OLYMON

COMPLETE PROBLEM SET

No solutions. See yearly files.

April, 2000 - March, 2004

PART 1

Problems 1-300

Notes: The *inradius* of a triangle is the radius of the *incircle*, the circle that touches each side of the polygon. The *circumradius* of a triangle is the radius of the *circumcircle*, the circle that passes through its three vertices.

A set of lines of *concurrent* if and only if they have a common point of intersection.

The word *unique* means *exactly one*. A *regular octahedron* is a solid figure with eight faces, each of which is an equilateral triangle. You can think of gluing two square pyramids together along the square bases. The symbol $\lfloor u \rfloor$ denotes the greatest integer that does not exceed u.

An *acute triangle* has all of its angles less than $90°$. The *orthocentre* of a triangle is the intersection point of its altitudes. Points are *collinear* iff they lie on a straight line.

For any real number x, $\lfloor x \rfloor$ (the *floor* of x) is equal to the greatest integer that is less than or equal to x.

A real-valued function f defined on an interval is *concave* iff $f((1-t)u+tv) \geq (1-t)f(u)+tf(v)$ whenever $0 < t < 1$ and u and v are in the domain of definition of $f(x)$. If $f(x)$ is a one-one function defined on a domain into a range, then the *inverse* function $g(x)$ defined on the set of values assumed by f is determined by $g(f(x)) = x$ and $f(g(y)) = y$; in other words, $f(x) = y$ if and only if $g(y) = x$.

A sequence $\{x_n\}$ *converges* if and only if there is a number c, called its *limit*, such that, as n increases, the number x_n gets closer and closer to c. If the sequences is *increasing* (*i.e.*, $x_{n+1} \geq x_n$ for each index n) and *bounded above* (*i.e.*, there is a number M for which $x_n \leq M$ for each n, then it must converge. [Do you see why this is so?] Similarly, a decreasing sequence that is bounded below converges. [Supply the definitions and justify the statement.] An infinite *series* is an expression of the form $\sum_{k=a}^{\infty} x_k = x_a + x_{a+1} + x_{a+2} + \cdots + x_k + \cdots$, where a is an integer, usually 0 or 1. The nth *partial sum* of the series is $s_n \equiv \sum_{k=a}^{n} x_k$. The series has sum s if and only if its sequence $\{s_n\}$ of partial sums converges and has limit s; when this happens, the series *converges*. If the sequence of partial sums fails to converge, the series *diverges*. If every term in the series is nonnegative and the sequence of partial sums is bounded above, then the series converges. If a series of nonnegative terms converges, then it is possible to rearrange the order of the terms without changing the value of the sum.

A *rectangular hyperbola* is an hyperbola whose asymptotes are at right angles.

A function $f : A \rightarrow B$ is a *bijection* iff it is one-one and onto; this means that, if $f(u) = f(v)$, then $u = v$, and, if w is some element of B, then A contains an element t for which $f(t) = w$. Such a function has an *inverse* f^{-1} which is determined by the condition

$$f^{-1}(b) = a \Leftrightarrow b = f(a) \ .$$

The sides of a right-angled triangle that are adjacent to the right angle are called *legs*. The *centre of gravity* or *centroid* of a collection of n mass particles is the point where the cumulative mass can be regarded as concentrated so that the motion of this point, when exposed to outside forces such as gravity, is identical to that of the whole collection. To illustrate this point, imagine that the mass particles are connected to a point by rigid non-material sticks (with mass 0) to form a structure. The point where the tip of a needle could be put so that this structure is in a state of balance is its centroid. In addition, there is an intuitive definition of a centroid of a lamina, and of a solid: The centroid of a lamina is the point, which would cause equilibrium (balance) when the tip of a needle is placed underneath to support it. Likewise, the centroid of a solid is the point, at which the solid "balances", *i.e.*, it will not revolve if force is applied. The centroid, G of a set of points is defined vectorially by

$$\overrightarrow{OG} = \frac{\sum_{i=1}^{n} m_i \cdot \overrightarrow{OM}_i}{\sum_{i=1}^{n} m_i}$$

where m_i is the mass of the particle at a position M_i (the summation extending over the whole collection). Problem 181 is related to the centroid of an assembly of three particles placed at the vertices of a given triangle. The *circumcentre* of a triangle is the centre of its circumscribed circle. The *orthocentre* of a triangle is the intersection point of its altitudes. An *unbounded* region in the plane is one not contained in the interior of any circle.

An *isosceles* tetrahedron is one for which the three pairs of oppposite edges are equal. For integers a, b and n, $a \equiv b$, modulo n, iff $a - b$ is a multiple of n.

A real-valued function on the reals is *increasing* if and only if $f(u) \leq f(v)$ whenever $u < v$. It is *strictly increasing* if and only if $f(u) < f(v)$ whenever $u < v$.

The inverse tangent function is denoted by $\tan^{-1} x$ or $\arctan x$. It is defined by the relation $y = \tan^{-1} x$ if and only if $\pi/2 < y < \pi/2$ and $x = \tan y/$

The absolute value $|x|$ is equal to x when x is nonnegative and $-x$ when x is negative; always $|x| \geq 0$. The *floor of* x, denoted by $\lfloor x \rfloor$ is equal to the greatest integer that does not exceed x. For example, $\lfloor 5.34 \rfloor = 5$, $\lfloor -2.3 \rfloor - -3$ and $\lfloor 5 \rfloor = 5$. A geometric figure is said to be *convex* if the segment joining any two points inside the figure also lies inside the figure.

Given a triangle, extend two nonadjacent sides. The circle tangent to these two sides and to the third side of the triangle is called an *excircle*, or sometimes an *escribed circle*. The centre of the circle is called the *excentre* and lies on the angle bisector of the opposite angle and the bisectors of the external angles formed by the extended sides with the third side. Every triangle has three excircles along with their excentres.

The *incircle* of a polygon is a circle inscribed inside of the polygon that is tangent to all of the sides of a polygon. While every triangle has an incircle, this is not true of all polygons.

The *greatest common divisor* of two integers m, n, denoted by *gcd* (m, n) is the largest positive integer which divides (evenly) both m and n. The *least common multiple* of two integers m, n, denoted by *lcm* (m, n) is the smallest positive integer which is divisible by both m and n.

Let n be a positive integer. It can be written uniquely as a sum of powers of 2, *i.e.* in the form

$$n = \epsilon_k \cdot 2^k + \epsilon_{k-1} \cdot 2^{k-1} + \cdots + \epsilon_1 \cdot 2 + \epsilon_0$$

where each ϵ_i takes one of the values 0 and 1. This is known as the *binary representation* of n and is denoted $(\epsilon_k, \epsilon_{k-1}, \cdots, \epsilon_0)_2$. The numbers ϵ_i are known as the *(binary) digits* of n.

The *circumcircle* of a triangle is the centre of the circle that passes through the three vertices of the triangle; the *incentre* of a triangle is centre of the circle within the triangle that is tangent to the three sides; the *orthocentre* of a triangle is the intersection point of its three altitudes.

The function f defined on the real numbers and taking real values is *increasing* if and only if, for $x < y$, $f(x) \leq f(y)$.

$\lfloor x \rfloor$, the *floor of* x, is the largest integer n that does not exceed x, *i.e.*, that integer n for which $n \leq x < n + 1$. $\{x\}$, the *fractional part of* x, is equal to $x - \lfloor x \rfloor$. The notation $[PQR]$ denotes the area of the triangle PQR. A *geometric progression* is a sequence for which the ratio of two successive terms is always the same; its nth term has the general form ar^{n-1}. A *truncated pyramid* is a pyramid with a similar pyramid on a base parallel to the base of the first pyramid removed. A polyhedron is inscribed in a sphere if each of its vertices lies on the surface of the sphere.

1. Let M be a set of eleven points consisting of the four vertices along with seven interior points of a square of unit area.

   (a) Prove that there are three of these points that are vertices of a triangle whose area is at most $1/16$.

   (b) Give an example of a set M for which no four of the interior points are collinear and each nondegenerate triangle formed by three of them has area at least $1/16$.

2. Let a, b, c be the lengths of the sides of a triangle. Suppose that $u = a^2 + b^2 + c^2$ and $v = (a + b + c)^2$. Prove that
$$\frac{1}{3} \le \frac{u}{v} < \frac{1}{2}$$
   and that the fraction $1/2$ on the right cannot be replaced by a smaller number.

3. Suppose that $f(x)$ is a function satisfying
$$|f(m+n) - f(m)| \le \frac{n}{m}$$
   for all rational numbers n and m. Show that, for all natural numbers k,
$$\sum_{i=1}^{k} |f(2^k) - f(2^i)| \le \frac{k(k-1)}{2} \ .$$

4. Is it true that any pair of triangles sharing a common angle, inradius and circumradius must be congruent?

5. Each point of the plane is coloured with one of 2000 different colours. Prove that there exists a rectangle all of whose vertices have the same colour.

6. Let n be a positive integer, P be a set of n primes and M a set of at least $n+1$ natural numbers, each of which is divisible by no primes other than those belonging to P. Prove that there is a nonvoid subset of M, the product of whose elements is a square integer.

7. Let
$$S = \frac{1^2}{1 \cdot 3} + \frac{2^2}{3 \cdot 5} + \frac{3^2}{5 \cdot 7} + \cdots + \frac{500^2}{999 \cdot 1001} \ .$$
   Find the value of S.

8. The sequences $\{a_n\}$ and $\{b_n\}$ are such that, for every positive integer n,
$$a_n > 0 \ , \quad b_n > 0 \ , \quad a_{n+1} = a_n + \frac{1}{b_n} \ , \quad b_{n+1} = b_n + \frac{1}{a_n} \ .$$
   Prove that $a_{50} + b_{50} > 20$.

9. There are six points in the plane. Any three of them are vertices of a triangle whose sides are of different length. Prove that there exists a triangle whose smallest side is the largest side of another triangle.

10. In a rectangle, whose sides are 20 and 25 units of length, are placed 120 squares of side 1 unit of length. Prove that a circle of diameter 1 unit can be placed in the rectangle, so that it has no common points with the squares.

11. Each of nine lines divides a square into two quadrilaterals, such that the ratio of their area is 2:3. Prove that at least three of these lines are concurrent.

12. Each vertex of a regular 100-sided polygon is marked with a number chosen from among the natural numbers $1, 2, 3, \cdots, 49$. Prove that there are four vertices (which we can denote as A, B, C, D with respective numbers a, b, c, d) such that $ABCD$ is a rectangle, the points A and B are two adjacent vertices of the rectangle and $a + b = c + d$.

13. Suppose that $x_1, x_2, \cdots, x_n$ are nonnegative real numbers for which $x_1 + x_2 + \cdots + x_n < \frac{1}{2}$. Prove that

$$(1 - x_1)(1 - x_2) \cdots (1 - x_n) > \frac{1}{2} \ ,$$

14. Given a convex quadrilateral, is it always possible to determine a point in its interior such that the four line segments joining the point to the midpoints of the sides divide the quadrilateral into four regions of equal area? If such a point exists, is it unique?

15. Determine all triples (x, y, z) of real numbers for which

$$x(y + 1) = y(z + 1) = z(x + 1) \ .$$

16. Suppose that $ABCDEZ$ is a regular octahedron whose pairs of opposite vertices are (A, Z), (B, D) and (C, E). The points F, G, H are chosen on the segments AB, AC, AD respectively such that $AF = AG = AH$.

