CHAPTER THREE

LOCATING ZEROS OF POLYNOMIALS

§1. APPROXIMATION OF ZEROS

Since the determination of the zeros of a polynomial in exact form is impractical for polynomials of degrees 3 and 4 and generally not feasible for polynomials of higher degree, it is necessary to get information about the zeros in other ways. We have seen in Chapter 2, that Newton's Method is a good approximating technique provided we start close to the desired zero. However, in order to apply it, we need to know roughly where in the complex plane it resides. More generally, there are applications of polynomials that depend on knowing the character of the roots, specifically whether they lie within the unit disc (in the analysis of the stability of recursions) or in the left half plane (in the analysis of the stability of solutions to differential equations).

In the case of real polynomials, we can apply the Intermediate Value Theorem that provides that a polynomial $p(x)$ has at least one zero in any interval (a, b) for which $p(a)$ and $p(b)$ are nonzero numbers of opposite sign. There are other tests of increasing sophistication. One of the simplest and most convenient is *Descartes' Rule of Signs* that asserts that for a real polynomial, the number of positive zeros of $p(x)$ cannot exceed the number of changes of signs in its nonzero coefficients when read from left to right, and the number of negative zeros cannot exceed the number of changes of signs in $p(-x)$.

A finer version of this is the *Theorem of Fourier and Budan*. Let u and v be two real numbers with $u < v$ and $p(u)p(v) \neq 0$. Form the sequences $\{p(u), p'(u), p''(u), \cdots\}$ and $\{p(v), p'(v), p''(v), \cdots\}$. If A is the number of sign changes in the former and B the number of sign changes in the latter, then the number of zeros of $p(z)$ in the interval $[u, v]$ cannot exceed $A - B$ and differs from it by an even number. A more complicated test which gives the exact number of zeros in a real interval is given by *Sturm's Theorem* [**1**, pp. 179-182].

One can approach the question by comparing the polynomial $p(z)$ under investigation with a second polynomial $q(z)$ whose zeros are known. For complex polynomials, we recall the classical result of Rouché: *Suppose that the polynomials p and q satisfy the condition*

$$|p(z) - q(z)| < |q(z)|$$

for z belonging to a closed path gamma in the complex plane. Then $p(z)$ and $q(z)$ have the same number of zeros counting multiplicity within the region surrounded by the path.

In Chapter 2, we have seen that the condition of apolarity on a pair of polynomials entails a relationship that intertwines their zeros. However, the expression of that relationship there was not useful, so we return to this idea in the present chapter and obtain Grace's Theorem, a tool to give us information about zeros of a polynomial.

§2. THE ZEROS OF A POLYNOMIAL AND ITS DERIVATIVE

For a real polynomial, the most elementary theorem that relates the zeros of a polynomial to those of its derivatives (the *critical points* of the polynomial) is *Rolle's Theorem*, that provides that between any two real zeros of a real polynomial is at least one zero of its derivative.

If all the zeros of a polynomial are real, then we have a "zero-shifting" property that applies to the zeros of itself and the derivative: *Suppose that the polynomial $p(x)$ of degree n has all real distinct roots $x_1 < x_2 < \cdots < x_n$, and the polynomial is altered by replacing one of its roots x_i by $x'_i \in (x_i, x_{i+1})$. Then all the roots of $p'(x)$ increase their value.* [**2**]

In studying the zeros of the derivative, we begin with the observation that, when $p(z) = \prod_{1\le i\le r}(z - z_i)^{m_i}$, then $p'(z) = \sum_i m_i(z-z_1)^{m_1}\cdots(z-z_i)^{m_i-1}\cdots(z-z_r)^{m_r}$ and

$$P(z) \equiv \frac{p'(z)}{p(z)} = \sum_{i=1}^{r} \frac{m_i}{z - z_i} \ . \tag{3.1}$$

The zeros of the function $P(z)$ are the zeros of the derivative that are not shared by $p(z)$.

Applying this to the root-shifting result, let $p(x) = [\prod_{j\ne i}(x-x_j)](x-x_i)$ and $q(x) = [\prod_{j\ne i}(x-x_j)](x-x_i')$, where $x_1 < x_2 < \cdots < x_i < x_i' < x_{i+1} < \cdots < x_n$. Then

$$P(x) - Q(x) \equiv \frac{p'(x)}{p(x)} - \frac{q'(x)}{q(x)} = \frac{1}{x-x_i} - \frac{1}{x-x_i'} = \frac{x_i - x_i'}{(x-x_i)(x-x_i')} \ ,$$

so that $Q(x) > P(x)$ whenever the functions are defined for $x < x_i$ and $x > x_i'$, and $Q(x) < P(x)$ for $x_i < x < x_i'$. One can then be persuaded of the result by analyzing the graphs of $P(x)$ and $Q(x)$. ♠

A complex analogue of Rolle's Theorem is the *Gauss-Lucas Theorem*, which provides that *all of the critical points of* $p(z)$ *lie in the closed convex hull of the zeros of* $p(z)$. This is certainly true if a critical point is actually a zero of $p(z)$.

