each case, prove that $G_{n+3}+G_n = 2(G_{n+2}+G_{n+1})$.

(a) *Solution.* Since $F_0 = F_2 - F_1 = 0$, the result holds for $n = 0$. Since $F_{-1} = F_1 - F_0 = 1$, the result holds for $n = 1$. Suppose that we have established the result for $n = 0, 1, 2, \cdots r$. Then

$$F_{-(r+1)} = F_{-r-1} = F_{-r+1} - F_{-r} = (-1)^r F_{r-1} - (-1)^{r+1} F_r = (-1)^{r+2}(F_{r-1} + F_r) = (-1)^{r+2} F_{r+1} \ .$$

The result follows by induction.

(b) *Solution 1.* The result holds for $n = 0$, $n = 1$ and $n = 2$. Suppose that it holds for $n = 0, 1, 2, \cdots, r$. Then

$$\tau^{r+1} = \tau^r + \tau^{r-1} = (F_r + F_{r-1})\tau + (F_{r-1} + F_{r-2}) = F_{r+1}\tau + F_r\tau \ .$$

This establishes the result for positive values of $n$. Now $\tau^{-1} = \tau - 1 = F_{-1}\tau + F_{-2}$, so the result holds for $n = -1$. Suppose that we have established the result for $n = 0, -1, -2, \cdots, -r$. Then

$$\tau^{-(r+1)} = \tau^{-(r-1)} - \tau^{-r} = (F_{-(r-1)} - F_{-r})\tau + (F_{-r} - F_{-(r+1)}) = F_{-(r+1)}\tau + F_{-(r+2)} \ .$$

*Solution 2.* The result holds for $n = 1$. Suppose that it holds for $n = r \geq 0$. Then

$$\tau^{r+1} = \tau^r \cdot \tau = (F_r\tau + F_{r-1})\tau = F_r\tau^2 + F_{r-1}\tau$$

$$= (F_r + F_{r-1})\tau + F_r = F_{r+1}\tau + F_r \ .$$

Now consider nonpositive values of $n$. We have that $\tau^0 = 1$, $\tau^{-1} = \tau - 1$, $\tau^{-2} = 1 - \tau^{-1} = 2 - \tau$. Suppose that we have shown for $r \geq 0$ that $\tau^{-r} = F_{-r}\tau + F_{-r-1}$. Then

$$\tau^{-(r+1)} = \tau^{-1}\tau^{-r} = F_{-r} + F_{-r-1}(\tau - 1) = F_{-r-1}\tau + (F_{-r} - F_{-r-1})$$

$$= F_{-r-1}\tau + F_{-r-2} = F_{-(r+1)}\tau + F_{-(r+1)-1} \ .$$

By induction, it follows that the result holds for both positive and negative values of $n$.

(c) *Solution.* Let $G_n = F_nF_{n+1}$. Then

$$\begin{aligned} G_{n+3} + G_n &= F_{n+4}F_{n+3} + F_{n+1}F_n \\ &= (F_{n+3} + F_{n+2})(F_{n+2} + F_{n+1}) + (F_{n+3} - F_{n+2})(F_{n+2} - F_{n+1}) \\ &= 2(F_{n+3}F_{n+2} + F_{n+2}F_{n+1}) = 2(G_{n+2} + G_{n+1}) \ . \end{aligned}$$

Let $G_n = F_{n+1}F_{n-1}$. Then

$$\begin{aligned} G_{n+3} + G_n &= F_{n+4}F_{n+2} + F_{n+1}F_{n-1} \\ &= (F_{n+3} + F_{n+2})(F_{n+1} + F_n) + (F_{n+3} - F_{n+2})(F_{n+1} - F_n) \\ &= 2(F_{n+3}F_{n+1} + F_{n+2}F_n) = 2(G_{n+2} + G_{n+1}) \ . \end{aligned}$$

Let $G_n = F_n^2$. Then

$$\begin{aligned} G_{n+3} + G_n &= F_{n+3}^2 + F_n^2 = (F_{n+2} + F_{n+1})^2 + (F_{n+2} - F_{n+1})^2 \\ &= F_{n+2}^2 + 2F_{n+2}F_{n+1} + F_{n+1}^2 + F_{n+2}^2 - 2F_{n+2}F_{n+1} + F_{n+1}^2 = 2(G_{n+2} + G_{n+1}) \ . \end{aligned}$$

*Comments.* Since $F_n^2 = F_nF_{n-1} + F_nF_{n-2}$, the third result of (c) can be obtained from the first two. J. Chui observed that, more generally, we can take $G_n = F_{n+u}F_{n+v}$ where $u$ and $v$ are integers. Then

$$\begin{aligned} G_{n+3} + G_n &- 2(G_{n+1} + G_{n+2}) \\ &= (F_{n+3+u}F_{n+3+v} + F_{n+u}F_{n+v}) - 2(F_{n+2+u}F_{n+2+v} + F_{n+1+u}F_{n+1+v}) \\ &= (2F_{n+1+u} + F_{n+u})(2F_{n+1+v} + F_{n+v}) + F_{n+u}F_{n+v} \\ &\qquad - 2(F_{n+1+u} + F_{n+u})(F_{n+1+v} + F_{n+v}) - 2F_{n+1+u}F_{n+1+v} \\ &= 0 \ , \end{aligned}$$

so that $G_{n+3} + G_n = 2(G_{n+2} + G_{n+1})$.

**673.** $ABC$ is an isosceles triangle with $AB = AC$. Let $D$ be the point on the side $AC$ for which $CD = 2AD$. Let $P$ be the point on the segment $BD$ such that $\angle APC = 90°$. Prove that $\angle ABP = \angle PCB$.

*Solution 1.* Produce $BA$ to $E$ so that $BA = AE$ and join $EC$. Then $D$ is the centroid of $\Delta BEC$ and $BD$ produced meets $EC$ at its midpoint $F$. Since $AE = AC$, $\Delta CAE$ is isosceles and so $AF \perp EC$. Also, since $A$ and $F$ are midpoints of their respective segments, $AF \| BC$ and so $\angle AFB = \angle DBC$. Because $\angle AFC$ and $\angle APC$ are both right, $APCF$ is concyclic so that $\angle AFP = \angle ACP$.