    (a) Show that EF and DG must intersect in a point K, and that BG and EH must intersect in a point L.

    (b) Let EG meet the plane of AKL in M. Show that $AKML$ is a square.

17. Suppose that r is a real number. Define the sequence x_n recursively by $x_0 = 0$, $x_1 = 1$, $x_{n+2} = rx_{n+1} - x_n$ for $n \geq 0$. For which values of r is it true that

$$x_1 + x_3 + x_5 + \cdots + x_{2m-1} = x_m^2$$

    for $m = 1, 2, 3, 4, \cdots$.

18. Let a and b be integers. How many solutions in real pairs (x, y) does the system

$$\lfloor x \rfloor + 2y = a$$

$$\lfloor y \rfloor + 2x = b$$

    have?

19. Is it possible to divide the natural numbers $1, 2, \cdots, n$ into two groups, such that the squares of the members in each group have the same sum, if (a) $n = 40000$; (b) $n = 40002$? Explain your answer.

20. Given any six irrational numbers, prove that there are always three of them, say a, b, c, for which $a+b$, $b + c$ and $c + a$ are irrational.

21. The natural numbers x_1, x_2, $\cdots$, x_{100} are such that

$$\frac{1}{\sqrt{x_1}} + \frac{1}{\sqrt{x_2}} + \cdots + \frac{1}{\sqrt{x_{100}}} = 20 \ .$$

Prove that at least two of the numbers are equal.

22. Let $\mathbf{R}$ be a rectangle with dimensions 11×12. Find the least natural number n for which it is possible to cover $\mathbf{R}$ with n rectangles, each of size 1×6 or 1×7, with no two of these having a common interior point.

23. Given 21 points on the circumference of a circle, prove that at least 100 of the arcs determined by pairs of these points subtend an angle not exceeding $120°$ at the centre.

24. ABC is an acute triangle with orthocentre H. Denote by M and N the midpoints of the respective segments AB and CH, and by P the intersection point of the bisectors of angles CAH and CBH. Prove that the points M, N and P are collinear.

25. Let a, b, c be non-negative numbers such that $a+b+c=1$. Prove that

$$\frac{ab}{c+1}+\frac{bc}{a+1}+\frac{ca}{b+1}\le\frac{1}{4}\ .$$

When does equality hold?

26. Each of m cards is labelled by one of the numbers $1, 2, \cdots, m$. Prove that, if the sum of labels of any subset of cards is not a multiple of $m+1$, then each card is labelled by the same number.

27. Find the least number of the form $|36^m - 5^n|$ where m and n are positive integers.

28. Let A be a finite set of real numbers which contains at least two elements and let $f : A \longrightarrow A$ be a function such that $|f(x) - f(y)| < |x - y|$ for every $x, y \in A$, $x \neq y$. Prove that there is $a \in A$ for which $f(a) = a$. Does the result remain valid if A is not a finite set?

29. Let A be a nonempty set of positive integers such that if $a \in A$, then $4a$ and $\lfloor\sqrt{a}\rfloor$ both belong to A. Prove that A is the set of all positive integers.

30. Find a point M within a regular pentagon for which the sum of its distances to the vertices is minimum.

31. Let x, y, z be positive real numbers for which $x^2+y^2+z^2=1$. Find the minimum value of

$$S=\frac{xy}{z}+\frac{yz}{x}+\frac{zx}{y}\ .$$

32. The segments BE and CF are altitudes of the acute triangle ABC, where E and F are points on the segments AC and AB, respectively. ABC is inscribed in the circle $\mathbf{Q}$ with centre O. Denote the orthocentre of ABC by H, and the midpoints of BC and AH be M and K, respectively. Let $\angle CAB = 45°$.

    (a) Prove, that the quadrilateral $MEKF$ is a square.

    (b) Prove that the midpoint of both diagonals of $MEKF$ is also the midpoint of the segment OH.

    (c) Find the length of EF, if the radius of $\mathbf{Q}$ has length 1 unit.

33. Prove the inequality $a^2+b^2+c^2+2abc<2$, if the numbers a, b, c are the lengths of the sides of a triangle with perimeter 2.

34. Each of the edges of a cube is 1 unit in length, and is divided by two points into three equal parts. Denote by $\mathbf{K}$ the solid with vertices at these points.

    (a) Find the volume of $\mathbf{K}$.

    (b) Every pair of vertices of $\mathbf{K}$ is connected by a segment. Some of the segments are coloured. Prove that it is always possible to find two vertices which are endpoints of the same number of coloured segments.

35. There are n points on a circle whose radius is 1 unit. What is the greatest number of segments between two of them, whose length exceeds $\sqrt{3}$?

36. Prove that there are not three rational numbers x, y, z such that

$$x^2 + y^2 + z^2 + 3(x + y + z) + 5 = 0 \ .$$

37. Let ABC be a triangle with sides a, b, c, inradius r and circumradius R (using the conventional notation). Prove that

$$\frac{r}{2R} \le \frac{abc}{\sqrt{2(a^2+b^2)(b^2+c^2)(c^2+a^2)}} \ .$$

When does equality hold?

38. Let us say that a set S of nonnegative real numbers if *hunky-dory* if and only if, for all x and y in S, either $x + y$ or $|x - y|$ is in S. For instance, if r is positive and n is a natural number, then $S(n, r) = \{0, r, 2r, \cdots, nr\}$ is hunky-dory. Show that every hunky-dory set with finitely many elements is $\{0\}$, is of the form $S(n, r)$ or has exactly four elements.

39. (a) $ABCDEF$ is a convex hexagon, each of whose diagonals AD, BE and CF pass through a common point. Must each of these diagonals bisect the area?

(b) $ABCDEF$ is a convex hexagon, each of whose diagonals AD, BE and CF bisects the area (so that half the area of the hexagon lies on either side of the diagonal). Must the three diagonals pass through a common point?

40. Determine all solutions in integer pairs (x, y) to the diophantine equation $x^2 = 1 + 4y^3(y + 2)$.

41. Determine the least positive number p for which there exists a positive number q such that

$$\sqrt{1+x} + \sqrt{1-x} \le 2 - \frac{x^p}{q}$$

for $0 \le x \le 1$. For this least value of p, what is the smallest value of q for which the inequality is satisfied for $0 \le x \le 1$?

42. G is a connected graph; that is, it consists of a number of vertices, some pairs of which are joined by edges, and, for any two vertices, one can travel from one to another along a chain of edges. We call two vertices *adjacent* if and only if they are endpoints of the same edge. Suppose there is associated with each vertex v a nonnegative integer $f(v)$ such that all of the following hold:
(1) If v and w are adjacent, then $|f(v) - f(w)| \le 1$.
(2) If $f(v) > 0$, then v is adjacent to at least one vertex w such that $f(w) < f(v)$.
(3) There is exactly one vertex u such that $f(u) = 0$.
Prove that $f(v)$ is the number of edges in the chain with the fewest edges connecting u and v.

43. Two players pay a game: the first player thinks of n integers x_1, x_2, $\cdots$, x_n, each with one digit, and the second player selects some numbers a_1, a_2, $\cdots$, a_n and asks what is the value of the sum $a_1x_1 + a_2x_2 + \cdots + a_nx_n$. What is the minimum number of questions used by the second player to find the integers x_1, x_2, $\cdots$, x_n?

44. Find the permutation $\{a_1, a_2, \cdots, a_n\}$ of the set $\{1, 2, \cdots, n\}$ for which the sum

$$S = |a_2 - a_1| + |a_3 - a_2| + \cdots + |a_n - a_{n-1}|$$

has maximum value.

45. Prove that there is no polynomial $p(x) = a_nx^n + a_{n-1}x^{n-1} + \cdots + a_0$ with integer coefficients a_i for which $p(m)$ is a prime number for every integer m.

46. Let $a_1 = 2$, $a_{n+1} = \frac{a_n+2}{1-2a_n}$ for $n = 1, 2, \cdots$. Prove that

(a) $a_n \neq 0$ for each positive integer n;

(b) there is no integer $p \geq 1$ for which $a_{n+p} = a_n$ for every integer $n \geq 1$ (*i.e.*, the sequence is not periodic).

47. Let $a_1, a_2, \cdots, a_n$ be positive real numbers such that $a_1 a_2 \cdots a_n = 1$. Prove that

$$\sum_{k=1}^{n} \frac{1}{s - a_k} \leq 1$$

where $s = 1 + a_1 + a_2 + \cdots + a_n$.

48. Let $A_1 A_2 \cdots A_n$ be a regular $n-$gon and d an arbitrary line. The parallels through A_i to d intersect its circumcircle respectively at B_i $(i = 1, 2, \cdots, n$. Prove that the sum

$$S = |A_1 B_1|^2 + \cdots + |A_n B_n|^2$$

is independent of d.

49. Find all ordered pairs (x, y) that are solutions of the following system of two equations (where a is a parameter):

$$x - y = 2$$

$$\left(x - \frac{2}{a}\right)\left(y - \frac{2}{a}\right) = a^2 - 1 \ .$$

Find all values of the parameter a for which the solutions of the system are two pairs of nonnegative numbers. Find the minimum value of $x + y$ for these values of a.

50. Let n be a natural number exceeding 1, and let A_n be the set of *all* natural numbers that are not relatively prime with n (*i.e.*, $A_n = \{x \in \mathbf{N} : \ \gcd(x, n) \neq 1\}$. Let us call the number n *magic* if for each two numbers $x, y \in A_n$, their sum $x + y$ is also an element of A_n (*i.e.*, $x + y \in A_n$ for $x, y \in A_n$).

(a) Prove that 67 is a magic number.

(b) Prove that 2001 is **not** a magic number.

(c) Find all magic numbers.

51. In the triangle ABC, $AB = 15$, $BC = 13$ and $AC = 12$. Prove that, for this triangle, the angle bisector from A, the median from B and the altitude from C are concurrent (*i.e.*, meet in a common point).

52. One solution of the equation $2x^3 + ax^2 + bx + 8 = 0$ is $1 + \sqrt{3}$. Given that a and b are rational numbers, determine its other two solutions.

53. Prove that among any 17 natural numbers chosen from the sets $\{1, 2, 3, \cdots, 24, 25\}$, it is always possible to find two whose product is a perfect square.

54. A circle has exactly one common point with each of the sides of a $(2n + 1)-$sided polygon. None of the vertices of the polygon is a point of the circle. Prove that at least one of the sides is a tangent of the circle.

55. A textbook problem has the following form: *A man is standing in a line in front of a movie theatre. The fraction x of the line is in front of him, and the fraction y of the line is behind him, where x and y are rational numbers written in lowest terms. How many people are there in the line?* Prove that, if the problem has an answer, then that answer must be the least common multiple of the denominators of x and y.

56. Let n be a positive integer and let $x_1, x_2, \cdots, x_n$ be integers for which

$$x_1^2 + x_2^2 + \cdots + x_n^2 + n^3 \leq (2n-1)(x_1 + x_2 + \cdots + x_n) + n^2 \ .$$

Show that

(a) $x_1, x_2, \cdots, x_n$ are all nonnegative;

(b) $x_1 + x_2 + \cdots + x_n + n + 1$ is not a perfect square.

57. Let $ABCD$ be a rectangle and let E be a point in the diagonal BD with $\angle DAE = 15°$. Let F be a point in AB with $EF \perp AB$. It is known that $EF = \frac{1}{2}AB$ and $AD = a$. Find the measure of the angle $\angle EAC$ and the length of the segment EC.