Suppose that z does not belong to the convex hull of the zeros of $p(z)$. Then there is a line that separates z from the zeros z_i of $p(z)$, so that for some α, $\alpha < \arg\,(z - z_i) < \alpha + \pi$ for each i. Therefore, $-(\alpha+\pi) < \arg\, m_i/(z-z_i) < -\alpha$ for each i. Hence, each of the terms in $P(z)$ lies strictly on one side of a line through the origin and so $P(z)$ cannot vanish. ♠

Another generalization of Rolle's theorem applies to the nonreal critical points of a real polynomial. *Jensen's Theorem* can be formulated this way. Suppose that $p(z)$ is a real polynomial that has a complex conjugate pair $(w, \overline{w})$ of zeros. Let D_w be the closed disc whose diameter joins w and $\overline{w}$. Then *every nonreal zero of* $p'(z)$ *lies on one of the* D_w.

The idea of the proof is to show that, for each complex z lying outside of all of the D_w, we can express $P(z)$ as the sum of terms whose imaginary parts are nonzero and have sign opposite to that of the imaginary part of z. One looks at the imaginary part of $(z-w)^{-1} + (z-\overline{w})^{-1}$ when the zero w is nonreal and of $(z-w)^{-1}$ when w is real. [**7**, p. 15].

If the polynomial has three distinct zeros, then the geometry of the zeros and those of the derivative is quite interesting. If all the zeros are simple, so that the polynomial is a cubic, then the zeros of the derivative are the foci of the unique ellipse that touches, at the midpoints of its sides, the triangle formed by the zeros in the complex plane, If the zeros have general multiplicity, then we still get a triangle formed by the zeros, but the ellipse whose foci are zeros of the derivative, touches the triangle at different points in its sides. [**5, 6**]

§3. THE ENESTROM-KAKEYA THEOREM

The Enestrom-Kakeya Theorem holds that, if $p(z) = a_n z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0$ is a polynomial with nonnegative real coefficients for which

$$a_n \ge a_{n-1} \ge a_{n-2} \ge \cdots \ge a_0 \ge 0 \ ,$$

then all of its zeros z satisfy $|z| \le 1$. A slight generalization of this was provided by Joyal, Labelle and Rahman [**3**], in that the condition $a_0 \ge 0$ is dropped and the conclusion becomes that all of the zeros satisfy the inequality

$$|z| \le \frac{a_n - a_0 + |a_0|}{|a_n|} \ .$$

Let $q(z) = (1-z)p(z) = -a_n z^{n+1} + f(z)$, where

$$f(z) = (a_n - a_{n-1})z^n + (a_{n-1} - a_{n-2})z^{n-1} + \cdots + (a_1 - a_0)z + a_0 \ .$$

We need to use the complex variables result that the maximum modulus of an analytic function on a closed disc is attained on the circumference. When $|z| = 1$, we have that

$$|f(z)| \leq (a_n - a_{n-1}) + (a_{n-1} - a_{n-2}) + \cdots + (a_1 - a_0) + |a_0| = a_n - a_0 + |a_0| \ .$$

When $|z| = 1$, also $|1/z| = 1$ and we find that

$$|z^n f(1/z)| \leq a_n - a_0 + |a_0| \ .$$

Since $z^n f(1/z)$ is a polynomial, and so analytic inside and on the unit disc, this inequality holds for all z with $|z| \leq 1$. Thus, when $0 < |z| \leq 1$,

$$\left| f\left(\frac{1}{z}\right) \right| \leq \frac{a_n - a_0 + |a_0|}{|z|^n} \ .$$

Replacing z by $1/z$, we deduce that

$$|f(z)| \leq (a_n - a_0 + |a_0|)|z|^n$$

when $|z| \geq 1$.