Hence $\angle ABP = \angle ABC - \angle DBC = \angle ABC - \angle AFB = \angle ACB - \angle ACP = \angle PCB$.

*Solution 2.* Let $E$ be the midpoint of $BC$ and let $F$ be a point on $BD$ produced so that $AF \| BC$. Since triangle $ADF$ and $CDB$ are similar and $CD = 2AD$, then $AF = EC$ and $AECF$ is a rectangle.

Since $\angle APC = \angle AFC = 90°$, the quadrilateral $APCF$ is concyclic, so that $\angle AFB = \angle ACP$. Since $AF \| BC$, $\angle AFB = \angle FBC$. Therefore

$$\angle ABP = \angle ABC - \angle PBC = \angle ABC - \angle FBC = \angle ACB - \angle ACP = \angle PCB \ .$$

*Solution 3.* [S. Sun] The circle with diameter $AC$ has as its centre the midpoint $O$ of $AC$. It intersects $BC$ at the midpoint $E$ (since $AB = AC$ and $AE \perp BC$). Let $EO$ produced meet the circle again at $F$; then $AECF$ is concyclic.

Suppose $FB$ meets $AC$ at $G$. A rotation of 180° about $O$ takes $A \leftrightarrow C$, $F \leftrightarrow E$, so that $BC = 2EC = 2AF$ and $AF \| BC$. The triangles $AGF$ and $CGB$ are similar. Since $BC = 2AF$, then $CG = 2GA$, so that $G$ and $D$ coincide. Because $AF \| BC$ and $AFCP$ is concyclic, $\angle DBC = \angle DFA = \angle PFA = \angle PCA$. Therefore

$$\angle ABP = \angle ABC - \angle DBC = \angle ABC - \angle PCA = \angle PCB \ .$$

*Solution 4.* Assign coordinates: $A \sim (0, a)$, $B \sim (-1, 0)$, $C \sim (1, 0)$. Then $D \sim (\frac{1}{3}, \frac{2a}{3})$. Let $P \sim (p, q)$. Then, since $P$ lies on the lines $y = \frac{a}{2}(x + 1)$, $q = \frac{a}{2}(p + 1)$. The relation $AP \perp PC$ implies that

$$-1 = \left(\frac{q-a}{p}\right)\left(\frac{q}{p-1}\right) = \left[\frac{a(p-1)}{2p}\right]\left[\frac{a(p+1)}{2(p-1)}\right] = \frac{a^2(p+1)}{4p} = \frac{aq}{2p}$$

whence $p = -a^2/(a^2 + 4)$ and $q = 2a/(a^2 + 4)$. Now

$$\tan \angle ABP = \frac{a - (a/2)}{1 + (a^2/2)} = \frac{a}{2 + a^2}$$

while

$$\tan \angle PCB = \frac{-q}{p-1} = \frac{-2a}{-a^2 - (a^2 + 4)} = \frac{a}{a^2 + 2} = \tan \angle ABP \ .$$

The result follows.

*Solution 5.* [C. Deng] Let $A \sim (0, b)$, $B \sim (-a, 0)$, $C \sim (a, 0)$ so that $D \sim (a/3, 2b/3)$. The midpoint $M$ of $AC$ has coordinates $(a/2, b/2)$. It can be checked that the point with coordinates

$$\left(\frac{-ab^2}{4a^2 + b^2}, \frac{2a^2 b}{4a^2 + b^2}\right)$$

is the same distance from $M$ as the points $AB$ so that it is on the circle with diameter $AC$ and $AP \| CP$. Since this point also lies on the line with equation $2ay = bx + ba$ through $B$ and $D$, it is none other than the point $P$. The circle with equation

$$x^2 + \left(y + \frac{a^2}{b}\right)^2 = a^2 + \frac{a^4}{b^2}$$

is tangent to $AB$ and $AC$ at $B$ and $C$ respectively and contains the point $P$. Hence $\angle PCB = \angle PBA = \angle DBA$, as desired.

**674.** The sides $BC$, $CA$, $AB$ of triangle $ABC$ are produced to the poins $R$, $P$, $Q$ respectively, so that $CR = AP = BQ$. Prove that triangle $PQR$ is equilateral if and only if triangle $ABC$ is equilateral.

*Solution .* Suppose that triangle $ABC$ is equilateral. A rotation of $60°$ about the centroid of $\Delta ABC$ will rotate the points $R$, $P$ and $Q$. Hence $\Delta PQR$ is equilateral. On the other hand, suppose, wolog, that $a \geq b \geq c$, with $a > c$. Then, for the internal angles of $\Delta ABC$, $A \geq B \geq C$. Suppose that $|PQ| = r$, $|QR| = p$ and $|PR| = q$, while $s$ is the common length of the extensions. Then

$$p^2 = s^2 + (a + s)^2 + 2s(a + s) \cos B$$

and

$$r^2 = s^2 + (c + s)^2 + 2s(c + s) \cos A \ .$$

Since $a > c$ and $\cos B \geq \cos A$, we find that $p > r$, and so $\Delta PQR$ is not equilateral.

**675.** $ABC$ is a triangle with circumcentre $O$ such that $\angle A$ exceeds $90°$ and $AB < AC$. Let $M$ and $N$ be the midpoints of $BC$ and $AO$, and let $D$ be the intersection of $MN$ and $AC$. Suppose that $AD = \frac{1}{2}(AB + AC)$. Determine $\angle A$.

