# BUILDING BONDS; TEACHERS, PUPILS, PARENTS

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Below is a fairly long list of problems, many of which have been tested by me with children of various ages, including those in elementary school. I will have time to discuss very few of them in the time available.

However, I would be very grateful if you would give them a try with your pupils, either in class or sending them home to be looked at by the pupils and perhaps their parents as well. You should, of course, try them first, and, if you run into difficulties, I would be very happy to discuss them with you my email; my address is above after my name. If you do use them with children or adolescents, please send me a message and tell me what happened. I would be interested in knowing which problem you used, how you used it (homework, class discussion, bulletin board, group work, *etc.*), how the students fared with the problem, what sort of solutions they presented, conceptual problems they may have encountered. If you sent it home and there was a parental response, it would be interesting to have that.

I do not pretend that all of these problems are easy, and some of them are exploratory. But all can be used for school children, although you may have to step them down a bit for some classes or deal with specific cases rather than more general cases. Many of them involve mathematical approaches that are important, although pupils will not see them in their normal work. However, they do support the syllabus, either by giving an occasion for practising skills or by developing reasoning and communication skills.

You may have to "live" with some of the problems for a while. It helps if you can involve some of your colleagues. Not only does this help with the solutions, but it would help generate ideas on the best way of presenting them and to discuss what actually goes on with the children.

## §1. Arithmetic

A1. Using the digits 1, 2, 3, 4, 5, 6, 7, 8 each once, construct two numbers and take their difference. Make the difference as small as possible.

A2. Using the ten digits 1, 2, 3, 4, 5, 6, 7, 8, 9, 0 each once, construct three positive integers such that the largest of the three is the sum of the other two. What is the smallest possible sum? the largest possible sum?

A3. Select an odd and an even digit. Using only these two digits, construct a ten-digit number that is (evenly) divisible by 1024.

A4. The sexton was posting the hymn numbers for the Sunday service. He noted that there were three hymns, and that he required one each of the digits 1, 2, 3, 4, 5, 6, 7, 8, 9. He further observed that the hymn numbers were in the ratio 1 : 3 : 5. What were the hymns that Sunday?

A5. Are there any numbers except 1 that are the sum of the squares of their digits? the sum of the cubes of their digits? the sum of the higher powers of their digits? What happens if you work to a base other than 10?

A6. An ordinary die is a cube whose faces are numbered from one to six. An octahedral die is the shape of a regular octahedron (formed by gluing two square-based pyramids together). It has eight triangular faces, with four of them meeting at each of its six vertices. Show how the faces can be numbered from one to eight in such a way that the sum of the numbers of the four faces meeting at each vertex is always the same. It might help to begin by deciding what this sum should be.

A7. The nine slots indicated by X in this triangular array are to be filled in by the digits 1, 2, 3, 4, 5, 6, 7, 8, 9 in such a way that the sums of the numbers on each side are all the same, and the sums of the squares of the numbers on each side are all the same.

$$\begin{array}{ccccccc} & & & X & & & \\ & & X & & X & & \\ & X & & & & X & \\ X & & X & & X & & X \end{array}$$

A8. Consider the following array of numbers:

$$\begin{array}{ccccccccc} 11 & & 15 & & 1 & & 13 & & 8 \\ & 4 & & 14 & & 12 & & 5 & \\ & & 10 & & 2 & & 7 & & \\ & & & 8 & & 5 & & & \\ & & & & 3 & & & & \end{array}$$

In each row except the top, the number is the (positive) difference of the two numbers immediately above it in teh previous row. The numbers have been chosed from 1 to 15 inclusive. However, in this example, the numbers 8 and 5 appear twice and 6 and 9 do not appear at all. Find a triangular array with the same subtraction property for which each of the numbers from 1 to 15 inclusive appears exactly once.

You might want to warm up with triangular arrays with 2 numbers are the top (using the numbers 1, 2, 3), three numbers at the top (using the numbers from 1 to 6) and four numbers at the top (using the numbers from 1 to 10).

A9. An amusing pastime is to start with any six different integers between 1 and 25 inclusive, and use any five of them along with the arithmetic operations of addition, subtraction, multiplication, division and raising to a power (the exponent being one of the numbers), with suitable bracketing, to obtain the sixth. For example, consider the set $\{3, 5, 6, 9, 18, 21\}$. For example, we can write

$$18 = (9 \times (21 + 5)) - 6^3 \ .$$

Can you obtain each of the others from the rest?

Do the problem for the set $\{2, 3, 5, 6, 7, 24\}$.