Suppose that $|z| > (a_n - a_0 + |a_0|)|a_n|^{-1}$. Then

$$\begin{aligned} |q(z)| &= |(1-z)p(z)| = |-a_n z^{n+1} + f(z)| \\ &\geq |z|^n(|a_n z| - (a_n - a_0 + |a_0|) > 0 \ . \end{aligned}$$

The result follows. ♠

§4. ANALYSIS OF A TRINOMIAL

The trinomial polynomial $p(z) \equiv 1 - z + cz^n$, where $c \neq 0$ and n is an integer exceeding 1, has a zero z for which $|z-1| \leq 1$ and a zero for which $|z-1| \geq 1$. Joyal, Labelle and Rahman provide a nice elementary proof of this fact. [**4**]

First, consider the case $n = 2$. Then the transformation $w = z - 1$ converts the quadratic polynomial to $cw^2 + (2c-1)w + c$. Since the product of the zeros of this quadratic is equal to 1, either both its zeros lie on the unit circle or one lies inside and the other outside. The desired result follows.

Now let $n \geq 3$. The derivative of the polynomial $p(z) = 1 - z + cz^n$, namely $ncz^{n-1} - 1$, has at least one zero in the left half plane. But this zero must be contained in the closed convex hull of the zeros of $p(z)$; hence, there must be a zero of $p(z)$ in the left half plane and this zero satisfies $|z-1| > 1$.

It remains to show that there is at least one zero for which $|z-1| \leq 1$. With $w = z - 1$, it is equivalent to show that $-w + c(w+1)^n = 0$ for some w with $|w| \leq 1$. If we make the transformation $v = 1/w$, then the equation becomes equivalent to

$$-v^{n-1} + c(v+1)^n = 0 \ .$$

We need to show that there is a solution with $|v| \geq 1$. Finally, if $v = u - 1$, the equation becomes

$$-(u-1)^{n-1} + cu^n = 0 \ .$$

Let K be the closed convex hull of the roots of this equation in u. By the Gauss-Lucas Theorem, K must contain all of the zeros of the $(n-2)$th derivative

$$-(n-1)!(u-1)+(cn!/2)u^2 \ ,$$

or equivalently, of

$$1-u+\frac{cn}{2}u^2 \ .$$

However, as we have seen, this quadratic has a zero that satisfies $|u-1|\geq 1$, and so this must be true of

$$-(u-1)^{n-1}+cu^n=0 \ .$$

Tracking back, we see that there must be a zero of $1-z+cz^n$ that satisfies $|z-1|\leq 1$.

§5. CENTRE OF MASS

For the $n-$tple $\mathbf{z}=(z_1,z_2,\cdots,z_n)$ of complex numbers, the centre of mass $G(\mathbf{z})$ is defined by

$$G(\mathbf{z})=\frac{1}{n}(z_1+z_2+\cdots+z_n) \ .$$

This can be generalized to the centre of mass with respect to an arbitrary complex ζ. Let $w(z)=a(z-\zeta)^{-1}+b$ be a linear fractional mapping that carries ζ to infinity. Compute $G(w(\mathbf{z}))$ and then map the result back via w^{-1}. The result is independent of the parameters a and b, and we find it to be

$$G_\zeta(\mathbf{z})\equiv\zeta+n\left(\sum_{i=1}^{n}\frac{1}{z_i-\zeta}\right)^{-1}=\zeta-n\left(\sum_{i=1}^{n}\frac{1}{\zeta-z_i}\right)^{-1} \ .$$

When $z_1,z_2,\cdots,z_n$ are the zeros, not necessarily distinct, of the nth degree polynomial $p(z)$, then

$$G_\zeta(p)\equiv G_\zeta(\mathbf{z})=\zeta-n\frac{p(\zeta)}{p'(\zeta)} \ .$$

Observe that $\lim_{\zeta\to\infty}G_\zeta(\mathbf{z})=G(\mathbf{z})$.

When r is a zero of $p(z)$, then its multiplicity exceeds its multiplicity as a zero of $p'(z)$ and $\lim_{\zeta\to r}G_\zeta(p)=r$. When r is nonreal, this entails that when ζ is close to r, then $G_\zeta(p)$ and ζ are on the same side of the real axis, so that Im $\zeta\cdot$ Im $G_\zeta(p)>0$.