*Solution.* Assign coordinates: $A \sim (0, 0)$, $B \sim (2 \cos \theta, 2 \sin \theta)$, $C \sim (2u, 0)$ where $90° < \theta < 180°$ and $u > 1$. First, we determine $O$ as the intersection of the right bisectors of $AB$ and $AC$. The centre of $AB$ has coordinates $(\cos \theta, \sin \theta)$ and its right bisector has equation

$$(\cos \theta)x + (\sin \theta)y = 1 \ .$$

The centre of segment $AC$ has coordinates $(u, 0)$ and its right bisector has equation $x = u$. Hence, we find that

$$O \sim \left(u, \frac{1 - u \cos \theta}{\sin \theta}\right)$$

$$N \sim \left(\frac{1}{2}u, \frac{1 - u \cos \theta}{2 \sin \theta}\right)$$

$$M \sim (u + \cos \theta, \sin \theta)$$

and

$$D \sim (u + 1, 0) \ .$$

The slope of $MD$ is $(\sin \theta)/(\cos \theta - 1)$. The slope of $ND$ is $(u \cos \theta - 1)/((u + 2) \sin \theta)$. Equating these two leads to the equation

$$u(\cos^2 \theta - \sin^2 \theta - \cos \theta) = 2 \sin^2 \theta + \cos \theta - 1$$

which reduces to

$$(u+1)(2\cos^2\theta - \cos\theta - 1) = 0 \ .$$

Since $u+1>0$, we have that $0 = 2\cos^2\theta - \cos\theta - 1 = (2\cos\theta+1)(\cos\theta-1)$. Hence $\cos\theta = -1/2$ and so $\angle A = 120°$.

**676.** Determine all functions $f$ from the set of reals to the set of reals which satisfy the functional equation

$$(x-y)f(x+y) - (x+y)f(x-y) = 4xy(x^2-y^2)$$

for all real $x$ and $y$.

*Solution.* Let $u$ and $v$ be any pair of real numbers. We can solve $x+y=u$ and $x-y=v$ to obtain

$$(x,y) = \left(\frac{1}{2}(u+v), \frac{1}{2}(u-v)\right) \ .$$

From the functional equation, we find that $vf(u) - uf(v) = (u^2-v^2)uv$, whence

$$\frac{f(u)}{u} - u^2 = \frac{f(v)}{v} - v^2 \ .$$

Thus $(f(x)/x) - x^2$ must be some constant $a$, so that $f(x) = x^3 + ax$. This checks out for any constant $a$.

**677.** For vectors in three-dimensional real space, establish the identity

$$[\mathbf{a}\times(\mathbf{b}-\mathbf{c})]^2 + [\mathbf{b}\times(\mathbf{c}-\mathbf{a})]^2 + [\mathbf{c}\times(\mathbf{a}-\mathbf{b})]^2 = (\mathbf{b}\times\mathbf{c})^2 + (\mathbf{c}\times\mathbf{a})^2 + (\mathbf{a}\times\mathbf{b})^2 + (\mathbf{b}\times\mathbf{c}+\mathbf{c}\times\mathbf{a}+\mathbf{a}\times\mathbf{b})^2 \ .$$

*Solution 1.* Let $\mathbf{u} = \mathbf{b}\times\mathbf{c}$, $\mathbf{v} = \mathbf{c}\times\mathbf{a}$ and $\mathbf{w} = \mathbf{a}\times\mathbf{b}$. Then, for example, $\mathbf{a}\times(\mathbf{b}-\mathbf{c}) = \mathbf{a}\times\mathbf{b} - \mathbf{a}\times\mathbf{c} = \mathbf{a}\times\mathbf{b} + \mathbf{c}\times\mathbf{a} = \mathbf{v}+\mathbf{w}$. The left side is equal to

$$(\mathbf{v}+\mathbf{w})\cdot(\mathbf{v}+\mathbf{w}) + (\mathbf{u}+\mathbf{w})\cdot(\mathbf{u}+\mathbf{w}) + (\mathbf{u}+\mathbf{v})\cdot(\mathbf{u}+\mathbf{v}) = 2[(\mathbf{u}\cdot\mathbf{u}) + (\mathbf{v}\cdot\mathbf{v}) + (\mathbf{w}\cdot\mathbf{w}) + (\mathbf{u}\cdot\mathbf{v}) + (\mathbf{v}\cdot\mathbf{w}) + (\mathbf{w}\cdot\mathbf{u})]$$

while the right side is equal to

$$(\mathbf{u}\cdot\mathbf{u}) + (\mathbf{v}\cdot\mathbf{v}) + (\mathbf{w}\cdot\mathbf{w}) + (\mathbf{u}+\mathbf{v}+\mathbf{w})^2$$

which expands to the final expression for the left side.

*Solution 2.* For vectors $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$, we have the identities

$$(\mathbf{u}\times\mathbf{v})\times\mathbf{w} = (\mathbf{u}\cdot\mathbf{w})\mathbf{v} - (\mathbf{v}\cdot\mathbf{w})\mathbf{u}$$

and

$$\mathbf{u}\cdot(\mathbf{v}\times\mathbf{w}) = (\mathbf{u}\times\mathbf{v})\cdot\mathbf{w} \ .$$

Using these, we find for example that

$$\begin{aligned}
[\mathbf{a}\times(\mathbf{b}-\mathbf{c})]\cdot[\mathbf{a}\times(\mathbf{b}-\mathbf{c})] &= [\mathbf{a}\times(\mathbf{b}-\mathbf{c})\times\mathbf{a}]\cdot(\mathbf{b}-\mathbf{c}) \\
&= \{(\mathbf{a}\cdot\mathbf{a})(\mathbf{b}-\mathbf{c}) - [(\mathbf{b}-\mathbf{c})\cdot\mathbf{a}]\mathbf{a}\}\cdot(\mathbf{b}-\mathbf{c}) \\
&= |\mathbf{a}|^2[|\mathbf{b}|^2 + |\mathbf{c}|^2 - 2(\mathbf{b}\cdot\mathbf{c})] - [(\mathbf{b}\cdot\mathbf{a} - \mathbf{c}\cdot\mathbf{a}]^2 \\
&= |\mathbf{a}|^2[|\mathbf{b}|^2 + |\mathbf{c}|^2 - 2(\mathbf{b}\cdot\mathbf{c})] - (\mathbf{b}\cdot\mathbf{a})^2 - (\mathbf{c}\cdot\mathbf{a})^2 + 2(\mathbf{b}\cdot\mathbf{a})(\mathbf{c}\cdot\mathbf{a}) \ .
\end{aligned}$$