58. Find integers a, b, c such that $a \neq 0$ and the quadratic function $f(x) = ax^2 + bx + c$ satisfies

$$f(f(1)) = f(f(2)) = f(f(3)) \ .$$

59. Let $ABCD$ be a concyclic quadrilateral. Prove that

$$|AC - BD| \leq |AB - CD| \ .$$

60. Let $n \geq 2$ be an integer and $M = \{1, 2, \cdots, n\}$. For every integer k with $1 \leq k \leq n-1$, let

$$x_k = \sum \{\min A + \max A : A \subseteq M, A \text{ has } k \text{ elements}\}$$

where $\min A$ is the smallest and $\max A$ is the largest number in A. Determine $\sum_{k=1}^{n} (-1)^{k-1} x_k$.

61. Let $S = 1!2!3! \cdots 99!100!$ (the product of the first 100 factorials). Prove that there exists an integer k for which $1 \leq k \leq 100$ and $S/k!$ is a perfect square. Is k unique? (*Optional:* Is it possible to find such a number k that exceeds 100?)

62. Let n be a positive integer. Show that, with three exceptions, $n! + 1$ has at least one prime divisor that exceeds $n + 1$.

63. Let n be a positive integer and k a nonnegative integer. Prove that

$$n! = (n+k)^n - \binom{n}{1}(n+k-1)^n + \binom{n}{2}(n+k-2)^n - \cdots \pm \binom{n}{n} k^n \ .$$

64. Let M be a point in the interior of triangle ABC, and suppose that D, E, F are points on the respective side BC, CA, AB. Suppose AD, BE and CF all pass through M. (In technical terms, they are *cevians.*) Suppose that the areas and the perimeters of the triangles BMD, CME, AMF are equal. Prove that triangle ABC must be equilateral.

65. Suppose that XTY is a straight line and that TU and TV are two rays emanating from T for which $\angle XTU = \angle UTV = \angle VTY = 60°$. Suppose that P, Q and R are respective points on the rays TY, TU and TV for which $PQ = PR$. Prove that $\angle QPR = 60°$.

66. (a) Let $ABCD$ be a square and let E be an arbitrary point on the side CD. Suppose that P is a point on the diagonal AC for which $EP \perp AC$ and that Q is a point on AE produced for which $CQ \perp AE$. Prove that B, P, Q are collinear.

(b) Does the result hold if the hypothesis is weakened to require only that $ABCD$ is a rectangle?

67. (a) Consider the infinite integer lattice in the plane (*i.e.*, the set of points with integer coordinates) as a graph, with the edges being the lines of unit length connecting nearby points. What is the minimum number of colours that can be used to colour all the vertices and edges of this graph, so that

(i) each pair of adjacent vertices gets two distinct colours; AND

(ii) each pair of edges that meet at a vertex get two distinct colours; AND

(iii) an edge is coloured differently that either of the two vertices at the ends?

(b) Extend this result to lattices in real $n-$dimensional space.

68. Let $a, b, c > 0$, $a < bc$ and $1 + a^3 = b^3 + c^3$. Prove that $1 + a < b + c$.

69. Let n, a_1, a_2, $\cdots$, a_k be positive integers for which $n \geq a_1 > a_2 > a_3 > \cdots > a_k$ and the least common multiple of a_i and a_j does not exceed n for all i and j. Prove that $ia_i \leq n$ for $i = 1, 2, \cdots, k$.

70. Let $f(x)$ be a concave strictly increasing function defined for $0 \leq x \leq 1$ such that $f(0) = 0$ and $f(1) = 1$. Suppose that $g(x)$ is its inverse. Prove that $f(x)g(x) \leq x^2$ for $0 \leq x \leq 1$.

71. Suppose that lengths a, b and i are given. Construct a triangle ABC for which $|AC| = b$. $|AB| = c$ and the length of the bisector AD of angle A is i (D being the point where the bisector meets the side BC).

72. The centres of the circumscribed and the inscribed spheres of a given tetrahedron coincide. Prove that the four triangular faces of the tetrahedron are congruent.

73. Solve the equation:

$$\left(\sqrt{2+\sqrt{2}}\right)^x + \left(\sqrt{2-\sqrt{2}}\right)^x = 2^x .$$

74. Prove that among any group of $n + 2$ natural numbers, there can be found two numbers so that their sum or their difference is divisible by $2n$.

75. Three consecutive natural numbers, larger than 3, represent the lengths of the sides of a triangle. The area of the triangle is also a natural number.

(a) Prove that one of the altitudes "cuts" the triangle into two triangles, whose side lengths are natural numbers.

(b) The altitude identified in (a) divides the side which is perpendicular to it into two segments. Find the difference between the lengths of these segments.

76. Solve the system of equations:

$$\log x + \frac{\log(xy^8)}{\log^2 x + \log^2 y} = 2 ,$$

$$\log y + \frac{\log(x^8/y)}{\log^2 x + \log^2 y} = 0 .$$

(The logarithms are taken to base 10.)

77. n points are chosen from the circumference or the interior of a regular hexagon with sides of unit length, so that the distance between any two of them is **not** less that $\sqrt{2}$. What is the largest natural number n for which this is possible?

78. A truck travelled from town A to town B over several days. During the first day, it covered $1/n$ of the total distance, where n is a natural number. During the second day, it travelled $1/m$ of the remaining

distance, where m is a natural number. During the third day, it travelled $1/n$ of the distance remaining after the second day, and during the fourth day, $1/m$ of the distance remaining after the third day. Find the values of m and n if it is known that, by the end of the fourth day, the truck had travelled $3/4$ of the distance between A and B. (Without loss of generality, assume that $m < n$.)

79. Let x_0, x_1, x_2 be three positive real numbers. A sequence $\{x_n\}$ is defined, for $n \geq 0$ by

$$x_{n+3} = \frac{x_{n+2} + x_{n+1} + 1}{x_n} \ .$$

Determine all such sequences whose entries consist solely of positive integers.

80. Prove that, for each positive integer n, the series

$$\sum_{k=1}^{\infty} \frac{k^n}{2^k}$$

converges to twice an odd integer not less than $(n+1)!$.

81. Suppose that $x \geq 1$ and that $x = \lfloor x \rfloor + \{x\}$, where $\lfloor x \rfloor$ is the greatest integer not exceeding x and the fractional part $\{x\}$ satisfies $0 \leq x < 1$. Define

$$f(x) = \frac{\sqrt{\lfloor x \rfloor} + \sqrt{\{x\}}}{\sqrt{x}} \ .$$

(a) Determine the small number z such that $f(x) \leq z$ for each $x \geq 1$.

(b) Let $x_0 \geq 1$ be given, and for $n \geq 1$, define $x_n = f(x_{n-1})$. Prove that $\lim_{n\to\infty} x_n$ exists.

82. (a) A regular pentagon has side length a and diagonal length b. Prove that

$$\frac{b^2}{a^2} + \frac{a^2}{b^2} = 3 \ .$$

(b) A regular heptagon (polygon with seven equal sides and seven equal angles) has diagonals of two different lengths. Let a be the length of a side, b be the length of a shorter diagonal and c be the length of a longer diagonal of a regular heptagon (so that $a < b < c$). Prove that:

$$\frac{a^2}{b^2} + \frac{b^2}{c^2} + \frac{c^2}{a^2} = 6$$

and

$$\frac{b^2}{a^2} + \frac{c^2}{b^2} + \frac{a^2}{c^2} = 5 \ .$$

83. Let $\mathfrak{C}$ be a circle with centre O and radius 1, and let $\mathfrak{F}$ be a closed convex region inside $\mathfrak{C}$. Suppose from each point $\mathfrak{C}$, we can draw two rays tangent to $\mathfrak{F}$ meeting at an angle of $60°$. Describe $\mathfrak{F}$.

84. Let ABC be an acute-angled triangle, with a point H inside. Let U, V, W be respectively the reflected image of H with respect to axes BC, AC, AB. Prove that H is the orthocentre of ΔABC if and only if U, V, W lie on the circumcircle of ΔABC,

85. Find all pairs (a, b) of positive integers with $a \neq b$ for which the system

$$\cos ax + \cos bx = 0$$

$$a \sin ax + b \sin bx = 0$$

has a solution. If so, determine its solutions.

86. Let $ABCD$ be a convex quadrilateral with $AB = AD$ and $CB = CD$. Prove that

    (a) it is possible to inscribe a circle in it;

    (b) it is possible to circumscribe a circle about it if and only if $AB \perp BC$;

    (c) if $AB \perp AC$ and R and r are the respective radii of the circumscribed and inscribed circles, then the distance between the centres of the two circles is equal to the square root of $R^2 + r^2 - r\sqrt{r^2 + 4R^2}$.

87. Prove that, if the real numbers a, b, c, satisfy the equation

$$\lfloor na \rfloor + \lfloor nb \rfloor = \lfloor nc \rfloor$$

    for each positive integer n, then at least one of a and b is an integer.

88. Let I be a real interval of length $1/n$. Prove that I contains no more than $\frac{1}{2}(n+1)$ irreducible fractions of the form p/q with p and q positive integers, $1 \leq q \leq n$ and the greatest common divisor of p and q equal to 1.

89. Prove that there is only one triple of positive integers, each exceeding 1, for which the product of any two of the numbers plus one is divisible by the third.

90. Let m be a positive integer, and let $f(m)$ be the smallest value of n for which the following statement is true:

    given any set of n integers, it is always possible to find a subset of m integers whose sum is divisible by m

    Determine $f(m)$.

91. A square and a regular pentagon are inscribed in a circle. The nine vertices are all distinct and divide the circumference into nine arcs. Prove that at least one of them does not exceed $1/40$ of the circumference of the circle.

92. Consider the sequence 200125, 2000125, 20000125, $\cdots$, $200\cdots00125$, $\cdots$ (in which the nth number has $n+1$ digits equal to zero). Can any of these numbers be the square or the cube of an integer?

93. For any natural number n, prove the following inequalities:

$$2^{(n-1)/(2^n-2)} \leq \sqrt{2}\sqrt[4]{4}\sqrt[8]{8}\cdots\sqrt[2^n]{2^n} < 4\ .$$

94. ABC is a right triangle with arms a and b and hypotenuse $c = |AB|$; the area of the triangle is s square units and its perimeter is $2p$ units. The numbers a, b and c are positive integers. Prove that s and p are also positive integers and that s is a multiple of p.

95. The triangle ABC is isosceles is isosceles with equal sides AC and BC. Two of its angles measure $40°$. The interior point M is such that $\angle MAB = 10°$ and $\angle MBA = 20°$. Determine the measure of $\angle CMB$.

96. Find all prime numbers p for which all three of the numbers $p^2 - 2$, $2p^2 - 1$ and $3p^2 + 4$ are also prime.

97. A triangle has its three vertices on a rectangular hyperbola. Prove that its orthocentre also lies on the hyperbola.

98. Let $a_1, a_2, \cdots, a_{n+1}, b_1, b_2, \cdots, b_n$ be nonnegative real numbers for which
    (i) $a_1 \geq a_2 \geq \cdots \geq a_{n+1} = 0$,

(ii) $0 \le b_k \le 1$ for $k = 1, 2, \cdots, n$.
Suppose that $m = \lfloor b_1 + b_2 + \cdots + b_n \rfloor + 1$. Prove that

$$\sum_{k=1}^{n} a_k b_k \le \sum_{k=1}^{m} a_k \ .$$

99. Let E and F be respective points on sides AB and BC of a triangle ABC for which $AE = CF$. The circle passing through the points B, C, E and the circle passing through the points A, B, F intersect at B and D. Prove that BD is the bisector of angle ABC.