On the other hand, suppose that all of the zeros of $p(z)$ are real, that there are at least two distinct zeros, and that ζ is nonreal. We will suppose that ζ is in the upper half plane, so that Im $\zeta>0$. We can relate the centre of mass with respect to ζ to the standard centre of mass through the transformation $z\longrightarrow(z-\zeta)^{-1}$. Geometrically, this is the composite of a translation $z\longrightarrow z-\zeta$, inversion in the unit circle of the complex plane and reflection about the real axis. The real line is carried first to a line in the lower half plane, then a circle in the lower half plane tangent to the real axis at the origin and finally, the reflected image of this circle in the upper half plane. The zeros of $p(z)$ are carried to points on this circle and the centre of mass (which is the image of $G_\zeta(p)$) of these zeros is in the interior of this circle. But the interior of this circle is the image of the lower half plane so that Im $G_\zeta(p)<0$. Therefore, Im $\zeta\cdot$ Im $G_\zeta(p)<0$. A similar argument gives the same inequality when ζ is in the lower half plane. Thus, we obtain the following result:

For a polynomial with real coefficients and at least two distinct zeros, all the zeros are real if and only if Im $\zeta\cdot$ Im $G_\zeta(p)<0$ *for every nonreal* ζ [**7**, Theorem 1.1.9, p. 6-7].

Since $G_\zeta(p)$ is a generalized centre of mass of the zeros of p, it is interesting to see under what circumstances it is a zero itself.

It is straightforward to see that, if ζ is already a zero of a polynomial $p(z)$, then $G_\zeta(p) = \zeta$. If the polynomial has exactly one zero r of arbitrary multiplicity, then $G_\zeta(p) = r$ for all ζ (so the foregoing result fails in this case). If $p(z)$ has exactly two distinct zeros, we may suppose that $p(z)$ is monic so that it is equal to $(z-r)^m(z-s)^k$, where $m+k=n$, the degree of the polynomial. Suppose that $G_\zeta(p) = r$. Then

$$\zeta - r = \frac{n(\zeta - r)(\zeta - s)}{n\zeta - (ms + kr)} \ ,$$

whence

$$0 = (\zeta - r)[n(\zeta - s) - n\zeta + (ms + kr)] = (\zeta - r)k(r - s) \ .$$

Therefore, $G_\zeta(p)$ is a zero of $p(z)$ only when ζ itself is a zero of $p(z)$.

Suppose that $p(z)$ is a polynomial of degree n with more than two distinct zeros and that one of the zeros is r with multiplicity m. Then $p(z) = (z-r)^m q(z)$ for some polynomial $q(z)$ of degree $n - m \geq 2$ which does ot vanish at r and has at least two distinct zeros. The condition $G_\zeta(p) = r$ leads to

$$(\zeta - r)q'(\zeta) = (n - m)q(\zeta) \ .$$

This is equivalent to

$$\frac{n-m}{\zeta - r} = \frac{q'(\zeta)}{q(\zeta)} = \sum_{i=1}^{s} \frac{m_i}{\zeta - r_i} \ ,$$

where the r_i are the zeros of $q(z)$, m_i are positive integers and $s \geq 2$. There is a nontrivial value of ζ distinct from the zeros of $p(z)$ that satisfies this equation.

Example 1. When $p(z) = z^3 - z = (z-1)z(z+1)$, then $G_\zeta(p) = 2\zeta(3\zeta^2 - 1)^{-1}$. While $G_\zeta(p) = 0$ if and only if $\zeta = 0$, we have that $G_\zeta(p) = 1$ when $\zeta = 1, -1/3$ and $G_\zeta(p) = -1$ when $\zeta = -1, 1/3$.

Example 2. When $p(z) = z^4 - 10z^2 + 9 = (z-3)(z-1)(z+1)(z+3)$, the condition that $G_\zeta(p) = 1$ leads to the equation

$$(\zeta - 1)(\zeta^3 - 5\zeta) = \zeta^4 - 10\zeta^2 + 9 \ ,$$

which reduces to

$$0 = (\zeta - 1)(\zeta^2 - 4\zeta - 9) \ .$$

The zeros of the quadratic factor are real.

§6. APOLARITY AND GRACE'S THEOREM

Let $p(z)$ be a polynomial of degree n and ζ be a number in the extended complex plane. We define the derivative of p with respect to ζ by

$$A_\zeta p(z) = \begin{cases} np(z) + (\zeta - z)p'(z) & \text{if } \zeta \neq \infty; \\ p'(z) & \text{if } \zeta = \infty. \end{cases}$$

When $p(z) = \sum_{k=0}^{n} \binom{n}{k} a_k z^k$ and $\zeta \neq \infty$, we have that

$$A_\zeta p(z) = n \sum_{k=0}^{n-1} \binom{n-1}{k} (a_k + a_{k+1}\zeta) z^k \ .$$