Also

$$\begin{aligned}
(\mathbf{b}\times\mathbf{c})\cdot(\mathbf{b}\times\mathbf{c}) &= [(\mathbf{b}\cdot\mathbf{b})\mathbf{c} - (\mathbf{c}\cdot\mathbf{b})\mathbf{b}]\cdot\mathbf{c} \\
&= |\mathbf{b}|^2|\mathbf{c}|^2 - (\mathbf{c}\cdot\mathbf{b})^2
\end{aligned}$$

and

$$(\mathbf{b} \times \mathbf{c}) \cdot (\mathbf{c} \times \mathbf{a}) = [(\mathbf{b} \times \mathbf{c}) \times \mathbf{c}] \cdot \mathbf{a} = (\mathbf{b} \cdot \mathbf{c})(\mathbf{c} \cdot \mathbf{a}) - (\mathbf{c} \cdot \mathbf{c})(\mathbf{b} \cdot \mathbf{a}) \ .$$

From these the identity can be checked.

**678.** For $a, b, c > 0$, prove that

$$\frac{1}{a(b+1)} + \frac{1}{b(c+1)} + \frac{1}{c(a+1)} \geq \frac{3}{1+abc} \ .$$

*Solution 1.* It is easy to verify the following identity

$$\frac{1}{a(1+b)} + \frac{1}{1+abc} = \frac{1}{1+abc}\left(\frac{1+a}{a(1+b)} + \frac{b(1+c)}{1+b}\right) \ .$$

This and its analogues imply that

$$\frac{1}{a(b+1)} + \frac{1}{b(c+1)} + \frac{1}{c(a+1)} + \frac{3}{1+abc} =$$

$$\frac{1}{1+abc}\left(\frac{1+a}{a(1+b)} + \frac{b(1+c)}{1+b} + \frac{1+b}{b(1+c)} + \frac{c(1+a)}{1+c} + \frac{1+c}{c(1+a)} + \frac{a(1+b)}{1+a}\right) \ .$$

The arithmetic-geometric means inequality yields

$$\frac{1}{a(b+1)} + \frac{1}{b(c+1)} + \frac{1}{c(a+1)} + \frac{3}{1+abc} \geq 6 \times \frac{1}{1+abc} \ .$$

Miraculously, subtracting $3/(1+abc)$ from both sides yields the required inequality. ♡

*Solution 2.* Multiplying the desired inequality by $(1+abc)a(b+1)b(c+1)c(a+1)$, after some manipulation, produces the equivalent inequality:

$$\begin{aligned} &abc(bc^2 + ca^2 + ab^2) + (bc + ca + ab) + (abc)^2(a + b + c) + (bc^2 + ca^2 + ab^2) \\ &\qquad \geq 2abc(a + b + c) + 2abc(bc + ca + ab) \ . \end{aligned}$$

Pairing off the terms of the left side and applying the arithemetic-geometric means inequality, we get

$$\begin{aligned} (a^2b^3c + bc) + (ab^2c^3 + ac) + (a^3bc^2 + ab) + (a^3b^2c^2 + ab^2) \\ + (a^2b^3c^2 + bc^2) + (a^2b^2c^3 + ca^2) \\ \geq 2ab^2c + 2abc^2 + 2a^2bc + 2a^2b^2c + 2ab^2c^2 + 2a^2bc^2 \\ = 2abc(a + b + c) + 2abc(ab + bc + ca) \end{aligned}$$

as required.

*Solution 3.* [C. Deng] Taking the difference between the two sides yields, where the summation is a cyclic one,

$$\begin{aligned}
\sum \left( \frac{1}{a(b+1)} - \frac{1}{1+abc} \right) &= \sum \frac{1+abc-a(b+1)}{a(b+1)(1+abc)} \\
&= \frac{1}{1+abc} \sum \left( \frac{b}{b+1}(c-1) - \frac{1}{a(b+1)}(a-1) \right) \\
&= \frac{1}{1+abc} \sum \left( \frac{c}{c+1}(a-1) - \frac{1}{a(b+1)}(a-1) \right) \\
&= \frac{1}{1+abc} \sum (a-1) \left( \frac{c}{c+1} - \frac{1}{a(b+1)} \right) \\
&= \frac{1}{1+abc} \sum \left( \frac{a^2-1}{a} \right) \left( \frac{abc+ac-c-1}{(a+1)(b+1)(c+1)} \right) \\
&= \frac{1}{(1+abc)(1+a)(1+b)(1+c)} \sum \left( a^2bc + a^2c + \frac{c}{a} + \frac{1}{a} - ac - a - bc - c \right) \\
&= \frac{1}{(1+abc)(1+a)(1+b)(1+c)} \sum \left( a^2bc + a^2c - 2ab - 2a + \frac{b}{c} + \frac{1}{c} \right) \\
&= \frac{1}{(1+abc)(1+a)(1+b)(1+c)} \sum \frac{b+1}{c}(a^2c^2 - 2ac + 1) \\
&= \frac{1}{(1+abc)(1+a)(1+b)(1+c)} \sum \frac{b+1}{c}(ac-1)^2 \ge 0 \ ,
\end{aligned}$$

as desired.