100. If 10 equally spaced points around a circle are joined consecutively, a convex regular inscribed decagon P is obtained; if every third point is joined, a self-intersecting regular decagon Q is formed. Prove that the difference between the length of a side of Q and the length of a side of P is equal to the radius of the circle. [*With thanks to Ross Honsberger.*]

101. Let a, b, u, v be nonnegative. Suppose that $a^5 + b^5 \le 1$ and $u^5 + v^5 \le 1$. Prove that

$$a^2u^3 + b^2v^3 \le 1 \ .$$

[*With thanks to Ross Honsberger.*]

102. Prove that there exists a tetrahedron $ABCD$, all of whose faces are similar right triangles, each face having acute angles at A and B. Determine which of the edges of the tetrahedron is largest and which is smallest, and find the ratio of their lengths.

103. Determine a value of the parameter θ so that

$$f(x) \equiv \cos^2 x + \cos^2(x + \theta) - \cos x \cos(x + \theta)$$

is a constant function of x.

104. Prove that there exists exactly one sequence $\{x_n\}$ of positive integers for which

$$x_1 = 1 \ , \quad x_2 > 1 \ , \quad x_{n+1}^3 + 1 = x_n x_{n+2}$$

for $n \ge 1$.

105. Prove that within a unit cube, one can place two regular unit tetrahedra that have no common point.

106. Find all pairs (x, y) of positive real numbers for which the least value of the function

$$f(x, y) = \frac{x^4}{y^4} + \frac{y^4}{x^4} - \frac{x^2}{y^2} - \frac{y^2}{x^2} + \frac{x}{y} + \frac{y}{x}$$

is attained. Determine that minimum value.

107. Given positive numbers a_i with $a_1 < a_2 < \cdots < a_n$, for which permutation $(b_1, b_2, \cdots, b_n)$ of these numbers is the product

$$\prod_{i=1}^{n} \left(a_i + \frac{1}{b_i} \right)$$

maximized?

108. Determine all real-valued functions $f(x)$ of a real variable x for which

$$f(xy) = \frac{f(x) + f(y)}{x + y}$$

for all real x and y for which $x + y \neq 0$.

109. Suppose that

$$\frac{x^2 + y^2}{x^2 - y^2} + \frac{x^2 - y^2}{x^2 + y^2} = k \ .$$

Find, in terms of k, the value of the expression

$$\frac{x^8 + y^8}{x^8 - y^8} + \frac{x^8 - y^8}{x^8 + y^8} \ .$$

110. Given a triangle ABC with an area of 1. Let $n > 1$ be a natural number. Suppose that M is a point on the side AB with $AB = nAM$, N is a point on the side BC with $BC = nBN$, and Q is a point on the side CA with $CA = nCQ$. Suppose also that $\{T\} = AN \cap CM$, $\{R\} = BQ \cap AN$ and $\{S\} = CM \cap BQ$, where $\cap$ signifies that the singleton is the intersection of the indicated segments. Find the area of the triangle TRS in terms of n.

111. (a) Are there four different numbers, not exceeding 10, for which the sum of any three is a prime number?

(b) Are there five different natural numbers such that the sum of every three of them is a prime number?

112. Suppose that the measure of angle BAC in the triangle ABC is equal to α. A line passing through the vertex A is perpendicular to the angle bisector of $\angle BAC$ and intersects the line BC at the point M. Find the other two angles of the triangle ABC in terms of α, if it is known that $BM = BA + AC$.

113. Find a function that satisfies all of the following conditions:

(a) f is defined for every positive integer n;

(b) f takes only positive values;

(c) $f(4) = 4$;

(d)

$$\frac{1}{f(1)f(2)} + \frac{1}{f(2)f(3)} + \cdots + \frac{1}{f(n)f(n+1)} = \frac{f(n)}{f(n+1)} \ .$$

114. A natural number is a multiple of 17. Its binary representation (*i.e.*, when written to base 2) contains exactly three digits equal to 1 and some zeros.

(a) Prove that there are at least six digits equal to 0 in its binary representation.

(b) Prove that, if there are exactly seven digits equal to 0 and three digits equal to 1, then the number must be even.

115. Let U be a set of n distinct real numbers and let V be the set of all sums of distinct pairs of them, *i.e.*,

$$V = \{x + y : x, y \in U, x \neq y\} \ .$$

What is the smallest possible number of distinct elements that V can contain?

116. Prove that the equation

$$x^4 + 5x^3 + 6x^2 - 4x - 16 = 0$$

has exactly two real solutions.

117. Let a be a real number. Solve the equation

$$(a - 1)\left(\frac{1}{\sin x} + \frac{1}{\cos x} + \frac{1}{\sin x \cos x}\right) = 2 \ .$$

118. Let a, b, c be nonnegative real numbers. Prove that

$$a^2(b+c-a)+b^2(c+a-b)+c^2(a+b-c) \leq 3abc \; .$$

When does equality hold?

119. The medians of a triangle ABC intersect in G. Prove that

$$|AB|^2+|BC|^2+|CA|^2 = 3(|GA|^2+|GB|^2+|GC|^2) \; .$$

120. Determine all pairs of nonnull vectors $\mathbf{x}$, $\mathbf{y}$ for which the following sequence $\{a_n : n = 1, 2, \cdots\}$ is (a) increasing, (b) decreasing, where

$$a_n = |\mathbf{x} - n\mathbf{y}| \; .$$

121. Let n be an integer exceeding 1. Let $a_1, a_2, \cdots, a_n$ be posive real numbers and $b_1, b_2, \cdots, b_n$ be arbitrary real numbers for which

$$\sum_{i \neq j} a_i b_j = 0 \; .$$

Prove that $\sum_{i \neq j} b_i b_j < 0$.

122. Determine all functions f from the real numbers to the real numbers that satisfy

$$f(f(x)+y) = f(x^2-y)+4f(x)y$$

for any real numbers x, y.

123. Let a and b be the lengths of two opposite edges of a tetrahedron which are mutually perpendicular and distant d apart. Determine the volume of the tetrahedron.

124. Prove that

$$\frac{(1^4+\frac{1}{4})(3^4+\frac{1}{4})(5^4+\frac{1}{4})\cdots(11^4+\frac{1}{4})}{(2^4+\frac{1}{4})(4^4+\frac{1}{4})(6^4+\frac{1}{4})\cdots(12^4+\frac{1}{4})} = \frac{1}{313} \; .$$

125. Determine the set of complex numbers z which satisfy

$$\text{Im } (z^4) = (\text{Re } (z^2))^2 \; ,$$

and sketch this set in the complex plane. (*Note:* Im and Re refer respectively to the imaginary and real parts.)

126. Let n be a positive integer exceeding 1, and let n circles (*i.e.*, circumferences) of radius 1 be given in the plane such that no two of them are tangent and the subset of the plane formed by the union of them is connected. Prove that the number of points that belong to at least two of these circles is at least n.

127. Let

$$A = 2^n + 3^n + 216(2^{n-6} + 3^{n-6})$$

and

$$B = 4^n + 5^n + 8000(4^{n-6} + 5^{n-6})$$

where $n > 6$ is a natural number. Prove that the fraction A/B is reducible.

128. Let n be a positive integer. On a circle, n points are marked. The number 1 is assigned to one of them and 0 is assigned to the others. The following operation is allowed: Choose a point to which 1 is assigned and then assign $(1-a)$ and $(1-b)$ to the two adjacent points, where a and b are, respectively,

the numbers assigned to these points before. Is it possible to assign 1 to all points by applying this operation several times if (a) $n = 2001$ and (b) $n = 2002$?

129. For every integer n, a nonnegative integer $f(n)$ is assigned such that

(a) $f(mn) = f(m) + f(n)$ for each pair m, n of natural numbers;

(b) $f(n) = 0$ when the rightmost digit in the decimal representation of the number n is 3; and

(c) $f(10) = 0$.

Prove that $f(n) = 0$ for any natural number n.

130. Let $ABCD$ be a rectangle for which the respective lengths of AB and BC are a and b. Another rectangle is circumscribed around $ABCD$ so that each of its sides passes through one of the vertices of $ABCD$. Consider all such rectangles and, among them, find the one with a maximum area. Express this area in terms of a and b.

131. At a recent winter meeting of the Canadian Mathematical Society, some of the attending mathematicians were friends. It appeared that every two mathematicians, that had the same number of friends among the participants, did not have a common friend. Prove that there was a mathematician who had only one friend.

132. Simplify the expression

$$\sqrt[5]{3\sqrt{2} - 2\sqrt{5}} \cdot \sqrt[10]{\frac{6\sqrt{10} + 19}{2}} \ .$$

133. Prove that, if a, b, c, d are real numbers, $b \neq c$, both sides of the equation are defined, and

$$\frac{ac - b^2}{a - 2b + c} = \frac{bd - c^2}{b - 2c + d} \ ,$$

then each side of the equation is equal to

$$\frac{ad - bc}{a - b - c + d} \ .$$

Give two essentially different examples of quadruples (a, b, c, d), not in geometric progression, for which the conditions are satisfied. What happens when $b = c$?

134. Suppose that

$$a = zb + yc$$

$$b = xc + za$$

$$c = ya + xb \ .$$

Prove that

$$\frac{a^2}{1 - x^2} = \frac{b^2}{1 - y^2} = \frac{c^2}{1 - z^2} \ .$$

Of course, if any of x^2, y^2, z^2 is equal to 1, then the conclusion involves undefined quantities. Give the proper conclusion in this situation. Provide two essentially different numerical examples.

135. For the positive integer n, let $p(n) = k$ if n is divisible by 2^k but not by 2^{k+1}. Let $x_0 = 0$ and define x_n for $n \geq 1$ recursively by

$$\frac{1}{x_n} = 1 + 2p(n) - x_{n-1} \ .$$

Prove that every nonnegative rational number occurs exactly once in the sequence $\{x_0, x_1, x_2, \cdots, x_n, \cdots\}$.∎

136. Prove that, if in a semicircle of radius 1, five points A, B, C, D, E are taken in consecutive order, then

$$|AB|^2 + |BC|^2 + |CD|^2 + |DE|^2 + |AB||BC||CD| + |BC||CD||DE| < 4 .$$

137. Can an arbitrary convex quadrilateral be decomposed by a polygonal line into two parts, each of whose diameters is less than the diameter of the given quadrilateral?

138. (a) A room contains ten people. Among any three. there are two (mutual) acquaintances. Prove that there are four people, any two of whom are acquainted.

(b) Does the assertion hold if "ten" is replaced by "nine"?

139. Let A, B, C be three pairwise orthogonal faces of a tetrahedran meeting at one of its vertices and having respective areas a, b, c. Let the face D opposite this vertex have area d. Prove that

$$d^2 = a^2 + b^2 + c^2 .$$

140. Angus likes to go to the movies. On Monday, standing in line, he noted that the fraction x of the line was in front of him, while $1/n$ of the line was behind him. On Tuesday, the same fraction x of the line was in front of him, while $1/(n+1)$ of the line was behind him. On Wednesday, the same fraction x of the line was in front of him, while $1/(n+2)$ of the line was behind him. Determine a value of n for which this is possible.