More generally,

$$A_{\zeta_1} A_{\zeta_2} \cdots A_{\zeta_n} f(z) = n![a_0 + a_1\sigma_1 + a_2\sigma_2 + \cdots + a_n\sigma_n]$$

where σ_i is the ith elementary symmetric function of the ζ_i. If the ζ_i are the zeros of the polynomial $g(z) = \sum_{k=0}^{n} \binom{n}{k} b_k z^k$, then the equality $A_{\zeta_1} A_{\zeta_2} \cdots A_{\zeta_n} f(z) = 0$ is equivalent to

$$a_0 b_n - \binom{n}{1} a_1 b_{n-1} + \binom{n}{2} a_2 b_{n-2} + \cdots + (-1)^n a_n b_0 = 0 \ . \qquad (3.3)$$

Two polynomials of degree n satisfying this condition are said to be *apolar*.

As was noted in Section 2.3, when $f(z)$ and $g(z)$ are two polynomials of degree not exceeding n, the polynomial

$$\langle f, g\rangle = f(z)g^{(n)}(z) - f'(z)g^{(n-1)}(z) + f''(z)g^{(n-2)}(z) - \cdots + (-1)^n f^{(n)}(z)g(z)$$

is a constant. By examining the Taylor expansion of the polynomials at 0, we see that the apolarity condition (3.3) is equivalent to $\langle f, g\rangle = 0$.

As an example, to see how the computations work out, it is convenient to look at the cubic situation. It can be checked that, for ζ finite,

$$A_\zeta(az + b) = (az + b) + (\zeta - z)a = a\zeta + b \ ,$$

$$A_\zeta(az^2 + 2bz + c) = 2(az^2 + 2bz + c) + (\zeta - z)(2az + 2b) = 2[(b + a\zeta)z + (c + b\zeta)] \ ,$$

and

$$\begin{aligned} A_\zeta(az^3 + 3bz^2 + 3cz + d) &= 3(az^2 + 3bz^2 + 3cz + d) + 3(\zeta - z)(az^2 + 2bz + c) \\ &= 3[(b + a\zeta)z^2 + 2(c + b\zeta)z + (d + c\zeta)] \ . \end{aligned}$$

Taking $\zeta = \zeta_1, \zeta_2, \zeta_3$ successively, we find that

$$A_{\zeta_2}A_{\zeta_1}(az^3 + 3bz^2 + 3cz + d) = 6[(c + b\zeta_1 + b\zeta_2 + a\zeta_1\zeta_2)z + (d + c\zeta_1 + c\zeta_2 + b\zeta_1\zeta_2)]$$

and

$$\begin{aligned} A_{\zeta_3}A_{\zeta_2}A_{\zeta_1}(az^3 + 3bz^2 + 3cz^2 + d) &= 6[c\zeta_3 + b\zeta_1\zeta_3 + b\zeta_2\zeta_3 + a\zeta_1\zeta_2\zeta_3 + d + c\zeta_1 + c\zeta_2 + b\zeta_1\zeta_2] \\ &= 6[d + c(\zeta_1 + \zeta_2 + \zeta_3) + b(\zeta_1\zeta_2 + \zeta_1\zeta_3 + \zeta_2\zeta_3) + a\zeta_1\zeta_2\zeta_3] \ . \end{aligned}$$

If ζ_1, ζ_2, ζ_3 are the zeros of the polynomial $q(z) = \alpha z^3 + 3\beta z^2 + 3\gamma z + \delta$ and $p(z) = az^3 + 3bz^2 + 3cz + d$, then

$$\begin{aligned} A_{\zeta_3}A_{\zeta_2}A_{\zeta_1}p(z) &= 6\left[d - \frac{3c\beta}{\alpha} + \frac{3b\gamma}{\alpha} - \frac{a\delta}{\alpha}\right] \\ &= (6/\alpha)[\alpha d - 3\beta c + 3\gamma b - \delta a] \ . \end{aligned}$$

The McLaurin Expansion of these polynomials can be written

$$p(z) = p(0) + 3\left[\frac{p'(0)}{3}\right]z + 3\left[\frac{p''(0)}{6}\right]z^2 + \left[\frac{p'''(0)}{6}\right]z^3$$

and

$$q(z) = q(0) + 3\left[\frac{q'(0)}{3}\right]z + 3\left[\frac{q''(0)}{6}\right]z^2 + \left[\frac{q'''(0)}{6}\right]z^3 \ ,$$

whence

$$A_{\zeta_3}A_{\zeta_2}A_{\zeta_1}p(z) = \frac{6}{q'''(0)}[p(0)q'''(0) - p'(0)q''(0) + p''(0)q'(0) - p'''(0)q(0)] \ .$$

Where a *circular domain* signifies either an open half plane or the interior or exterior of a disk, we have the following result for helping to locate the zeros of a polynomial:

Grace's Theorem. *Let f and g be apolar polynomials. If all the zeros of f belong to a circular domain, then at least one of the zeros of g also belong to the same circular domain.*

We can deduce from this that when f and g are apolar, any convex region that encloses all the zeros of $f(z)$ must intersect any convex region that encloses all the zeros of $g(z)$.