*Solution 4.* [S. Seraj] Using the Arithmetic-Geometric Means Inequality, we obtain $a^2c + a^2b^2c^3 \ge 2a^2bc^2$ and $ab + a^3bc^2 \ge 2a^2bc$ and the two cyclic variants of each. Adding the six inequalities yields that

$$\begin{aligned}
&a^2c + a^2b^2c^3 + ab^2 + a^3b^2c^2 + bc^2 + a^2b^3c^2 + ab + a^3bc^2 + bc + a^2b^3c + ac + ab^2c^3 \\
&\qquad \ge 2a^2bc^2 + 2a^2b^2c + 2ab^2c^2 + 2a^2bc + 2ab^2c + 2abc^2 \ .
\end{aligned}$$

Adding the same terms to both sides of the equations, and then factoring the two sides leads to

$$\begin{aligned}
&(1+abc)(3abc + a^2bc + ab^2c + abc^2 + a^2c + ab^2 + bc^2 + ab + bc + ca) \\
&\qquad \ge 3abc(abc + ac + bc + ab + a + b + c + 1) = 3abc(a+1)(b+1)(c+1) \ .
\end{aligned}$$

Carrying out some divisions and strategically grouping terms in the numerator yields that

$$\frac{(abc^2 + bc^2 + abc + bc) + (a^2bc + a^2c + abc + ac) + (ab^2c + ab^2 + abc + ab)}{abc(a+1)(b+1)(c+1)} \ge \frac{3}{1+abc} \ .$$

Factoring each bracket and simplifying leads to the desired inequality.

**679.** Let $F_1$ and $F_2$ be the foci of an ellipse and $P$ be a point in the plane of the ellipse. Suppose that $G_1$ and $G_2$ are points on the ellipse for which $PG_1$ and $PG_2$ are tangents to the ellipse. Prove that $\angle F_1PG_1 = \angle F_2PG_2$.

*Solution.* Let $H_1$ be the reflection of $F_1$ in the tangent $PG_1$, and $H_2$ be the reflection of $F_2$ in the tangent $PG_2$. We have that $PH_1 = PF_1$ and $PF_2 = PH_2$. By the reflection property, $\angle PG_1F_2 = \angle F_1G_1Q = \angle H_1G_1Q$, where $Q$ is a point on $PG_1$ produced. Therefore, $H_1F_2$ intersects the ellipse in $G_1$. Similarly, $H_2F_1$ intersects the ellipse in $K_2$. Therefore

$$\begin{aligned}
H_1F_2 &= H_1G_1 + G_1F_2 = F_1G_1 + G_1F_2 \\
&= F_1G_2 + G_2F_2 = F_1G_2 + G_2H_2 = H_2F_1 \ .
\end{aligned}$$

Therefore, triangle $PH_1F_2$ and $PF_1H_2$ are congruent (SSS), so that $\angle H_1PF_2 = \angle H_2PF_1$. It follows that

$$2\angle F_1PG_1 = \angle H_1PF_1 = \angle H_2PF_2 = 2\angle F_2PG_2$$

and the desired result follows.

**680.** Let $u_0 = 1$, $u_1 = 2$ and $u_{n+1} = 2u_n + u_{n-1}$ for $n \geq 1$. Prove that, for every nonnegative integer $n$,

$$u_n = \sum \left\{ \frac{(i+j+k)!}{i!j!k!} : i,j,k \geq 0, i+j+2k = n \right\} .$$

*Solution 1.* Suppose that we have a supply of white and of blue coaches, each of length 1, and of red coaches, each of length 2; the coaches of each colour are indistinguishable. Let $v_n$ be the number of trains of total length $n$ that can be made up of red, white and blue coaches of total length $n$. Then $v_0 = 1$, $v_1 = 2$ and $v_2 = 5$ (R, WW, WB, BW, BB). In general, for $n \geq 1$, we can get a train of length $n+1$ by appending either a white or a blue coach to a train of length $n$ or a red coach to a train of length $n-1$, so that $v_{n+1} = 2v_n + v_{n-1}$. Therefore $v_n = u_n$ for $n \geq 0$.

We can count $v_n$ in another way. Suppose that the train consists of $i$ white coaches, $j$ blue coaches and $k$ red coaches, so that $i + j + 2k = n$. There are $(i + j + k)!$ ways of arranging the coaches in order; any permutation of the $i$ white coaches among themselves, the $j$ blue coaches among themselves and $k$ red coaches among themselves does not change the train. Therefore

$$u_n = \sum \left\{ \frac{(i+j+k)!}{i!j!k!} : i,j,k \geq 0, i+j+2k = n \right\} .$$

*Solution 2.* Let $f(t) = \sum_{n=0}^{\infty} u_n t^n$. Then

$$\begin{aligned} f(t) &= u_0 + u_1 t + (2u_1 + u_0)t^2 + (2u_2 + u_1)t^3 + \cdots \\ &= u_0 + u_1 t + 2t(f(t) - u_0) + t^2 f(t) = u_0 + (u_1 - 2u_0)t + (2t + t^2)f(t) \\ &= 1 + (2t + t^2)f(t) , \end{aligned}$$

whence

$$\begin{aligned} f(t) &= \frac{1}{1 - 2t - t^2} = \frac{1}{1 - t - t - t^2} \\ &= \sum_{n=0}^{\infty} (t + t + t^2)^n = \sum_{n=0}^{\infty} t^n \left[ \sum \left\{ \frac{(i+j+k)!}{i!j!k!} : i,j,k \geq 0, i+j+2k = n \right\} \right] . \end{aligned}$$

*Solution 3.* Let $w_n$ be the sum in the problem. It is straightforward to check that $u_0 = w_0$ and $u_1 = w_1$. We show that, for $n \geq 1$, $w_{n+1} = 2w_n + w_{n-1}$ from which it follows by induction that $u_n = w_n$ for each $n$. By convention, let $(-1)! = \infty$. Then, for $i, j, k \geq 0$ and $i + j + 2k = n + 1$, we have that

$$\begin{aligned} \frac{(i+j+k)!}{i!j!k!} &= \frac{(i+j+k)(i+j+k-1)!}{i!j!k!} \\ &= \frac{(i+j+k-1)!}{(i-1)!j!k!} + \frac{(i+j+k-1)!}{i!(j-1)!k!} + \frac{(i+j+k-1)!}{i!j!(k-1)!} , \end{aligned}$$

whence

$$\begin{aligned} w_{n+1} &= \sum \left\{ \frac{(i+j+k-1)!}{(i-1)!j!k!} : i,j,k \geq 0, (i-1)+j+2k = n \right\} \\ &\quad + \sum \left\{ \frac{(i+j+k-1)!}{i!(j-1)!k!} : i,j,k \geq 0, i+(j-1)+2k = n \right\} \\ &\quad + \sum \left\{ \frac{(i+j+k-1)!}{i!j!(k-1)!} : i,j,k \geq 0, i+j+2(k-1) = n-1 \right\} \\ &= w_n + w_n + w_{n-1} = 2w_n + w_{n-1} \end{aligned}$$

as desired.

