Palindromic products and other digital novelties

Some time ago, I drew the attention of readers to the equation $3516 \times 8274 = 4728 \times 6153$, where the digits on the right side appear in reverse order to those on the left, and challenged them to come up with other examples. J. Kerry Skipper rose to the task, fired up his computer, and discovered no fewer than 615 pairs of four digit numbers that exhibit the same phenomenon. If one of the numbers is 3516, there are eighteen other mates beyond 8274. One of them is the obvious 6153. The others are 2121, 2331, 2541, 2751, 2961, 4032, 4242, 4452, 4662, 4872, 6363, 6573, 6783, 6993, 8064, 8484, 8694. (Elementary teachers wishing to add a little spice to an otherwise mundane multiplication exercise may ask their pupils to check that these work. Secondary students can be asked to make an efficient computer program that will churn these examples out.)

The job of finding pairs with two digits each can be accomplished without a calculator by using a little algebra. Let the two numbers be $10a+b$ and $10c+d$, where a, b, c, d are single digits and a and c are not zero. Then we set up the equation

$$0 = (10a+b)(10c+d) - (10d+c)(10b+a) = 99(ac-bd)$$

and arrange that the product of a and c is the same as the product of b and d. In this way, we can get, for example, $62 \times 13 = 31 \times 26$.

It is also possible to find pairs with one multiplier having two and the other three digits. The product of 12 and 231 is a palindrome, 2772, so we can run the digits backwards to find $12 \times 231 = 132 \times 21$. To see why this works, write out the complete multiplication algorithm for each product. We can derive other examples by multiplying either factor by a digit that does not introduce carries: For example, $24 \times 693 = 396 \times 42 = 2 \times 3 \times 2772 = 16632$. Other examples are $13 \times 682 = 286 \times 31$ and $28 \times 451 = 154 \times 82$.

For a bit of icing on the cake, Skipper produced the intriguing equation $1596 \times 6951 = 2793 \times 3972 = 4 \times 9 \times 49 \times 331$, where on each side, the second factor is a palindrome of the first.

Over the holidays, readers were given five numerical puzzles to solve. The first was to find a number whose third and fourth powers used all ten digits, each exactly once. A single-digit number would not have enough digits for these two powers. So the fourth power would have more digits than the third power. Since the cube of a three-digit number has at least seven digits, we conclude that the number we are looking for has exactly two digits. Now argue that its cube must have four digits and the fourth power six. At this point, we need some inspired guess work. The square of 17 is less than 300, so its fourth power is less than 90000, and so has at most five digits. So the number we are looking for is at least 18. The square of 23 is greater than 500, so its cube is greater than 10000, and so has more than four digits.

Thus, we are looking for a number between 18 and 22 inclusive. We can rule out 20 and 21 immediately, and so need to check only 18, 19 and 22. Indeed, $18^3 = 5832$ and $18^4 = 104976$.

The second puzzle was to find three numbers in the ratio 1:3:5 that used all the nonzero digits, each once. Again, we start by narrowing down the search, and I will sketch the reasoning. First, the smallest number, call it N, must have exactly three digits. Secondly, it must be less than 200. Thirdly, since the largest number, $5N$, must end in 5, neither $3N$ nor $5N$ can begin with 5, so that N lies between 120 and 166. Thus N must be a number within these bounds that ends in one of the digits 3, 9. (7 is ruled out since $3N$ cannot end in 1.) Without too much arithmetic, we are lead to the answer $N = 129$, with $3N = 387$ and $5N = 645$.

We will discuss the remaining three holiday problems in a future column.