A10. Write down a square array of integers.(You can start with a $3 \times 3$ array, and work up to arrays with more elements.) Let $S$ be a number that exceeds the sum of the numbers in each row and in each column. Replace each entry by an integer that is at least as great to obtain an array in which each row and each column add up to $S$. For example, consider the square array:

$$\begin{pmatrix} 5 & -2 & 3 & 0 \\ -7 & 4 & 1 & -1 \\ 8 & 2 & 1 & 3 \\ 4 & 0 & 0 & 1 \end{pmatrix}$$

The row sums are $6, -3, 14, 5$ and the column sums are $10, 4, 5, 3$. The number 15 exceeds all of these. So, by replacing, the entries, try to make all the rows and column sum to 15. Can you in fact make them all sum to 14?

A11. A calculator is defective. All the usual buttons work except for the multiplication button. However there is a reciprocal button that outputs $1/x$ when you input $x$. Show, however, that you can input a pair $a$, $b$ of numbers and output their product $a \times b$. (Do not use continued addition; this is too slow.)

A12. Suppose that we take four of the first eight positive integers, say $1, 3, 4, 5$, and write them in increasing order. Write the remaining four integers in decreasing order, $8, 7, 6, 2$. Now pair them off and write the sum of the positive differences of the corresponding pairs:

$$(8-1)+(7-3)+(6-4)+(5-2)=16 \ .$$

Now split the set of eight into two other sets of four and perform the same operation. What do you observe? Try the same thing with other sets of evenly many integers, splitting it into two sets with the same number of elements. Can you account for what happens?

A13. Find two pairs of positive whole numbers (using each number only once) for which the sum of each pair is equal to the product of the other pair. How many examples are there?

Try the same problem where the product of each pair is twice the sum of the other. Or have the product of each pair the same multiple of the sum of the other.

A14. We make up a square grid of integers as follows. Write a number at the head of each column and at the left of each row. The number to be entered in the $r$th row and $s$th column of the grid is the sum of the numbers at the left of the $r$th row and the head of the $s$th column. Thus, for example, if the number at the left of the third row is 4 and at the head of the second column is 3, then we will enter 7 in the position in the third row and second column.

Now select a subset of these integers in such a way that there is exactly one of them selected from each row and exactly one from each column. Sum the numbers that you have selected. Do this in several different ways. Account for what happens.

A15. What is the longest number that you can make that has all digits distinct and the property that the number formed by the leftmost $k$ digits is divisible by $k$, for $k = 1, 2, \cdots$. Can you get a ten digits number with this property? What happens if your work with another base? (For example, in the case of the number 12360, observe that 1 is divisible by 1, 12 is divisible by 2, 123 is divisible by 3, 1236 is divisible by 4 and 12360 is divisible by 5.)

A16. Is it possible for a positive integer with at least two digits to be equal to the sum of its digits? to be equal to the product of its digits? Justify your answer.

## §2. Ratio and rates

R1. At noon, Iphigenia set off on a bike from her home in the Ottawa Valley, maintaining a leisurely pace of 20 km per hour on the pleasantly level terrain. Later, her mother noticed that she forgot her dinner, and sent Electra off on her bike to meet her; Electra maintained a steady pace of 30 km per hour. But then the sky darkened and the storm clouds gathered. So, exactly a half hour after Electra left, Orestes was sent off to meet the others with rain gear. Orestes rode at a steady pace of 40 km per hour. All three followed the same route. As it happened, the three siblings met at exactly the same time. What time was that?

R2. Olga and Tamara are two peasant ladies living repectively in the towns of Aigrad and Bigrad. One morning at sunrise, each set out on foot for the town of the other, each travelling at a constant speed (different for each lady) along the same route. The passed at noon; Olga arrived at Bigrad at 4 in the afternoon, while Tamara did not get to Aigrad until 9 in the evening. What time was sunrise?

R3. James is on a railway bridge joining points **A** and **B**. He is $3/8$ of the way across from **A**. He hears a train approaching **A**; it is travelling 60 km per hour. If he runs towards **A**, he will arrive there exactly when the train does. If he runs towards **B**, the train will overtake him at **B**. How fast can James run?

R4. Your age should properly be referred to as your age last birthday, the number of complete years that you have lived. You have a certain integer age for a year, and this integer increases by 1 on your birthday.

Consider any two people, perhaps you and some friend or relative, and assume that you both live indefinitely. For how long a period is the age of the older exactly twice the age of the younger? What happens for twins?