Before getting to the proof of Grace's Theorem, we need a preliminary result which is a generalization of the Gauss-Lucas Theorem: *Let K be a circular domain that contains all the zeros of the polynomial $f(z)$ and suppose that $\zeta \notin K$. Then all the zeros of $A_\zeta f(z)$ also lie in K.* Since the case $\zeta = \infty$ is the Gauss-Lucas Theorem, we will suppose that ζ is in the complex plane. Suppose that $A_\zeta f(\alpha) = 0$. Then this means that

$$(\zeta - \alpha)f'(\alpha) + nf(\alpha) = 0$$

whence

$$\zeta = \alpha - \frac{nf(\alpha)}{f'(\alpha)} = G_\alpha(f) \ ,$$

so that ζ is the $\alpha-$centre of mass of the zeros of f.

Suppose, if possible, that $\alpha \notin K$. Then for any set $\mathbf{z}$, $G_\alpha(\mathbf{z})$ lies in the closed convex hull of $\mathbf{z}$. To see this, consider a line that separates α and K; the linear fractional mapping $w(z)$ used to define G_α maps α to ∞ and the line to a circle containing $w(\mathbf{z})$. The regular centre of mass of $w(\mathbf{z})$ lies inside this circle, and so its image under w^{-1} lies in the same half-plane determined by the line as $\mathbf{z}$. This means that $\zeta = G_\alpha(f)$ must be in K, contrary to hypothesis.

Now we can establish Grace's Theorem. Suppose that $\zeta_1, \zeta_2, \cdots, \zeta_n$ are the zeros of $g(z)$, and suppose, if possible, that all of these lie outside of K. Then, by the foregoing, all the zeros of the generalized derivatives of $f(z)$ must be contained in K. In particular,

$$A_{\zeta_2}A_{\zeta_3}\cdots A_{\zeta_n}f(z) = c(z-u) \ ,$$

a linear polynomial with $u \in K$. Because f and g are apolar,

$$A_{\zeta_1}A_{\zeta_2}A_{\zeta_3}\cdots A_{\zeta_n}f(z) = 0 \ ,$$

so that $0 = A_{\zeta_1}(z-u) = \zeta_1 - u$. Thus $u = \zeta_1 \notin K$, which contradicts the previous statement. Hence, at least one of the ζ_i belongs to K. ♠

Example. We can apply this theorem to affirming what was earlier shown about the zeros of $p(z) = 1 - z + cz^n$. Suppose that

$$q(z) = z^n + \binom{n}{1}b_{n-1}z^{n-1} + \cdots + b_0 \ .$$

Then p and q are apolar if and only if

$$0 = 1 - n(-1/n)b_{n-1} + (-1)^n cb_0 = 1 + b_{n-1} + (-1)^n cb_0 \ .$$

Specifically, let $q(z) = (z-1)^n + (-1)^{n-1}$. Then $q(z) = 0$ is equivalent to $(z-1)^n = (-1)^n$, so that all the zeros of $q(z)$ are on the circumference of the circle defined by $|z-1| = 1$. Hence $q(z)$ has all its zeros satisfying each of the conditions $|z-1| \le 1$ and $|z-1| \ge 1$. Since $q(z) = z^n - nz^{n-1} + \cdots + (-1)^{n-1}nz$, p and q are apolar and we find that p must have at least one zero satisfying each of the conditions $|z-1| \le 1$ and $|z-1| \ge 1$. ♣

A result that is an almost immediate consequence of Grace's Theorem is Szegös's Theorem: *Suppose that f and g are both polynomials of degree n and that all of the zeros of f lie within a circular domain K. Let*

$$f(z) = \sum_{k=0}^{n} \binom{n}{k} a_k z^k \qquad \text{and} \qquad g(z) = \sum_{k=0}^{n} \binom{n}{k} b_k z^k \ ,$$

and define the polynomial

$$h(z) = \sum_{k=0}^{n} \binom{n}{k} a_k b_k z^k \ .$$

Then every zero of $h(z)$ is of the form $-ws$, where w is a zero of $g(z)$ and $s \in K$.