141. In how many ways can the rational 2002/2001 be written as the product of two rationals of the form $(n+1)/n$, where n is a positive integer?

142. Let $x, y > 0$ be such that $x^3 + y^3 \leq x - y$. Prove that $x^2 + y^2 \leq 1$.

143. A sequence whose entries are 0 and 1 has the property that, if each 0 is replaced by 01 and each 1 by 001, then the sequence remains unchanged. Thus, it starts out as $010010101001\cdots$. What is the 2002th term of the sequence?

144. Let a, b, c, d be rational numbers for which $bc \neq ad$. Prove that there are infinitely many rational values of x for which $\sqrt{(a+bx)(c+dx)}$ is rational. Explain the situation when $bc = ad$.

145. Let ABC be a right triangle with $\angle A = 90°$. Let P be a point on the hypotenuse BC, and let Q and R be the respective feet of the perpendiculars from P to AC and AB. For what position of P is the length of QR minimum?

146. Suppose that ABC is an equilateral triangle. Let P and Q be the respective midpoint of AB and AC, and let U and V be points on the side BC with $4BU = 4VC = BC$ and $2UV = BC$. Suppose that PV are joined and that W is the foot of the perpendicular from U to PV and that Z is the foot of the perpendicular from Q to PV.

Explain how that four polygons $APZQ$, $BUWP$, $CQZV$ and UVW can be rearranged to form a rectangle. Is this rectangle a square?

147. Let $a > 0$ and let n be a positive integer. Determine the maximum value of

$$\frac{x_1 x_2 \cdots x_n}{(1+x_1)(x_1+x_2)\cdots(x_{n-1}+x_n)(x_n+a^{n+1})}$$

subject to the constraint that $x_1, x_2, \cdots, x_n > 0$.

148. For a given prime number p, find the number of distinct sequences of natural numbers (positive integers) $\{a_0, a_1, \cdots, a_n \cdots\}$ satisfying, for each positive integer n, the equation

$$\frac{a_0}{a_1} + \frac{a_0}{a_2} + \cdots + \frac{a_0}{a_n} + \frac{p}{a_{n+1}} = 1 .$$

149. Consider a cube concentric with a parallelepiped (rectangular box) with sides $a < b < c$ and faces parallel to that of the cube. Find the side length of the cube for which the difference between the volume of the union and the volume of the intersection of the cube and parallelepiped is minimum.

150. The area of the bases of a truncated pyramid are equal to S_1 and S_2 and the total area of the lateral surface is S. Prove that, if there is a plane parallel to each of the bases that partitions the truncated pyramid into two truncated pyramids within each of which a sphere can be inscribed, then

$$S = (\sqrt{S_1} + \sqrt{S_2})(\sqrt[4]{S_1} + \sqrt[4]{S_2})^2 .$$

151. Prove that, for any natural number n, the equation

$$x(x+1)(x+2)\cdots(x+2n-1) + (x+2n+1)(x+2n+2)\cdots(x+4n) = 0$$

does not have real solutions.

152. Andrew and Brenda are playing the following game. Taking turns, they write in a sequence, from left to right, the numbers 0 or 1 until each of them has written 2002 numbers (to produce a 4004-digit number). Brenda is the winner if the sequence of zeros and ones, considered as a binary number (*i.e.*, written to base 2), can be written as the sum of two integer squares. Otherwise, the winner is Andrew. Prove that the second player, Brenda, can always win the game, and explain her winning strategy (*i.e.*, how she must play to ensure winning every game).

153. (a) Prove that, among any 39 consecutive natural numbers, there is one the sum of whose digits (in base 10) is divisible by 11.

(b) Present some generalizations of this problem.

154. (a) Give as neat a proof as you can that, for any natural number n, the sum of the squares of the numbers $1, 2, \cdots, n$ is equal to $n(n+1)(2n+1)/6$.

(b) Find the least natural number n exceeding 1 for which $(1^2 + 2^2 + \cdots + n^2)/n$ is the square of a natural number.

155. Find all real numbers x that satisfy the equation

$$3^{[(1/2)+\log_3(\cos x+\sin x)]} - 2^{\log_2(\cos x-\sin x)} = \sqrt{2} .$$

[The logarithms are taken to bases 3 and 2 respectively.]

156. In the triangle ABC, the point M is from the inside of the angle BAC such that $\angle MAB = \angle MCA$ and $\angle MAC = \angle MBA$. Similarly, the point N is from the inside of the angle ABC such that $\angle NBA = \angle NCB$ and $\angle NBC = \angle NAB$. Also, the point P is from the inside of the angle ACB such that $\angle PCA = \angle PBC$ and $\angle PCB = \angle PAC$. (The points M, N and P each could be inside or outside of the triangle.) Prove that the lines AM, BN and CP are concurrent and that their intersection point belongs to the circumcircle of the triangle MNP.

157. Prove that if the quadratic equation $x^2 + ax + b + 1 = 0$ has nonzero integer solutions, then $a^2 + b^2$ is a composite integer.

158. Let $f(x)$ be a polynomial with real coefficients for which the equation $f(x) = x$ has no real solution. Prove that the equation $f(f(x)) = x$ has no real solution either.

159. Let $0 \le a \le 4$. Prove that the area of the bounded region enclosed by the curves with equations

$$y = 1 - |x - 1|$$

and

$$y = |2x - a|$$

cannot exceed $\frac{1}{3}$.

160. Let I be the incentre of the triangle ABC and D be the point of contact of the inscribed circle with the side AB. Suppose that ID is produced outside of the triangle ABC to H so that the length DH is equal to the semi-perimeter of ΔABC. Prove that the quadrilateral $AHBI$ is concyclic if and only if angle C is equal to $90°$.

161. Let a, b, c be positive real numbers for which $a + b + c = 1$. Prove that

$$\frac{a^3}{a^2 + b^2} + \frac{b^3}{b^2 + c^2} + \frac{c^3}{c^2 + a^2} \geq \frac{1}{2} \ .$$

162. Let A and B be fixed points in the plane. Find all positive integers k for which the following assertion holds:

among all triangles ABC with $|AC| = k|BC|$, the one with the largest area is isosceles.

163. Let R_i and r_i re the respective circumradius and inradius of triangle $A_iB_iC_i$ $(i = 1, 2)$. Prove that, if $\angle C_1 = \angle C_2$ and $R_1r_2 = r_1R_2$, then the two triangles are similar.

164. Let n be a positive integer and X a set with n distinct elements. Suppose that there are k distinct subsets of X for which the union of any four contains no more that $n-2$ elements. Prove that $k \leq 2^{n-2}$.

165. Let n be a positive integer. Determine all $n-$tples $\{a_1, a_2, \cdots, a_n\}$ of positive integers for which $a_1 + a_2 + \cdots + a_n = 2n$ and there is no subset of them whose sum is equal to n.

166. Suppose that f is a real-valued function defined on the reals for which

$$f(xy) + f(y - x) \geq f(y + x)$$

for all real x and y. Prove that $f(x) \geq 0$ for all real x.

167. Let $u = (\sqrt{5} - 2)^{1/3} - (\sqrt{5} + 2)^{1/3}$ and $v = (\sqrt{189} - 8)^{1/3} - (\sqrt{189} + 8)^{1/3}$. Prove that, for each positive integer n, $u^n + v^{n+1} = 0$.

168. Determine the value of

$$\cos 5° + \cos 77° + \cos 149° + \cos 221° + \cos 293° \ .$$

169. Prove that, for each positive integer n exceeding 1,

$$\frac{1}{2^n} + \frac{1}{2^{1/n}} < 1 \ .$$

170. Solve, for real x,

$$x \cdot 2^{1/x} + \frac{1}{x} \cdot 2^x = 4 \ .$$

171. Let n be a positive integer. In a round-robin match, n teams compete and each pair of teams plays exactly one game. At the end of the match, the ith team has x_i wins and y_i losses. There are no ties. Prove that

$$x_1^2 + x_2^2 + \cdots + x_n^2 = y_1^2 + y_2^2 + \cdots + y_n^2 \ .$$

172. Let a, b, c, d. e, f be different integers. Prove that

$$(a-b)^2 + (b-c)^2 + (c-d)^2 + (d-e)^2 + (e-f)^2 + (f-a)^2 \geq 18 \; .$$

173. Suppose that a and b are positive real numbers for which $a + b = 1$. Prove that

$$\left(a + \frac{1}{a}\right)^2 + \left(b + \frac{1}{b}\right)^2 \geq \frac{25}{2} \; .$$

Determine when equality holds.

174. For which real value of x is the function

$$(1-x)^5(1+x)(1+2x)^2$$

maximum? Determine its maximum value.

175. ABC is a triangle such that $AB < AC$. The point D is the midpoint of the arc with endpoints B and C of that arc of the circumcircle of ΔABC that contains A. The foot of the perpendicular from D to AC is E. Prove that $AB + AE = EC$.

176. Three noncollinear points A, M and N are given in the plane. Construct the square such that one of its vertices is the point A, and the two sides which do not contain this vertex are on the lines through M and N respectively. [*Note:* In such a problem, your solution should consist of a description of the construction (with straightedge and compasses) and a proof in correct logical order proceeding from what is given to what is desired that the construction is valid. You should deal with the feasibility of the construction.]

177. Let a_1, a_2, $\cdots$, a_n be nonnegative integers such that, whenever $1 \leq i$, $1 \leq j$, $i + j \leq n$, then

$$a_i + a_j \leq a_{i+j} \leq a_i + a_j + 1 \; .$$

(a) Give an example of such a sequence which is not an arithmetic progression.

(b) Prove that there exists a real number x such that $a_k = \lfloor kx \rfloor$ for $1 \leq k \leq n$.

178. Suppose that n is a positive integer and that $x_1, x_2, \cdots, x_n$ are positive real numbers such that $x_1 + x_2 + \cdots + x_n = n$. Prove that

$$\sum_{i=1}^{n} \sqrt[n]{ax_i + b} \leq a + b + n - 1$$

for every pair a, b of real numbers with each $ax_i + b$ nonnegative. Describe the situation when equality occurs.

179. Determine the units digit of the numbers a^2, b^2 and ab (in base 10 numeration), where

$$a = 2^{2002} + 3^{2002} + 4^{2002} + 5^{2002}$$

and

$$b = 3^1 + 3^2 + 3^3 + \cdots + 3^{2002} \; .$$

180. Consider the function f that takes the set of complex numbers into itself defined by $f(z) = 3z + |z|$. Prove that f is a bijection and find its inverse.

181. Consider a regular polygon with n sides, each of length a, and an interior point located at distances a_1, a_2, $\cdots$, a_n from the sides. Prove that

$$a\sum_{i=1}^{n}\frac{1}{a_i} > 2\pi\ .$$

182. Let M be an interior point of the equilateral triangle ABC with each side of unit length. Prove that

$$MA.MB + MB.MC + MC.MA \geq 1\ .$$

183. Simplify the expression

$$\frac{\sqrt{1+\sqrt{1-x^2}}((1+x)\sqrt{1+x}-(1-x)\sqrt{1-x})}{x(2+\sqrt{1-x^2})}\ ,$$

where $0 < |x| < 1$.

184. Using complex numbers, or otherwise, evaluate

$$\sin 10^\circ \sin 50^\circ \sin 70^\circ\ .$$

185. Find all triples of natural numbers a, b, c, such that all of the following conditions hold: (1) $a < 1974$; (2) b is less than c by 1575; (3) $a^2 + b^2 = c^2$.