R5. A farmer has two fields that need ploughing, one exactly twice the area of another. One day, a band of ploughmen showed up and they all worked on the larger field until the middle of the day. Then they split into two equal groups. One group remained in the larger field and finished the job at the end of the day. The second group moved to the smaller field, but were unable to finish by the end of the day. So the following day, one ploughman spent the whole day finishing the smaller field. All the ploughmen worked at the same rate for the whole time, the two days were of equal length and the first day was split into periods of equal length.

R6. A clock loses four minutes every hour. It was set to the correct time at 8:30 this morning. What will the actual time be when the clock shows that it is noon today?

R7. A man and his grandson have their birthdays on the same day. Several years ago, the man was twelve times the age of his grandson; now he is six times his age. How old is the grandson now?

R8. Castor and Pollux always walk to school each morning folowing the same route, and each walking at his own constant speed. However, they do not go together. It takes Castor 45 minutes to walk the distance, while Pollux requires only 30 minutes. One morning, Caster set out a 8:00 am, while Pollys left 10 minutes later. Did Pollus overtake Castor? If so, when did this occur?

R9. Goneril, Regan and Cordelia are going to visit their grandmother, who lives 60 km away. To get there, they have a tandem bicycle that accommodates only two of them and is capable of a maximum speed of 45 km per hour. Any one of the three can walk at a maximum speed of 5 km per hour. They leave at 8:50 am. Can they make it to grandmother's house by noon?

R10. Penelope and Ulysses like to have a bit of exercise each morning. They set off at exactly the same time each morning, and run or walk the same circuit. Both run at exactly the same speed, and both walk at exactly the same speed. However, they do not remain together. Penelope runs for half the *time* that she is out and walks for the other half of the time. Ulysses runs for half the *distance* and walks for the other half of the distance. The first one returning home has to make the coffee. Who makes the coffee?

R11. The Watsyorsis Mine has just filled an open tank with 4000 kg of slurry which is 99% water and 1% tailings by weight. Since the tank was too heavy to transport, the proprietors left the tank until some of the water could evaporate away. After a while, the slurry was 98% water. What did it weigh then?

R12. Zeus and Poseiden run a 100-meter race. Zeus wins the race, arriving at the finish line when Poseiden has run only 90 meters. Out of consideration for his brother's self-esteem, in a second race, Zeus starts five meters behind the starting line (so he has to run 105 meters to reach the finish line) while Poseiden runs the the 100 meters as before. If each runs as fast as in the first race, who wins the second race?

R13. A criminal, having escaped from prison, travelled for 10 hours before his escape was detected. He was then pursued and gained upon at 3 miles per hour. When his pursuers had been 8 hours on the way, they met an express (train) going in the opposite direction at the same rate as themselves, which had met the criminal 2 hours and 24 minutes earlier. In what time from the beginning of the pursuit will the criminal be overtaken? [from *The high school algebra* by Robertson and Birchard, approved for Ontario schools in 1886]

## §3. Geometry

G1. Show how to cut a square into three pieces so that all the pieces are the same shape (similar) but not all the same size (congruent). Do this where two of the three have the same size, and where all three pieces have different sizes.

G2. You have a $9 \times 16$ rectangle. Show how, with straight cuts, you can partition it into three polygons that can be rearranged without overlapping or gaps to form a square.

G3. Three of the corners of a square are clipped off by cuts joining the midpoints of adjacent sides to form a pentagon. The area of this pentagon is $5/8$ of the area of the original square.

Show how to make straight cuts to partition the pentagon into pieces that can be reconsituted into a square.

G4. A trapezoid $ABCD$ with $AB$ and $DC$ parallel, is partitioned into four triangles by its two diagonals. The two triangles that share a parallel side with the trapezoid of areas 9 and 16. What is the area of the trapezoid? (Replace 9 and 16 by $a$ and $b$ for a general problem.)

G5. Consider a $3 \times 5$ rectangular grid of squares. A straight line is drawn. What is the maximum number of squares in the grid whose interiors can be penetrated by the line?

Solve the same problem for other dimensions of the grid. What happens when there are $m$ rows and $n$ columns?

G6. You have a geoboard with nine pegs arranged in a $3 \times 3$ square array. Starting at one peg, a string goes from one peg to the next in some order, until the ninth peg is reached. The string should meet each peg exactly once and not cross itself; it has eight segments. The distance between horizontally or vertically adjacent pegs is 1, between diagonally adjacent pegs is $\sqrt{2}$), or about 1.41 and between a corner peg and one in the middle of an adjacent side is $\sqrt{5}$ or about 2.24.

What should be the configuration to make the path traced by the string as long as possible?