**681.** Let $\mathbf{a}$ and $\mathbf{b}$, the latter nonzero, be vectors in $\mathbf{R}^3$. Determine the value of $\lambda$ for which the vector equation

$$\mathbf{a} - (\mathbf{x} \times \mathbf{b}) = \lambda \mathbf{b}$$

is solvable, and then solve it.

*Solution 1.* If there is a solution, we must have $\mathbf{a} \cdot \mathbf{b} = \lambda |\mathbf{b}|^2$, so that $\lambda = (\mathbf{a} \cdot \mathbf{b})/|\mathbf{b}|^2$. On the other hand, suppose that $\lambda$ has this value. Then

$$\begin{aligned} \mathbf{0} &= \mathbf{b} \times \mathbf{a} - \mathbf{b} \times (\mathbf{x} \times \mathbf{b}) \\ &= \mathbf{b} \times \mathbf{a} - [(\mathbf{b} \cdot \mathbf{b})\mathbf{x} - (\mathbf{b} \cdot \mathbf{x})\mathbf{b}] \end{aligned}$$

so that

$$\mathbf{b} \times \mathbf{a} = |\mathbf{b}|^2 \mathbf{x} - (\mathbf{b} \cdot \mathbf{x})\mathbf{b} \ .$$

A particular solution of this equation is

$$\mathbf{x} = \mathbf{u} \equiv \frac{\mathbf{b} \times \mathbf{a}}{|\mathbf{b}|^2} \ .$$

Let $\mathbf{x} = \mathbf{z}$ be any other solution. Then

$$\begin{aligned} |\mathbf{b}|^2(\mathbf{z} - \mathbf{u}) &= |\mathbf{b}|^2 \mathbf{z} - |\mathbf{b}|^2 \mathbf{u} \\ &= (\mathbf{b} \times \mathbf{a} + (\mathbf{b} \cdot \mathbf{z})\mathbf{b}) - (\mathbf{b} \times \mathbf{a} + (\mathbf{b} \cdot \mathbf{u})\mathbf{b}) \\ &= (\mathbf{b} \cdot \mathbf{z})\mathbf{b} \end{aligned}$$

so that $\mathbf{z} - \mathbf{u} = \mu \mathbf{b}$ for some scalar $\mu$.

We check when this works. Let $\mathbf{x} = \mathbf{u} + \mu \mathbf{b}$ for some scalar $\mu$. Then

$$\begin{aligned} \mathbf{a} - (\mathbf{x} \times \mathbf{b}) = \mathbf{a} - (\mathbf{u} \times \mathbf{b}) &= \mathbf{a} - \frac{(\mathbf{b} \times \mathbf{a}) \times \mathbf{b}}{|\mathbf{b}|^2} \\ &= \mathbf{a} + \frac{\mathbf{b} \times (\mathbf{b} \times \mathbf{a})}{|\mathbf{b}|^2} \\ &= \mathbf{a} + \frac{(\mathbf{b} \cdot \mathbf{a})\mathbf{b} - (\mathbf{b} \cdot \mathbf{b})\mathbf{a}}{|\mathbf{b}|^2} \\ &= \mathbf{a} + \left( \frac{\mathbf{b} \cdot \mathbf{a}}{|\mathbf{b}|^2} \right) \mathbf{b} - \mathbf{a} = \lambda \mathbf{b} \ , \end{aligned}$$

as desired. Hence, the solutions is

$$\mathbf{x} = \frac{\mathbf{b} \times \mathbf{a}}{|\mathbf{b}|^2} + \mu \mathbf{b} \ ,$$

where $\mu$ is an arbitrary scalar.

*Solution 2.* [B. Yahagni] Suppose, to begin with, that $\{\mathbf{a}, \mathbf{b}\}$ is linearly dependent. Then $\mathbf{a} = [(\mathbf{a} \cdot \mathbf{b})/|\mathbf{b}|^2]\mathbf{b}$. Since $(\mathbf{x} \times \mathbf{b}) \cdot \mathbf{b} = 0$ for all $\mathbf{x}$, the equation has no solutions except when $\lambda = (\mathbf{a} \cdot \mathbf{b})/|\mathbf{b}|^2$. In this case, it becomes $\mathbf{x} \times \mathbf{b} = \mathbf{0}$ and is satisfied by $\mathbf{x} = \mu \mathbf{b}$, where $\mu$ is any scalar.

Otherwise, $\{\mathbf{a}, \mathbf{b}, \mathbf{a} \times \mathbf{b}\}$ is linearly independent and constitutes a basis for $\mathbb{R}^3$. Let a solution be

$$\mathbf{x} = \alpha \mathbf{a} + \mu \mathbf{b} + \beta(\mathbf{a} \times \mathbf{b}) \ .$$

Then

$$\mathbf{x} \times \mathbf{b} = \alpha(\mathbf{a} \times \mathbf{b}) + \beta[(\mathbf{a} \times \mathbf{b}) \times \mathbf{b}] = \alpha(\mathbf{a} \times \mathbf{b}) + \beta(\mathbf{a} \cdot \mathbf{b})\mathbf{b} - \beta(\mathbf{b} \cdot \mathbf{b})\mathbf{a}$$

and the equation becomes

$$(1 + \beta|\mathbf{b}|^2)\mathbf{a} - \beta(\mathbf{a} \cdot \mathbf{b})\mathbf{b} - \alpha(\mathbf{a} \times \mathbf{b}) = \lambda\mathbf{b} \ .$$

Therefore $\alpha = 0$, $\mu$ is arbitrary, $\beta = -1/|\mathbf{b}|^2$ and $\lambda = -\beta(\mathbf{a} \cdot \mathbf{b}) = (\mathbf{a} \cdot \mathbf{b})/|\mathbf{b}|^2$.