186. Find all natural numbers n such that there exists a convex $n-$sided polygon whose diagonals are all of the same length.

187. Suppose that p is a real parameter and that

$$f(x) = x^3 - (p+5)x^2 - 2(p-3)(p-1)x + 4p^2 - 24p + 36\ .$$

(a) Check that $f(3-p) = 0$.

(b) Find all values of p for which two of the roots of the equation $f(x) = 0$ (expressed in terms of p) can be the lengths of the two legs in a right-angled triangle with a hypotenuse of $4\sqrt{2}$.

188. (a) The measure of the angles of an acute triangle are α, β and γ degrees. Determine (as an expression of α, β, γ) what masses must be placed at each of the triangle's vertices for the centroid (centre of gravity) to coincide with (i) the orthocentre of the triangle; (ii) the circumcentre of the triangle.

(b) The sides of an arbitrary triangle are a, b, c units in length. Determine (as an expression of a, b, c) what masses must be placed at each of the triangle's vertices for the centroid (centre of gravity) to coincide with (i) the centre of the inscribed circle of the triangle; (ii) the intersection point of the three segments joining the vertices of the triangle to the points on the opposite sides where the inscribed circle is tangent (be sure to prove that, indeed, the three segments intersect in a common point).

189. There are n lines in the plane, where n is an integer exceeding 2. No three of them are concurrent and no two of them are parallel. The lines divide the plane into regions; some of them are closed (they have the form of a convex polygon); others are unbounded (their borders are broken lines consisting of segments and rays).

(a) Determine as a function of n the number of unbounded regions.

(b) Suppose that some of the regions are coloured, so that no two coloured regions have a common side (a segment or ray). Prove that the number of regions coloured in this way does not exceed $\frac{1}{3}(n^2+n)$.

190. Find all integer values of the parameter a for which the equation

$$|2x+1|+|x-2|=a$$

has exactly one integer among its solutions.

191. In **Olymonland** the distances between every two cities is different. When the transportation program of the country was being developed, for each city, the closest of the other cities was chosen and a highway was built to connect them. All highways are line segments. Prove that

(a) no two highways intersect;

(b) every city is connected by a highway to no more than 5 other cities;

(c) there is no closed broken line composed of highways only.

192. Let ABC be a triangle, D be the midpoint of AB and E a point on the side AC for which $AE = 2EC$. Prove that BE bisects the segment CD.

193. Determine the volume of an isosceles tetrahedron for which the pairs of opposite edges have lengths a, b, c. Check your answer independently for a regular tetrahedron.

194. Let ABC be a triangle with incentre I. Let M be the midpoint of BC, U be the intersection of AI produced with BC, D be the foot of the perpendicular from I to BC and P be the foot of the perpendicular from A to BC. Prove that

$$|PD||DM| = |DU||PM| \ .$$

195. Let $ABCD$ be a convex quadrilateral and let the midpoints of AC and BD be P and Q respectively, Prove that

$$|AB|^2+|BC|^2+|CD|^2+|DA|^2 = |AC|^2+|BD|^2+4|PQ|^2 \ .$$

196. Determine five values of p for which the polynomial $x^2+2002x-1002p$ has integer roots.

197. Determine all integers x and y that satisfy the equation $x^3+9xy+127=y^3$.

198. Let p be a prime number and let $f(x)$ be a polynomial of degree d with integer coefficients such that $f(0)=0$ and $f(1)=1$ and that, for every positive integer n, $f(n)\equiv 0$ or $f(n)\equiv 1$, modulo p. Prove that $d \geq p-1$. Give an example of such a polynomial.

199. Let A and B be two points on a parabola with vertex V such that VA is perpendicular to VB and θ is the angle between the chord VA and the axis of the parabola. Prove that

$$\frac{|VA|}{|VB|} = \cot^3\theta \ .$$

200. Let n be a positive integer exceeding 1. Determine the number of permutations $(a_1, a_2, \cdots, a_n)$ of $(1, 2, \cdots, n)$ for which there exists exactly one index i with $1 \leq i \leq n$ and $a_i > a_{i+1}$.

201. Let $(a_1, a_2, \cdots, a_n)$ be an arithmetic progression and $(b_1, b_2, \cdots, b_n)$ be a geometric progression, each of n positive real numbers, for which $a_1 = b_1$ and $a_n = b_n$. Prove that

$$a_1+a_2+\cdots+a_n \geq b_1+b_2+\cdots+b_n \ .$$

202. For each positive integer k, let $a_k = 1+(1/2)+(1/3)+\cdots+(1/k)$. Prove that, for each positive integer n,

$$3a_1+5a_2+7a_3+\cdots+(2n+1)a_n = (n+1)^2a_n - \frac{1}{2}n(n+1) \ .$$

203. Every midpoint of an edge of a tetrahedron is contained in a plane that is perpendicular to the opposite edge. Prove that these six planes intersect in a point that is symmetric to the centre of the circumsphere of the tetrahedron with respect to its centroid.

204. Each of $n \geq 2$ people in a certain village has at least one of eight different names. No two people have exactly the same set of names. For an arbitrary set of k names (with $1 \leq k \leq 7$), the number of people containing at least one of the $k(\geq 1)$ names among his/her set of names is even. Determine the value of n.

205. Let $f(x)$ be a convex realvalued function defined on the reals, $n \geq 2$ and $x_1 < x_2 < \cdots < x_n$. Prove that

$$x_1 f(x_2) + x_2 f(x_3) + \cdots + x_n f(x_1) \geq x_2 f(x_1) + x_3 f(x_2) + \cdots + x_1 f(x_n) \ .$$

206. In a group consisting of five people, among any three people, there are two who know each other and two neither of whom knows the other. Prove that it is possible to seat the group around a circular table so that each adjacent pair knows each other.

207. Let n be a positive integer exceeding 1. Suppose that $A = (a_1, a_2, \cdots, a_m)$ is an ordered set of $m = 2^n$ numbers, each of which is equal to either 1 or -1. Let

$$S(A) = (a_1 a_2, a_2 a_3, \cdots, a_{m-1} a_m, a_m a_1) \ .$$

Define, $S^0(A) = A$, $S^1(A) = S(A)$, and for $k \geq 1$, $S^{k+1} = S(S^k(A))$. Is it always possible to find a positive integer r for which $S^r(A)$ consists entirely of 1s?

208. Determine all positive integers n for which $n = a^2 + b^2 + c^2 + d^2$, where $a < b < c < d$ and a, b, c, d are the four smallest positive divisors of n.

209. Determine all positive integers n for which $2^n - 1$ is a multiple of 3 and $(2^n - 1)/3$ has a multiple of the form $4m^2 + 1$ for some integer m.

210. ABC and DAC are two isosceles triangles for which B and D are on opposite sides of AC, $AB = AC$, $DA = DC$, $\angle BAC = 20°$ and $\angle ADC = 100°$. Prove that $AB = BC + CD$.

211. Let ABC be a triangle and let M be an interior point. Prove that

$$\min \{MA, MB, MC\} + MA + MB + MC < AB + BC + CA \ .$$

212. A set S of points in space has at least three elements and satisfies the condition that, for any two distinct points A and B in S, the right bisecting plane of the segment AB is a plane of symmetry for S. Determine all possible finite sets S that satisfy the condition.

213. Suppose that each side and each diagonal of a regular hexagon $A_1A_2A_3A_4A_5A_6$ is coloured either red or blue, and that no triangle $A_iA_jA_k$ has all of its sides coloured blue. For each $k = 1, 2, \cdots, 6$, let r_k be the number of segments A_kA_j $(j \neq k)$ coloured red. Prove that

$$\sum_{k=1}^{6} (2r_k - 7)^2 \leq 54 \ .$$

214. Let S be a circle with centre O and radius 1, and let P_i $(1 \leq i \leq n)$ be points chosen on the (circumference of the) circle for which $\sum_{i=1}^{n} \overrightarrow{OP_i} = \mathbf{0}$. Prove that, for each point X in the plane, $\sum |XP_i| \geq n$.

215. Find all values of the parameter a for which the equation $16x^4 - ax^3 + (2a + 17)x^2 - ax + 16 = 0$ has exactly four real solutions which are in geometric progression.

216. Let x be positive and let $0 < a \leq 1$. Prove that

$$(1 - x^a)(1 - x)^{-1} \leq (1 + x)^{a-1} .$$

217. Let the three side lengths of a scalene triangle be given. There are two possible ways of orienting the triangle with these side lengths, one obtainable from the other by turning the triangle over, or by reflecting in a mirror. Prove that it is possible to slice the triangle in one of its orientations into finitely many pieces that can be rearranged using rotations and translations in the plane (but not reflections and rotations out of the plane) to form the other.

218. Let ABC be a triangle. Suppose that D is a point on BA produced and E a point on the side BC, and that DE intersects the side AC at F. Let $BE + EF = BA + AF$. Prove that $BC + CF = BD + DF$.

219. There are two definitions of an ellipse.

(1) An ellipse is the locus of points P such that the sum of its distances from two fixed points F_1 and F_2 (called *foci*) is constant.

(2) An ellipse is the locus of points P such that, for some real number e (called the *eccentricity*) with $0 < e < 1$, the distance from P to a fixed point F (called a *focus*) is equal to e times its perpendicular distance to a fixed straight line (called the *directrix*).

Prove that the two definitions are compatible.

220. Prove or disprove: A quadrilateral with one pair of opposite sides and one pair of opposite angles equal is a parallelogram.

221. A *cycloid* is the locus of a point P fixed on a circle that rolls without slipping upon a line u. It consists of a sequence of arches, each arch extending from that position on the locus at which the point P rests on the line u, through a curve that rises to a position whose distance from u is equal to the diameter of the generating circle and then falls to a subsequent position at which P rests on the line u. Let v be the straight line parallel to u that is tangent to the cycloid at the point furthest from the line u.

(a) Consider a position of the generating circle, and let P be on this circle and on the cycloid. Let PQ be the chord on this circle that is parallel to u (and to v). Show that the locus of Q is a similar cycloid formed by a circle of the same radius rolling (upside down) along the line v.

(b) The region between the two cycloids consists of a number of "beads". Argue that the area of one of these beads is equal to the area of the generating circle.

(c) Use the considerations of (a) and (b) to find the area between u and one arch of the cycloid using a method that does not make use of calculus.

222. Evaluate

$$\sum_{n=1}^{\infty} \tan^{-1}\left(\frac{2}{n^2}\right) .$$

223. Let a, b, c be positive real numbers for which $a + b + c = abc$. Prove that

$$\frac{1}{\sqrt{1+a^2}} + \frac{1}{\sqrt{1+b^2}} + \frac{1}{\sqrt{1+c^2}} \leq \frac{3}{2} .$$

224. For $x > 0$, $y > 0$, let $g(x, y)$ denote the minimum of the three quantities, x, $y + 1/x$ and $1/y$. Determine the maximum value of $g(x, y)$ and where this maximum is assumed.

225. A set of n lighbulbs, each with an *on-off* switch, numbered $1, 2, \cdots, n$ are arranged in a line. All are initially off. Switch 1 can be operated at any time to turn its bulb on of off. Switch 2 can turn bulb 2

on or off if and only if bulb 1 is off; otherwise, it does not function. For $k \geq 3$, switch k can turn bulb k on or off if and only if bulb $k-1$ is off and bulbs $1, 2, \cdots, k-2$ are all on; otherwise it does not function.