G7. Take a sheet of paper in the form of a quadrilateral (a polygon with four sides; you can have any shape that you want). Join the midpoints of adjacent sides, to get a smaller quadrilateral. This central quadrilateral turns out to be a parallelogram (do you see why?). Thus, you have partitioned the quadrilateral into a parallelogram and four triangular "ears". Show how you can cut off the ears and arrange them to exactly cover the parallelogram with no overlapping (thus showing that the area of the parallelogram is exactly half that of the quadrilateral).

## §4. Combinatorics

C1. Show how to place seven distinct points on a page so that, among any three of them, you can find two that are exactly three centimeters apart.

C2. Suppose that you are given any six distinct points on a page, and that you draw all 15 edges that join pairs of them. We will say that a *triangle* is a set of three edges that connect the three pairs of a triple of points. Each edge is coloured either red or blue.

Explain why, no matter how the colouring is made, you will always find a triangle with all edges red or a triangle with all edges blue.

This property leads to an interesting game called *Sim*. Two players move alternately, one with a red marker and the second with a blue. Each in turn joins two points, not already joined, with a segment in his own colour. The first one to complete a triangle in his own colour loses and the other wins. There must always be a winner in this game. What is the best way to play. Who has a winning strategy.

There is a rather surprising result about the twenty triangles that can be drawn for three of the six points. There is an edge joining two of the points which forms the longest side of one triangle while forming the shortest side of another. Why is this so? [Hint: Colour the shortest side of each triangle red and the remaining edges blue.]

## §5. Logic

L1. You have six balls that look identical. Two are coloured red, two white and two blue. One red, one white and one blue ball weight 100 grams, while the remaining three balls each weight 80 grams. Using an equal arms balance twice, determine which balls are heavier and which lighter.

L2. You have twelve balls that look identical. Eleven of them weight the same, but the twelfth has a different weight, although you do not know whether it is heavier or lighter than the rest. Using an equal arm balance the minimum number of times, determine the odd ball and whether it is heavier or lighter.

L3. Three husband-wife couples come to a river. There is only one boat available on the near bank and it can hold at most two people. If each husband insists that his wife cannot be in the presence of another man unless he is present, determine how the couples can get to the other side of the river.

L4. A traveller to a strange island discovers that it is inhabited by knights who can make only true statements and knaves who can make only false statements. One day, the traveller encounters three inhabitants, whom we will call **A**, **B** and **C**. He asked, "How many knights are among you three?" **A** gave an answer that the traveller did not catch, but which was understood by the other two. So he asked **B** what **A** said. **B** responded, "there is one knight among us." "Don't believe **B**," shot in **C**, "he is lying." What are **B** and **C**?

L5. Stephen's clock has stopped, and he has no way of getting the correct time at his home. So he winds up his clock to make it go, and sets it at random. He then walks to the house of his friend, Stockwell, who has a functioning clock that reads the correct time. Eventually, Stephen leaves and walks home following the way he came. Upon arrival at his house, he sets his clock to the correct time. How does he do this?

L6. On the table are three boxes. One contains two black marbles, another two white marbles and the third a white and a black marble. The boxes are labelled according to their contents: **BB**, **WW**, **BW**. Unfortunately, the cleaning lady comes along and switches the labels so that all the boxes are now incorrectly labelled. You know this, and are allowed to take one marble out of any box and replace it, without looking inside. What is the smallest number of drawings of this type that are necessary to determine the contents of all three boxes?

L7. Three knights are told by their king that, in recognition of their bravery in rescuing his beautiful daughter from a fierce dragon, he will give them a chance to win a great fortune. He tells them that he will seat them in as circle, and place on the head of each a hat that is either red or green. Each of them will be able to see the hats of the other, but not his own hat. Each will be asked to either guess the colour of his own hat or keep silent. If all those who speak guess correctly, then all will receive the fortune. Otherwise, he receives nothing. Of course, the king set some rules. Beforehand, the knights are permitted to discuss the strategy that they will use. However, once they are seated in the circle, there must be no further communication among them. You might think the best they can manage is to have an even chance of winning the fortune (by, for example, having a particular one take a guess). But they can increase the odds in their favour. How?

L8. Enough gas to allow a racing car to make one circuit of a race track is poured into a finite number of cans that are distributed randomly at various points around a track. Show that the race car, starting with an empty gas tank, can be place at a certain position in the track and proceed in one direction for a complete circuit, always reaching a can in time to replenish its supply and never running out of gas.

L9. Each of ten gossips has a piece of information not known by the others. They then make some two-way phone calls among themselves. At each phone call, each party tells the other all the information that she knows up to this point. What is the minimum number of phone calls necessary in order for all the gossips to know all of the information?