Suppose that $h(v) = 0$, so that

$$\sum_{k=0}^{n} \binom{n}{k} a_k b_k v^k = 0 \ . \tag{3.4}$$

Let

$$\begin{aligned} p(z) = z^n g\left(\frac{-v}{z}\right) &= z^n \left[\sum_{k=0}^{n} \binom{n}{k} (-1)^k b_k \left(\frac{v^k}{z^k}\right) \right] \\ &= (-1)^n b_n v^n + (-1)^{n-1} n b_{n-1} v^{n-1} z + (-1)^{n-2} \binom{n}{2} b_{n-2} v^{n-2} z^2 + \cdots \ . \end{aligned}$$

Since (3.4) is simply the condition that $p(z)$ and $f(z)$ are apolar, $p(z)$ has a zero u inside K. In other words, there exists $s \in K$ such that $-v/s = w$, a zero of $g(z)$. The result follows. ♠

An open question is the *Sendov-Ilieff Conjecture*: *Let $p(z)$ be a polynomial of degree exceeding 1 all of whose roots lie inside the closed unit disc defined by $|z| \leq 1$. Then, if w is one of the roots of $p(z)$, then the closed unit disc with centre w contains at least one roots of the derivative $p'(z)$.* This conjecture holds for polynomials of degree not exceeding 5 and for polynomials with only three distinct roots. [**7**. p. 18]

§7. STABILITY

In solving an nth order linear differential equation with constant coefficients, it is often an issue whether its solutions remain bounded as the variable tends to infinity. This is so, provided that all the zeros of its characteristic polynomial have negative real parts. A polynomial with this property is said to be *stable.* The Routh-Hurwitz Problem is to determine what conditions are equivalent to stability.

For real linear, quadratic and cubic polynomials, the criterion is reasonably straightforward. All that is required for a quadratic is that its coefficients have the same sign, all positive or all negative. For the cubic, $z^3 + bz^2 + cz + d$, it is necessary and sufficient that all coefficients are positive and also that $bc > d$.

We observe that it suffices to consider polynomials with real coefficients. For, if $p(z) = \sum_{k=0}^{n} a_k z^k$, the polynomial

$$f(z) = (\sum_{k=0}^{n} a_k z^k)(\sum_{k=0}^{n} \overline{a_k} z^k) = p(z)\overline{p(\overline{z})}$$

has real coefficients and its roots are those of $p(z)$ along with their complex conjugates. Thus, $p(z)$ is stable if and only if $f(z)$ is stable.

Theorem. *Suppose that the monic polynomial $p(z) = \sum_{k=0}^{n} a_k z^k$ is a polynomial with real coefficients and that $q(z) = \sum_{b=0}^{m} b_k z^k$ is that monic polynomial of degree $m = \binom{n}{2}$ whose zeros are the sums of all possible pairs of zeros of $p(z)$. Then $p(z)$ is stable if and only if all the coefficients of both p and q are positive.*

Proof. Suppose that $p(z)$ is stable. Then it can be factored as a product of real linear and quadratic polynomials of the form $z + \alpha$ and $(z + \alpha + i\beta)(z + \alpha - i\beta) = (z + \alpha)^2 + \beta^2$, whose coefficients are all positive. Hence, the coefficients of $p(z)$ are all positive. Since every zero of $p(z)$ has a negative real part, this is true of the zeros of $q(z)$, and it too has positive coefficients.

On the other hand, suppose that $p(z)$ and $q(z)$ have all coefficients positive. Then all the real zeros of both of these polynomials must be negative, by Descartes' Rule of Signs. Since each complex conjugate pair of nonreal zeros of $p(z)$ add to a real zero of $q(z)$, all the nonreal zeros of $p(z)$ have negative real parts. □

§8. PROBLEMS AND INVESTIGATIONS

1. Prove that, for each positive integer n, the polynomial

$$x(x+1)(x+2)\cdots(x+2n-1)+(x+2n+1)\cdots(x+4n)$$

has no real zeros.

2. Let $f(x)=x^3-3x+1$. How many distinct real roots does the equation $f(f(x))=0$ have?

3. Let $f(x)=x^4-4ax^3+6b^2x^2-4c^3x+d^4$ where $a,b,c,d>0$. Prove that, if $f(x)$ has four distinct real zeros, then $a>b>c>d$.

4. Let a,b,c be real numbers for which $0\leq c\leq b\leq a\leq 1$. Must every zero w of the polynomial z^3+az^2+bz+c satisfy $|w|\leq 1$?

