Horses and Egyptian fractions

A reader of the *Frontenac News* sent us a problem that related the story of a farmer with seventeen horses who willed half of them to one son, a third of them to another and one ninth to the third. When he passed on, the three sons were puzzled about how the division might be made and consulted a neighbour. The neighbour solved the riddle by introducing his own horse, to make eighteen in all, then assigning nine, six and two horses to the three sons. This, of course, only required seventeen horses, so the neighbour was able to reclaim his own horse.

This is one version of an old mathematical joke. In another version, a sheik with eleven camels leaves one half, one quarter and one sixth to his sons, and the problem is similarly solved by a passing Bedouin. In each case, the problem involves a representation of 1 by four distinct unit fractions, whose numerators are 1 and whose denominators are postive whole numbers:

$$1 = \frac{1}{2} + \frac{1}{3} + \frac{1}{9} + \frac{1}{18} = \frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \frac{1}{12}.$$

Unit fractions are also known as *Egyptian fractions*, because of an ancient Egyptian manuscript that came to light 1858 in which a number of arithmetic problems were posed and the fractions were represented as sums of distinct unit fractions. The interesting tale of this manuscript can be found online at: *http://en.wikipedia.org/wiki/Rhind_Mathematical_Papy*

An interesting pastime is to see in how many ways you can express 1 as the sum of distinct unit fractions. There are no possibilities with two fractions and only one with three, with denominators $(2, 3, 6)$. An equivalent problem is to express a number as the sum of some of its divisors. For example, the equation $1 = (1/2) + (1/3) + (1/6)$ is equivalent to $6 = 3 + 2 + 1$. You may find it easier to work with this formulation.

For four fractions, we have the denominators $(2, 3, 7, 42)$, $(2, 3, 8, 24)$, $(2, 3, 9, 18)$, $(2, 3, 10, 15)$, $(2, 4, 5, 20)$, $(2, 4, 6, 12)$. In your search, a useful formula to keep in mind is $1/n = 1/(n + 1) + 1/n(n + 1)$ which allows you, for example, to replace $1/7$ by $1/8 + 1/56$. For a given number of distinct unit fractions whose sum is 1, what is the set that has the largest denominator? Can you express 1 as the sum of distinct unit fractions whose denominators are all odd numbers?

I mentioned expressing numbers as the sum of some of their divisors. There are a few numbers, known as *perfect numbers*, which are the sums of all of their divisors except itself. The smallest three such numbers are 6, 28 and 496. Euclid knew how to find such numbers and it was proved in the eighteenth century that all even perfect numbers could be found in the following way. Pick any prime number p such that $2^p - 1$ is also prime (these are known as *Mersenne primes.* For example, 5 is a Mersenne prime since $31 = 2^5 - 1$ is also prime. Then the number $2^{p-1} \times (2^p - 1)$ is a perfect number. Nobody has ever found an odd perfect number and it is an open problem whether there are any.

Here is a further task for you to think about. Take any fraction whose numerator is 4 and whose denominator is bigger than 2. Express this as the sum of three distinct unit fractions. $4/3 = 1/1 + 1/4 + 1/12$, $4/5 = 1/2 + 1/5 + 1/10$, $4/6 = 1/3 + 1/4 + 1/12$, $4/7 = 1/2 + 1/15 + 1/210$. The Hungarian mathematician, Paul Erdös (1913-1996), conjectured in 1948 that this could always be done, and in fact, it is fairly straightforward to find a scheme to express $4/n$ as the sum of three unit fractions as long as n is not one more than a multiple of 24. It is easy for even values of n. For example, $4/10 = 2/5 = 1/5 + 1/5 = 1/5 + 1/6 + 1/30$. I have found that a typical high school class can pick up all the cases where the denominator does not exceed a multiple of 24 by 1 in less than an hour. However, it is still not known whether the result always holds in this exceptional case. Values of n have been checked to a very large number without a counterexample turning up. To find out more about this problem, you can Google *Erdos-Strauss Conjecture*.