This column is for your mathematical amusement. Its author is very happy to correspond with readers about mathematical matters, and hopes that the column will turn out to be a dialogue with readers of the Frontenac News. His email address is barbeau@math.utoronto.ca.

Divisibility

Those of you who make a hobby of mental arithmetic will know some divisibility tests, that is ways of telling whether a given whole number is divisible by some integer. The simplest of these is the test for divisibility by 2; a number is divisible by 2 exactly when its last digit is even. This can be generalized. To check for divisibility by 4, we just need to determine whether the number formed by its last two digits is divisible by 4. A number is divisible by 8 exactly when the number formed by its last three digits is divisible by 8. Your teen-aged child or grandchild might be able to explain to you why this is so.

Another simple test is for divisibility by 5 – just check whether the last digit is 5 or 0. For divisibiity by 25, we need the last two digits to be 00, 25, 50 or 75.

Next we turn to divisibility by 3, 6 or 9. The older among the readers of this column may have learned the technique of *casting out nines* when they were in public school. To perform this, we take a given number and determine its *digital sum*. This involves adding its digits, then adding the digits of the total and repeating this until you get a single digit answer. For example, to work out the digital sum of the number 3544789201, we keep adding the digits until we arrive at a single digit:

$$3544789201 \longrightarrow 43 = 3+5+4+4+7+8+9+2+0+1 \longrightarrow 7.$$

The digital sum of 3544789201 is 7. It turns out that any number has the same remainder as its digital sum when you divide by 9. Thus, if we divide 3544789201 by 9, we get remainder 7. Indeed,

$$3544789201 = 9 \times 393865466 \times 9 + 7.$$

So a number is divisible by 9 exactly when its digital sum is 9, is divisible by 3 when its digital sum is 3, 6 or 9 and is divisible by 6 when it is even and its digital sum is 3, 6 or 9.

This leaves divisibility by 7, and the checking rule is a little more complicated. Here the rule is to start with the number you want to check. Take away its last digit, and subtract twice this last digit from what remains. Keep on doing this until you get a number that you recognize either as a multiple or a nonmultiple of 7. This operation will convert a multiple of 7 into another multiple of 7 and a non-multiple of 7 into another non-multiple. Let us check the two numbers 81246 and 57064.

$$81246 \longrightarrow 8112 = 8124 - 12 \longrightarrow 807 = 811 - 4 \longrightarrow 66 = 80 - 14.$$

$$57064 \longrightarrow 5698 = 5706 - 8 \longrightarrow 553 = 569 - 16 \longrightarrow 49 = 55 - 6.$$

Since 66 is not a multiple of 7, neither is 81246. Since 49 is a multiple of 7, so is 57064. (Warning: unlike with casting out nines, this process does not necessarily give a number with the same remainder.)

Can you devise a divisibility test when 11 or 13 is the divisor?

Here are two problems for you to think about:

1. Take any even and any odd digit, say 6 and 5. Using only these two digits, construct a ten-digit number which is evenly divisible by 1024. (Hint: $1024 = 2^{10}$, the product of ten factors each equal to 2.)

2. The number 123654 has the property that 1 is divisible by 1; 12 is divisible by 2; 123 is divisible by 3; 1236 is divisible by 4; 12365 is divisible by 5; and 123654 is divisible by 6. Using the digits 1, 2, 3, 4, 5, 6, 7, each once, find a seven-digit number such that the $k-$digit number formed from its first k digits is divisible by k for each k from 1 to 7. Solve the analogous problem for an eight-digit number and for a nine-digit number.