5. If the quadratic polynomial z^2+az+b has two zeros that satisfy $|z|=1$, then $|a|\leq 2$ and $|b|=1$. Is the converse true?

6. The polynomial x^3-33x^2+216x has the property that it and all its derivatives have integer zeros. Investigate which polynomials of degree not excededing 5 that have this property.

7. Let $p(z)$ be a polynomial of degree n all of whose zeroes satisfy $|z|=1$, and suppose that $q(z)=np(z)-2zp'(z)$. Must all of the zeros of $q(z)$ lie on the unit circle?

8. Let $n\geq 2$ and

$$P_n(x)=x^{n^2+n-1}-3x^{n^2}+x^{n^2-1}+x^{n^2-n}+x^{2n-1}+x^n-3x^{n-1}+1\ .$$

Prove that $P_n(x)\geq 0$ when $x\geq 0$.

9. Let $\sum_{k=0}^{n}a_kz^k$ be a monic polynomial with complex coefficients and zeros $z_1,\ z_2,\ \cdots,\ z_n$. Prove that

$$\frac{1}{n}\sum_{k=1}^{n}|z_k|^2<1+\max_{1\leq k\leq n}|a_{n-k}|^2\ .$$

10. Let $a_0,\ a_1,\ \cdots,\ a_{2m}$ be real numbers with $a_{2m}\neq 0$. Define polynomials $p(z)$ and $q(z)$ by

$$p(z)=\sum_{i=0}^{2m}a_iz^i$$

and

$$q(z)=\sum_{i=0}^{2m}a_iz^{2m-i}\ .$$

Suppose that $p(z)$ is not a multiple of $z-1$ and that

$$zp(z)-q(z)=a_{2m}(z-1)^{2m+1}\ .$$

Prove that, if $(-1)^mp(1)/a_{2m}>0$, then $p(z)$ has m zeros satisfying $|z|>1$ and m zeros satisfying $|z|<1$. Prove that, if $(-1)^mp(1)/a_{2m}<0$, then $p(z)$ has $m+1$ zeros satisfying $|z|>1$ and $m-1$ zeros satisfying $|z|<1$.

11. Let $k\geq 4$ be an integer and $\alpha>2$ be a real. The equation $x^k-\alpha x^{k-1}+1=0$ has exactly one root u that satisfies $0<u<1$. Prove that every complex root z of this equation satisfies $|z|\leq u$.

12. Show that the equation $z^4-2cz^3+2\bar{c}z-1=0$ has a root not on the unit circle $|z|=1$ if and only of $u^{1/3}+v^{1/3}i$ lies outside the unit circle, where $c=u+vi$.

13. The polynomial $x^3 - 33x^2 + 216x$ has the property that all of the zeros of it and its derivatives are integers. Investigate which polynomials of degree not exceeding 5 have this property.

Hints and Comments

1. Let $t = x + 2n$.

2. Sketch the graph of the cubic and locate its zeros.

6. [AMM 106:10 (December, 1999), 959]

8. Use Descartes Rule of Signs to determine the number or positive roots, and then find what these are. [AMM #10978: 109:10 (December, 2002), 921; 111:7 (August-September, 2004), 629.]

9. [AMM #11008: 110:4 (April, 2003), 340; 112:1 (January, 2005), 91]

10. An example is $z^2 + 3$. [AMM #10995: 110:2 (February, 2003), 156; 111:10 (December, 2004), 917]

11. [AMM #10688: 105:8 (October, 1998), 769; 107:2 (February, 2000), 181]

12. [AMM #10253: 99:8 (October, 1992), 782; 102:3 (March, 1995), 277]

12. [AMM 106:10 (December, 1999), 959] See also [**3**].

References

1. Edward J. Barbeau, *Polynomials.* Springer, 2003
2. Matthew Boelkins, Jennifer Miller & Benjamin Vugteveen, *From Chebyshev to Bernstein. College Mathematics Journal* 37:3 (May, 2006), 194-204
3. C.E. Carroll, *Polynomials all of whose derivatives have integer roots. American Mathematical Monthly* 96:2 (February, 1989), 129-130.
4. A. Joyal, G. Labelle & Q.I. Rahman, *On the location of zeros of a polynomial. Canadian Mathematical Bulletin* 10 (1967), 53-63
5. Don Kalman, *An elementary proof of Marden's theorem. American Mathematical Monthly* 115:4 (April, 2008), 330-337
6. Morris Marden, *Geometry of polynomials.* American Mathematical Society, 1966
7. Victor V. Prasolov, *Polynomials.* Springer, 2000