(a) Prove that there is an algorithm that will turn all of the bulbs on.

(b) If x_n is the length of the shortest algorithm that will turn on all n bulbs when they are initially off, determine the largest prime divisor of $3x_n + 1$ when n is odd.

226. Suppose that the polynomial $f(x)$ of degree $n \geq 1$ has all real roots and that $\lambda > 0$. Prove that the set $\{x \in \mathbf{R} : |f(x)| \leq \lambda |f'(x)|\}$ is a finite union of closed intervals whose total length is equal to $2n\lambda$.

227. Let n be an integer exceeding 2 and let $a_0, a_1, a_2, \cdots, a_n, a_{n+1}$ be positive real numbers for which $a_0 = a_n$, $a_1 = a_{n+1}$ and

$$a_{i-1} + a_{i+1} = k_i a_i$$

for some positive integers k_i, where $1 \leq i \leq n$.

Prove that

$$2n \leq k_1 + k_2 + \cdots + k_n \leq 3n \ .$$

228. Prove that, if $1 < a < b < c$, then

$$\log_a(\log_a b) + \log_b(\log_b c) + \log_c(\log_c a) > 0 \ .$$

229. Suppose that n is a positive integer and that $0 < i < j < n$. Prove that the greatest common divisor of $\binom{n}{i}$ and $\binom{n}{j}$ exceeds 1.

230. Let f be a strictly increasing function on the closed interval $[0, 1]$ for which $f(0) = 0$ and $f(1) = 1$. Let g be its inverse. Prove that

$$\sum_{k=1}^{9} \left(f\left(\frac{k}{10}\right) + g\left(\frac{k}{10}\right) \right) \leq 9.9 \ .$$

231. For $n \geq 10$, let $g(n)$ be defined as follows: n is mapped by g to the sum of the number formed by taking all but the last three digits of its square and adding it to the number formed by the last three digits of its square. For example, $g(54) = 918$ since $54^2 = 2916$ and $2 + 916 = 918$. Is it possible to start with 527 and, through repeated applications of g, arrive at 605?

232. (a) Prove that, for positive integers n and positive values of x,

$$(1 + x^{n+1})^n \leq (1 + x^n)^{n+1} \leq 2(1 + x^{n+1})^n \ .$$

(b) Let $h(x)$ be the function defined by

$$h(x) = \begin{cases} 1, & \text{if } 0 \leq x \leq 1; \\ x, & \text{if } x > 1. \end{cases}$$

Determine a value N for which

$$|h(x) - (1 + x^n)^{\frac{1}{n}}| < 10^{-6}$$

whenever $0 \leq x \leq 10$ and $n \geq N$.

233. Let $p(x)$ be a polynomial of degree 4 with rational coefficients for which the equation $p(x) = 0$ has *exactly one* real solution. Prove that this solution is rational.

234. A square of side length 100 is divided into 10000 smaller unit squares. Two squares sharing a common side are called *neighbours*.

(a) Is it possible to colour an even number of squares so that each coloured square has an even number of coloured neighbours?

(b) Is it possible to colour an odd number of squares so that each coloured square has an odd number of coloured neighbours?

235. Find all positive integers, N, for which:

(i) N has exactly sixteen positive divisors: $1 = d_1 < d_2 < \cdots < d_{16} = N$;

(ii) the divisor with the *index* d_5 (namely, d_{d_5}) is equal to $(d_2 + d_4) \times d_6$ (the product of the two).

236. For any positive real numbers a, b, c, prove that

$$\frac{1}{b(a+b)} + \frac{1}{c(b+c)} + \frac{1}{a(c+a)} \geq \frac{27}{2(a+b+c)^2} \ .$$

237. The sequence $\{a_n : n = 1, 2, \cdots\}$ is defined by the recursion

$$a_1 = 20 \qquad\qquad a_2 = 30$$

$$a_{n+2} = 3a_{n+1} - a_n \qquad \text{for} \quad n \geq 1 \ .$$

Find all natural numbers n for which $1 + 5a_n a_{n+1}$ is a perfect square.

238. Let ABC be an acute-angled triangle, and let M be a point on the side AC and N a point on the side BC. The circumcircles of triangles CAN and BCM intersect at the two points C and D. Prove that the line CD passes through the circumcentre of triangle ABC if and only if the right bisector of AB passes through the midpoint of MN.

239. Find all natural numbers n for which the diophantine equation

$$(x + y + z)^2 = nxyz$$

has positive integer solutions x, y, z.

240. In a competition, 8 judges rate each contestant "yes" or "no". After the competition, it turned out, that for any two contestants, two judges marked the first one by "yes" and the second one also by "yes"; two judges have marked the first one by "yes" and the second one by "no"; two judges have marked the first one by "no" and the second one by "yes"; and, finally, two judges have marked the first one by "no" and the second one by "no". What is the greatest number of contestants?

241. Determine $\sec 40° + \sec 80° + \sec 160°$.

242. Let ABC be a triangle with sides of length a, b, c oppposite respective angles A, B, C. What is the radius of the circle that passes through the points A, B and the incentre of triangle ABC when angle C is equal to (a) $90°$; (b) $120°$; (c) $60°$. (With thanks to Jean Turgeon, Université de Montréal.)

243. The inscribed circle, with centre I, of the triangle ABC touches the sides BC, CA and AB at the respective points D, E and F. The line through A parallel to BC meets DE and DF produced at the respective points M and N. The modpoints of DM and DN are P and Q respectively. Prove that A, E, F, I, P, Q lie on a common circle.

244. Let $x_0 = 4$, $x_1 = x_2 = 0$, $x_3 = 3$, and, for $n \geq 4$, $x_{n+4} = x_{n+1} + x_n$. Prove that, for each prime p, x_p is a multiple of p.

245. Determine all pairs (m, n) of positive integers with $m \leq n$ for which an $m \times n$ rectangle can be tiled with congruent pieces formed by removing a 1×1 square from a 2×2 square.

246. Let $p(n)$ be the number of partitions of the positive integer n, and let $q(n)$ denote the number of finite sets $\{u_1, u_2, u_3, \cdots, u_k\}$ of positive integers that satisfy $u_1 > u_2 > u_3 > \cdots > u_k$ such that $n = u_1 + u_3 + u_5 + \cdots$ (the sum of the ones with odd indices). Prove that $p(n) = q(n)$ for each positive integer n.

For example, $q(6)$ counts the sets $\{6\}$, $\{6,5\}$, $\{6,4\}$, $\{6,3\}$, $\{6,2\}$, $\{6,1\}$, $\{5,4,1\}$, $\{5,3,1\}$, $\{5,2,1\}$, $\{4,3,2\}$, $\{4,3,2,1\}$.

247. Let $ABCD$ be a convex quadrilateral with no pairs of parallel sides. Associate to side AB a point T as follows. Draw lines through A and B parallel to the opposite side CD. Let these lines meet CB produced at B' and DA produced at A', and let T be the intersection of AB and $B'A'$. Let U, V, W be points similarly constructed with respect to sides BC, CD, DA, respectively. Prove that $TUVW$ is a parallelogram.

248. Find all real solutions to the equation

$$\sqrt{x + 3 - 4\sqrt{x-1}} + \sqrt{x + 8 - 6\sqrt{x-1}} = 1 \ .$$

249. The non-isosceles right triangle ABC has $\angle CAB = 90°$. Its inscribed circle with centre T touches the sides AB and AC at U and V respectively. The tangent through A of the circumscribed circle of triangle ABC meets UV in S. Prove that:

(a) $ST \| BC$;

(b) $|d_1 - d_2| = r$, where r is the radius of the inscribed circle, and d_1 and d_2 are the respective distances from S to AC and AB.

250. In a convex polygon $\mathfrak{P}$, some diagonals have been drawn so that no two have an intersection in the interior of $\mathfrak{P}$. Show that there exists at least two vertices of $\mathfrak{P}$, neither of which is an enpoint of any of these diagonals.

251. Prove that there are infinitely many positive integers n for which the numbers $\{1, 2, 3, \cdots, 3n\}$ can be arranged in a rectangular array with three rows and n columns for which (a) each row has the same sum, a multiple of 6, and (b) each column has the same sum, a multiple of 6.

252. Suppose that a and b are the roots of the quadratic $x^2 + px + 1$ and that c and d are the roots of the quadratic $x^2 + qx + 1$. Determine $(a - c)(b - c)(a + d)(b + d)$ as a function of p and q.

253. Let n be a positive integer and let $\theta = \pi/(2n + 1)$. Prove that $\cot^2\theta$, $\cot^2 2\theta$, $\cdots$, $\cot^2 n\theta$ are the solutions of the equation

$$\binom{2n+1}{1}x^n - \binom{2n+1}{3}x^{n-1} + \binom{2n+1}{5}x^{n-2} - \cdots = 0 \ .$$

254. Determine the set of all triples (x, y, z) of integers with $1 \le x, y, z \le 1000$ for which $x^2 + y^2 + z^2$ is a multiple of xyz.

255. Prove that there is no positive integer that, when written to base 10, is equal to its kth multiple when its initial digit (on the left) is transferred to the right (units end), where $2 \le k \le 9$ and $k \ne 3$.

256. Find the condition that must be satisfied by y_1, y_2, y_3, y_4 in order that the following set of six simultaneous equations in x_1, x_2, x_3, x_4 is solvable. Where possible, find the solution.

$$x_1 + x_2 = y_1y_2 \qquad x_1 + x_3 = y_1y_3 \qquad x_1 + x_4 = y_1y_4$$

$$x_2 + x_3 = y_2y_3 \qquad x_2 + x_4 = y_2y_4 \qquad x_3 + x_4 = y_3y_4 \ .$$

257. Let n be a positive integer exceeding 1. Discuss the solution of the system of equations:

$$ax_1 + x_2 + \cdots + x_n = 1$$
$$x_1 + ax_2 + \cdots + x_n = a$$
$$\cdots$$
$$x_1 + x_2 + \cdots + ax_i + \cdots + x_n = a^{i-1}$$
$$\cdots$$
$$x_1 + x_2 + \cdots + x_i + \cdots + ax_n = a^{n-1} \ .$$

258. The infinite sequence $\{a_n; n = 0, 1, 2, \cdots\}$ satisfies the recursion

$$a_{n+1} = a_n^2 + (a_n - 1)^2$$

for $n \geq 0$. Find all rational numbers a_0 such that there are four distinct indices p, q, r, s for which $a_p - a_q = a_r - a_s$.

259. Let ABC be a given triangle and let $A'BC$, $AB'C$, ABC' be equilateral triangles erected outwards on the sides of triangle ABC. Let Ω be the circumcircle of $A'B'C'$ and let A'', B'', C'' be the respective intersections of Ω with the lines AA', BB', CC'.

Prove that AA', BB', CC' are concurrent and that

$$AA'' + BB'' + CC'' = AA' = BB' = CC' \ .$$

260. $TABC$ is a tetrahedron with volume 1, G is the centroid of triangle ABC and O is the midpoint of TG. Reflect $TABC$ in O to get $T'A'B'C'$. Find the volume of the intersection of $TABC$ and $T'A'B'C'$.

261. Let $x, y, z > 0$. Prove that

$$\frac{x}{x + \sqrt{(x+y)(x+z)}} + \frac{y}{y + \sqrt{(x+y)(y+z)}} + \frac{z}{z + \sqrt{(x+z)(y+z)}} \leq 1 \ .$$

as above to get a linear polynomial with root r.

