PREFACE

This monograph is not a comprehensive treatment of the subject of polynomials, but rather a sequence of essays on topics related to polynomials that might interest undergraduate and graduate students of mathematics and indicate the scope of the area. It is based on a graduate course that was given in Halifax during the summer of 2007 under the auspices of AARMS (Atlantic Association for Research in the Mathematical Sciences) in Eastern Canada. The students who participated were at various stages in their mathematics education; most were graduate students from both Canada and abroad, but there were some undergraduates as well.

Polynomials is an active and broad area of research in many directions, and there exist excellent texts that give a comprehensive summary of the basic results. For an introduction based on extension material given to high school students, the reader can consult the author's *Polynomials*, which appeared in the Springer series of problems books.

A recent and excellent book that gives an overview of elementary results on polynomials and their connections to other areas of mathematics is *Uncommon mathematical excursions: Polynomial and related realms* by Dan Kalman. This can be considered as a companion to this volume.

Three recent books that give a systematic account of advanced results in polynomials that the reader can turn to for a more detailed treatment of the topics raised in this monograph:

P. Borwein & T. Erdélyi, *Polynomials and polynomial inequalities*

G.V. Milovanović, D.S. Matrinović & Th. M. Rassias, *Topics in polynomials, extremal problems, inequalities, zeros*

Victor V. Prasolov, *Polynomials*

These three books give a more detailed treatment of some of the work of the first six chapters of this book.

FOREWORD

In this book, I have tried to provide brief glimpses of work on polynomials that represents some historical material as well as more modern work. Because research into this area stretched over many centuries, there is much that is accessible to undergraduates, but also many areas that are deep and technical. Any attempt to give a sampling in about a hundred pages is bound to be idiosyncratic, and significant areas are going to be left out altogether. Galois theory is a conspicuous example in this book.

I assume that the reader has taken core courses in a standard undergraduate mathematical curriculum that include linear algebra, calculus, basic results of real and complex analysis, and introductory modern algebra. In particular, the reader should have had experience dealing with polynomials that includes solution of equations of low degree and the relationship between the coefficients and zeros of a polynomial.

In the first chapter, I discuss the result that every polynomial of positive degree has a complex zero, showing how this was attempted in the eighteenth century by providing an algorithm and how successful proofs rely on basic results of real and complex variables. The second chapter looks at Taylor expansion, and how they figure in Newton's method of approximation and the prefiguration of apolarity. Apolarity is treated more directly in the third chapter whose theme is the location of zeros of polynomials. The fourth chapter takes up the question of interpolating data by means of polynomials either through methods of Lagrange and Hermite or using finite differences. However, continuous functions can be approximated by polynomials , and this is the burden of Chapter 5, which provides proofs of the Weierstrass Approximation Theorem, introduces Bernstein polynomials and Bézier curves and concludes with a brief look at approximate integration. Chapter 6 is more algebraic, with a look at irreducibility and factorization of polynomials. In Chapter 7, there is a brief look at dynamical systems, particularly involving quadratic functions; this is merely an enticement to the more detailed treatment found in books such as *An introduction to chaotic dynamical systems* by Robert L. Devaney. The eighth chapter was motivated by generalizing a recursion due to the British mathematician Robert Lyness and examing seed which make it periodic. The resolution fo this problem relies on the theory of cubic cubic curves in the plane, upon its points with rational coordinates, a group operation can be defined. The recursion can be defined using the theory of the quadratic, and the ninth chapter gives a first pass at trying to develop a theory of such recursions, dubbed "asllemands". The tenth chapter treats the solution of norm form diophantine equations, in particular, Pell's equation and its generalizations to higher degree. These pellian equations can be formulated for polynomials as well as integers and there are polynomial values of the parameter that admit polynomial solutions. This appears to be an area where a more systematic treatment is needed. The final chapter continues the theme of diophantine equations for polynomials, and provides a proof of the *abc* Theorem and its corollary that the version of the Fermat theorem for polynomials is true with a simple argument.

Most chapters conclude with a short list of problems and investigations, along with a list of pertinent references. The problems are not provided with solutions, but many are taken from the journals of the *Mathematical Association of America*, particularly the *Monthly*, a rich source of material. These are referenced in the format *Journal abbreviation*, #(Number), Vol:No (Month, Year), page .

At the end of each chapter, a few references are given on material that is pertinent to that chapter. This may include books and papers that provide background as well as sources for the results in the chapter. At the end of the monograph is a list of books and papers that the reader can consult. In particular, the *American Mathematical Monthly* is a rich source of nice expository papers on polynomials that together provide a nice introduction. Most of the references are recent; a comprehensive list of older references can be found in my book *Polynomials*.