Therefore, the existence of a solution requires that $\lambda = (\mathbf{a} \cdot \mathbf{b})/|\mathbf{b}|^2$ and the solution then is

$$\mathbf{x} = \mu\mathbf{b} - \frac{1}{|\mathbf{b}|^2}(\mathbf{a} \times \mathbf{b}) \ .$$

*Solution 3.* Writing the equation in vector components yields the system

$$b_3x_2 - b_2x_3 = a_1 - \lambda b_1 \ ;$$

$$-b_3x_1 + b_1x_3 = a_2 - \lambda b_2 \ ;$$

$$b_2x_1 - b_1x_2 = a_3 - \lambda b_3 \ .$$

The matrix of coefficients of the left side is of rank 2, so that the corresponding homogeneous system of equations has a single infinity of solutions. Multiplying the three equations by $b_1$, $b_2$ and $b_3$ respectively and adding yields

$$0 = a_1b_1 + a_2b_2 + a_3b_3 - \lambda(b_1^2 + b_2^2 + b_3^2) \ .$$

Thus, for a solution to exist, we require that

$$\lambda = \frac{a_1b_1 + a_2b_2 + a_3b_3}{b_1^2 + b_2^2 + b_3^2)}.$$

In addition, we learn that the corresponding homogeneous system is satisfied by

$$(x_1, x_2, x_3) = \mu(b_1, b_2, b_3)$$

where $\mu$ is an arbitrary scalar.

It remains to find a particular solution for the nonhomogeneous system. Multiplying the third equation by $b_2$ and subtracting the second multiplied by $b_3$, we obtain that

$$(b_2^2 + b_3^2)x_1 = b_1(b_2x_2 + b_3x_3) + (a_3b_2 - a_2b_3) \ .$$

Therefore, setting $b_1^2 + b_2^2 + b_3^2 = b^2$, we have that

$$b^2x_1 = b_1(b_1x_1 + b_2x_2 + b_3x_3) + (a_3b_2 - a_2b_3) \ .$$

Similarly

$$b^2x_2 = b_2(b_1x_1 + b_2x_2 + b_3x_3) + (a_1b_3 - a_3b_1) \ ,$$

$$b^2x_3 = b_3(b_1x_1 + b_2x_2 + b_3x_3) + (a_2b_1 - a_1b_2) \ .$$

Observing that $b_1x_1 + b_2x_2 + b_3x_3$ vanishes when

$$(x_1, x_2, x_3) = (a_3b_2 - a_2b_3, a_1b_3 - a_3b_1, a_2b_1 - a_1b_2) \ ,$$

we obtain a particular solution to the system:

$$(x_1, x_2, x_3) = b^{-2}(a_3b_2 - a_2b_3, a_1b_3 - a_3b_1, a_2b_1 - a_1b_2) \ .$$

Adding to this the general solution of the homogeneous system yields the solution of the nonhomogeneous system.

**682.** The plane is partitioned into $n$ regions by three families of parallel lines. What is the least number of lines to ensure that $n \geq 2010$?

*Solution.* Suppose that there are $x$, $y$ and $z$ lines in the three families. Assume that no point is common to three distinct lines. The $x+y$ lines of the first two families partition the plane into $(x+1)(y+1)$ regions. Let $\lambda$ be one of the lines of the third family. It is cut into $x+y+1$ parts by the lines in the first two families, so the number of regions is increased by $x+y+1$. Since this happens $z$ times, the number of regions that the plane is partitioned into by the three families of

$$n = (x+1)(y+1) + z(x+y+1) = (x+y+z) + (xy+yz+zx) + 1 \ .$$

Let $u = x+y+z$ and $v = xy+yz+zx$. Then (by the Cauchy-Schwarz Inequality for example), $v \leq x^2+y^2+z^2$, so that $u^2 = x^2+y^2+z^2+2v \geq 3v$. Therefore, $n \leq u + \frac{1}{3}u^2 + 1$. This takes the value 2002 when $u = 76$. However, when $(x,y,z) = (26,26,25)$, then $u = 77$, $v = 1976$ and $n = 2044$. Therefore, we need at least 77 lines, but a suitably chosen set of 77 lines will suffice.

**683.** Let $f(x)$ be a quadratic polynomial. Prove that there exist quadratic polynomials $g(x)$ and $h(x)$ for which

$$f(x)f(x+1) = g(h(x)) \ ,$$

*Solution 1.* [A. Remorov] Let $f(x) = a(x-r)(x-s)$. Then

$$\begin{aligned} f(x)f(x+1) &= a^2(x-r)(x-s+1)(x-r+1)(x-s) \\ &= a^2(x^2+x-rx-sx+rs-r)(x^2+x-rx-sx+rs-s) \\ &= a^2[(x^2-(r+s-1)x+rs)-r][(x^2-(r+s-1)x+rs)-s] \\ &= g(h(x)) \ , \end{aligned}$$

where $g(x) = a^2(x-r)(x-s) = af(x)$ and $h(x) = x^2-(r+s-1)x+rs$.