262. Let ABC be an acute triangle. Suppose that P and U are points on the side BC so that P lies between B and U, that Q and V are points on the side CA so that Q lies between C and V, and that R and W are points on the side AB so that R lies between A and W. Suppose also that

$$\angle APU = \angle AUP = \angle BQV = \angle BVQ = \angle CRW = \angle CWR \ .$$

The lines AP, BQ and CR bound a triangle T_1 and the lines AU, BV and CW bound a triangle T_2. Prove that all six vertices of the triangles T_1 and T_2 lie on a common circle.

263. The ten digits $0, 1, 2, 3, 4, 5, 6, 7, 8, 9$ are each used exactly once altogether to form three positive integers for which the largest is the sum of the other two. What are the largest and the smallest possible values of the sum?

264. For the real parameter a, solve for real x the equation

$$x = \sqrt{a + \sqrt{a + x}} \ .$$

A complete answer will discuss the circumstances under which a solution is feasible.

265. Note that $959^2 = 919681$, $919 + 681 = 40^2$; $960^2 = 921600$, $921 + 600 = 39^2$; and $961^2 = 923521$, $923 + 521 = 38^2$. Establish a general result of which these are special instances.

266. Prove that, for any positive integer n, $\binom{2n}{n}$ divides the least common multiple of the numbers $1, 2, 3, \cdots, 2n-1, 2n$.

267. A non-orthogonal reflection in an axis a takes each point on a to itself, and each point P not on a to a point P' on the other side of a in such a way that a intersects PP' at its midpoint and PP' always makes a fixed angle θ with a. Does this transformation preserves lines? preserve angles? Discuss the image of a circle under such a transformation.

268. Determine all continuous real functions f of a real variable for which

$$f(x + 2f(y)) = f(x) + y + f(y)$$

for all real x and y.

269. Prove that the number

$$N = 2 \times 4 \times 6 \times \cdots \times 2000 \times 2002 + 1 \times 3 \times 5 \times \cdots \times 1999 \times 2001$$

is divisible by 2003.

270. A straight line cuts an acute triangle into two parts (not necessarily triangles). In the same way, two other lines cut each of these two parts into two parts. These steps repeat until all the parts are triangles. Is it possible for all the resulting triangle to be obtuse? (Provide reasoning to support your answer.)

271. Let x, y, z be natural numbers, such that the number

$$\frac{x - y\sqrt{2003}}{y - z\sqrt{2003}}$$

is rational. Prove that

(a) $xz = y^2$;

(b) when $y \neq 1$, the numbers $x^2 + y^2 + z^2$ and $x^2 + 4z^2$ are composite.

272. Let $ABCD$ be a parallelogram whose area is 2003 sq. cm. Several points are chosen on the sides of the parallelogram.

(a) If there are 1000 points in addition to A, B, C, D, prove that there always exist three points among these 1004 points that are vertices of a triangle whose area is less that 2 sq. cm.

(b) If there are 2000 points in addition to A, B, C, D, is it true that there always exist three points among these 2004 points that are vertices of a triangle whose area is less than 1 sq. cm?

273. Solve the logarithmic inequality

$$\log_4(9^x - 3^x - 1) \geq \log_2 \sqrt{5} \ .$$

274. The inscribed circle of an isosceles triangle ABC is tangent to the side AB at the point T and bisects the segment CT. If $CT = 6\sqrt{2}$, find the sides of the triangle.

275. Find all solutions of the trigonometric equation

$$\sin x - \sin 3x + \sin 5x = \cos x - \cos 3x + \cos 5x \ .$$

276. Let a, b, c be the lengths of the sides of a triangle and let $s = \frac{1}{2}(a + b + c)$ be its semi-perimeter and r be the radius of the inscribed circle. Prove that

$$(s - a)^{-2} + (s - b)^{-2} + (s - c)^{-2} \geq r^{-2}$$

and indicate when equality holds.

277. Let m and n be positive integers for which $m < n$. Suppose that an arbitrary set of n integers is given and the following operation is performed: select any m of them and add 1 to each. For which pairs (m, n) is it always possible to modify the given set by performing the operation finitely often to obtain a set for which all the integers are equal?

278. (a) Show that $4mn - m - n$ can be an integer square for infinitely many pairs (m, n) of integers. Is it possible for either m or n to be positive?

(b) Show that there are infinitely many pairs (m, n) of positive integers for which $4mn - m - n$ is one less than a perfect square.

279. (a) For which values of n is it possible to construct a sequence of abutting segments in the plane to form a polygon whose side lengths are $1, 2, \cdots, n$ exactly in this order, where two neighbouring segments are perpendicular?

(b) For which values of n is it possible to construct a sequence of abutting segments in space to form a polygon whose side lengths are $1, 2, \cdots, n$ exactly in this order, where any two of three sucessive segments are perpendicular?

280. Consider all finite sequences of positive integers whose sum is n. Determine $T(n, k)$, the number of times that the positive integer k occurs in all of these sequences taken together.

281. Let a be the result of tossing a black die (a number cube whose sides are numbers from 1 to 6 inclusive), and b the result of tossing a white die. What is the probability that there exist real numbers x, y, z for which $x + y + z = a$ and $xy + yz + zx = b$?

282. Suppose that at the vertices of a pentagon five integers are specified in such a way that the sum of the integers is positive. If not all the integers are non-negative, we can perform the following operation: suppose that x, y, z are three consecutive integers for which $y < 0$; we replace them respectively by the integers $x + y, -y, z + y$. In the event that there is more than one negative integer, there is a choice of how this operation may be performed. Given any choice of integers, and any sequence of operations, must we arrive at a set of nonnegative integers after a finite number of steps?

For example, if we start with the numbers $(2, -3, 3, -6, 7)$ around the pentagon, we can produce $(1, 3, 0, -6, 7)$ or $(2, -3, -3, 6, 1)$.

283. (a) Determine all quadruples (a, b, c, d) of positive integers for which the greatest common divisor of its elements is 1,

$$\frac{a}{b} = \frac{c}{d}$$

and $a + b + c = d$.

(b) Of those quadruples found in (a), which also satisfy

$$\frac{1}{b} + \frac{1}{c} + \frac{1}{d} = \frac{1}{a} \ ?$$

(c) For quadruples (a, b, c, d) of positive integers, do the conditions $a+b+c = d$ and $(1/b)+(1/c)+(1/d) = (1/a)$ together imply that $a/b = c/d$?

284. Suppose that $ABCDEF$ is a convex hexagon for which $\angle A + \angle C + \angle E = 360°$ and

$$\frac{AB}{BC} \cdot \frac{CD}{DE} \cdot \frac{EF}{FA} = 1 \ .$$

Prove that

$$\frac{AB}{BF} \cdot \frac{FD}{DE} \cdot \frac{EC}{CA} = 1 \ .$$

285. (a) Solve the following system of equations:

$$(1 + 4^{2x-y})(5^{1-2x+y}) = 1 + 2^{2x-y+1} \ ;$$

$$y^2 + 4x = \log_2(y^2 + 2x + 1) \ .$$

(b) Solve for real values of x:

$$3^x \cdot 8^{x/(x+2)} = 6 \ .$$

Express your answers in a simple form.

286. Construct inside a triangle ABC a point P such that, if X, Y, Z are the respective feet of the perpendiculars from P to BC, CA, AB, then P is the centroid (intersection of the medians) of triangle XYZ.

287. Let M and N be the respective midpoints of the sides BC and AC of the triangle ABC. Prove that the centroid of the triangle ABC lies on the circumscribed circle of the triangle CMN if and only if

$$4 \cdot |AM| \cdot |BN| = 3 \cdot |AC| \cdot |BC| \ .$$

288. Suppose that $a_1 < a_2 < \cdots < a_n$. Prove that

$$a_1a_2^4 + a_2a_3^4 + \cdots + a_na_1^4 \geq a_2a_1^4 + a_3a_2^4 + \cdots + a_1a_n^4 \ .$$

289. Let $n(r)$ be the number of points with integer coordinates on the circumference of a circle of radius $r > 1$ in the cartesian plane. Prove that

$$n(r) < 6\sqrt[3]{\pi r^2} \ .$$

290. The School of Architecture in the *Olymon* University proposed two projects for the new Housing Campus of the University. In each project, the campus is designed to have several identical dormitory buildings, with the same number of one-bedroom apartments in each building. In the first project, there are 12096 apartments in total. There are eight more buildings in the second project than in the first, and each building has more apartments, which raises the total of apartments in the project to 23625. How many buildings does the second project require?

291. The n-sided polygon $A_1, A_2, \cdots, A_n$ $(n \geq 4)$ has the following property: The diagonals from each of its vertices divide the respective angle of the polygon into $n-2$ equal angles. Find all natural numbers n for which this implies that the polygon $A_1A_2\cdots A_n$ is regular.

292. 1200 different points are randomly chosen on the circumference of a circle with centre O. Prove that it is possible to find two points on the circumference, M and N, so that:

- M and N are different from the chosen 1200 points;
- $\angle MON = 30°$;
- there are *exactly* 100 of the 1200 points inside the angle MON.

293. Two players, Amanda and Brenda, play the following game: Given a number n, Amanda writes n different natural numbers. Then, Brenda is allowed to erase several (including none, but not all) of them, and to write either $+$ or $-$ in front of each of the remaining numbers, making them positive or negative, respectively, Then they calculate their sum. Brenda wins the game is the sum is a multiple of 2004. Otherwise the winner is Amanda. Determine which one of them has a winning strategy, for the different choices of n. Indicate your reasoning and describe the strategy.

294. The number $N = 10101\cdots0101$ is written using $n+1$ ones and n zeros. What is the least possible value of n for which the number N is a multiple of 9999?

295. In a triangle ABC, the angle bisectors AM and CK (with M and K on BC and AB respectively) intersect at the point O. It is known that

$$|AO| \div |OM| = \frac{\sqrt{6}+\sqrt{3}+1}{2}$$

and

$$|CO| \div |OK| = \frac{\sqrt{2}}{\sqrt{3}-1} \ .$$

Find the measures of the angles in triangle ABC.

296. Solve the equation

$$5\sin x + \frac{5}{2\sin x} - 5 = 2\sin^2 x + \frac{1}{2\sin^2 x} \ .$$

297. The point P lies on the side BC of triangle ABC so that $PC = 2BP$, $\angle ABC = 45°$ and $\angle APC = 60°$. Determine $\angle ACB$.

298. Let O be a point in the interior of a quadrilateral of area S, and suppose that

$$2S = |OA|^2 + |OB|^2 + |OC|^2 + |OD|^2 \ .$$

Prove that $ABCD$ is a square with centre O.

299. Let $\sigma(r)$ denote the sum of all the divisors of r, including r and 1. Prove that there are infinitely many natural numbers n for which

$$\frac{\sigma(n)}{n} > \frac{\sigma(k)}{k}$$

whenever $1 \leq k \leq n$.

300. Suppose that ABC is a right triangle with $\angle B < \angle C < \angle A = 90°$, and let $\mathfrak{K}$ be its circumcircle. Suppose that the tangent to $\mathfrak{K}$ at A meets BC produced at D and that E is the reflection of A in the axis BC. Let X be the foot of the perpendicular for A to BE and Y the midpoint of AX. Suppose that BY meets $\mathfrak{K}$ again in Z. Prove that BD is tangent to the circumcircle of triangle ADZ.