*Solution 2.* Let $f(x) = ax^2+bx+c$, $g(x) = px^2+qx+r$ and $h(x) = ux^2+vx+w$. Then

$$f(x)f(x+1) = a^2x^4 + 2a(a+b)x^3 + (a^2+b^2+3ab+2ac)x^2 + (b+2c)(a+b)x + c(a+b-c)$$

$$\begin{aligned} g(h(x)) &= p(ux^2+vx+w)^2 + q(ux+vx+w) + r \\ &= pu^2x^4 + 2puvx^3 + (2puw+pv^2+qu)x^2 + (2pvw+qv)x + (pw^2+qw+r) \ . \end{aligned}$$

Equating coefficients, we find that $pu^2 = a^2$, $puv = a(a+b)$, $2puw+pv^2+qu = a^2+b^2+3ab+2ac$, $(b+2c)(a+b) = (2pw+q)v$ and $c(a+b+c) = pw^2+qw+r$. We need to find just one solution of this system. Let $p=1$ and $u=a$. Then $v=a+b$ and $b+2c = 2pw+q$ from the second and fourth equations. This yields the third equation automatically. Let $q=b$ and $w=c$. Then from the fifth equation, we find that $r = ac$.

Thus, when $f(x) = ax^2+bx+c$, we can take $g(x) = x^2+bx+ac$ and $h(x) = ax^2+(a+b)x+c$.

*Solution 3.* [S. Wang] Suppose that

$$f(x) = a(x+h)^2 + k = a(t-(1/2))^2 + k \ ,$$

where $t = x+h+\frac{1}{2}$. Then $f(x+1) = a(x+1+h)^2 + k = a(t+(1/2))^2 + k$, so that

$$\begin{aligned} f(x)f(x+1) &= a^2(t^2-(1/4))^2 + 2ak(t^2+(1/4)) + k^2 \\ &= a^2t^4 + \left(-\frac{a^2}{2} + 2ak\right)t^2 + \left(\frac{a^2}{16} + \frac{ak}{2} + k^2\right) \ . \end{aligned}$$

Thus, we can achieve the desired representation with $h(x) = t^2 = x^2 + (2h+1)x + \frac{1}{4}$ and $g(x) = a^2x^2 + (\frac{-a^2}{2} + 2ak)x + (\frac{a^2}{16} + \frac{ak}{2} + k^2)$.

*Solution 4.* [V. Krakovna] Let $f(x) = ax^2 + bx + c = au(x)$ where $u(x) = x^2 + dx + e$, where $b = ad$ and $c = ae$. If we can find functions $v(x)$ and $w(x)$ for which $u(x)u(x+1) = v(w(x))$, then $f(x)f(x+1) = a^2v(w(x))$, and we can take $h(x) = w(x)$ and $g(x) = a^2v(x)$.

Define $p(t) = u(x+t)$, so that $p(t)$ is a monic quadratic in $t$. Then, noting that $p''(t) = u''(x+t) = 2$, we have that

$$p(t) = u(x+t) = u(x) + u'(x)t + \frac{u''(x)}{2}t^2 = t^2 + u'(x)t + u(x) \ ,$$

from which we find that

$$\begin{aligned} u(x)u(x+1) = p(0)p(1) &= u(x)[u(x) + u'(x) + 1] \\ &= u(x)^2 + u'(x)u(x) + u(x) = p(u(x)) = u(x + u(x)) \ . \end{aligned}$$

Thus, $u(x)u(x+1) = v(w(x))$ where $w(x) = x + u(x)$ and $v(x) = u(x)$. Therefore, we get the desired representation with

$$h(x) = x + u(x) = x^2 + \left(1 + \frac{b}{a}\right)x + \frac{c}{a}$$

and

$$g(x) = a^2v(x) = a^2u(x) = af(x) = a^2x^2 + abx + ac \ .$$

*Solution 5.* [Generalization by J. Rickards.] The following statement is true: *Let the quartic polynomial $f(x)$ have roots $r_1, r_2, r_3, r_4$ (not necessarily distinct). Then $f(x)$ can be expressed in the form $g(h(x)$ for quadratic polynomials $g(x)$ and $h(x)$ if and only if the sum of two of $r_1$, $r_2$, $r_3$, $r_4$ is equal to the sum of the other two.*

Wolog, suppose that $r_1 + r_2 = r_3 + r_4$. Let the leading coefficient of $f(x)$ be $a$. Define $h(x) = (x - r_1)(x - r_2)$ and $g(x) = ax(x - r_3^2 + r_1r_3 + r_2r_3 - r_1r_2)$. Then

$$\begin{aligned} g(h(x)) &= a(x-r_1)(x-r_2)[(x-r_1)(x-r_2) - r_3^2 + r_1r_3 + r_2r_3 - r_1r_2 \\ &= a(x-r_1)(x-r_2)[x^2 - (r_1+r_2)x - r_3^2 + r_1r_3 + r_2r_3) \\ &= a(x-r_1)(x-r_2)[x^2 - (r_3+r_4)x + r_3(r_1+r_2-r_3)] \\ &= a(x-r_1)(x-r_2)(x^2 - (r_3+r_4)x + r_3r_4 \\ &= a(x-r_1)(x-r_2)(x-r_3)(x-r_4) \end{aligned}$$

as required.

Conversely, assume that we are given quadratic polynomials $g(x) = b(x - r_5)(x - r_6)$ and $h(x)$ and that $c$ is the leading coefficient of $h(x)$. Let $f(x) = g(h(x))$.

Suppose that

$$h(x) - r_5 = c(x - r_1)(x - r_2)$$

and that

$$h(x) - r_6 = c(x - r_3)(x - r_4) \ .$$

Then

$$f(x) = g(h(x)) = bc^2(x-r_1)(x-r_2)(x-r_3)(x-r_4) \ .$$

We have that

$$h(x) = c(x-r_1)(x-r_2) + r_5 = cx62 - c(r_1+r_2)x + cr_1r_2 + r_5$$

and

$$h(x) = c9x - r_3)(x - r_4) + r_6 = cx^2 - c(r_3 + r_4)x + cr_3r_4 + r_6 \ ,$$

whereupon it follows that $r_1 + r_2 = r_3 + r_4$ and the desired result follows.

*Comment.* The second solution can also be obtained by looking at special cases, such as when $a = 1$ or $b = 0$, getting the answer and then making a conjecture.