The Four Colour Theorem

In 1976, the American Mathematical Society and the Mathematical Association of America held their annual summer meeting at the University of Toronto. Shortly before the meeting, the AMS announced that a notorious longstanding conjecture had been solved after 124 years and that one of the two solvers would be making a presentation in Toronto.

In 1852, Francis Guthrie, an English geography student, noted that he could colour a map of the counties of Great Britain with at most four colours and wondered to his brother, a mathematics student, whether this was true in general. The brother mentioned this question to his supervisor, Augustus de Morgan, and thus began a long quest to establish the result. To make the question more precise, this applies to any map drawn on a plane or sphere where each country is connected (you can travel from one point in the country to another without leaving it). Two countries are adjacent if they share a common border and two adjacent countries are to be coloured differently.

In 1879, Alfred Kempe claimed a proof, but this was found to be flawed by Percy Heawood in 1890, who was nevertheless able to establish that five colours sufficed. Now the challenge was taken up in earnest. There was a succession of mistaken "proofs" and "counterexamples" and work on the problem led to the establishment of an important applicable mathematical area called **graph theory**. However, the final solution required a delicate and detailed analysis that could be handled only by a computer. It was this that solvers Kenneth Appel and Wolfgang Haken of the University of Illinois discussed in 1976 at Toronto.

The basic strategy of the argument can be illustrated by showing that six colours are enough. If you imagine a map on a sphere and make each country flat, you can consider it as a polyhedron (three dimensional solid figure) with faces, vertices and edges. We need a result given in the eighteenth century by the Swiss mathematician, Leonard Euler (1707-1783): *if F is the number of faces of a polyhedron, V is the number of vertices and E the number of edges, then $V - E + F = 2$.* For example, for a cube, $(V, E, F) = (8, 12, 6)$; for an Egyptian pyramid, $(V, E, F) = (5, 8, 5)$; for a dodecahedral desk calendar, $(V, E, F) = (20, 30, 12)$; and for a soccer ball covered with 12 pentagonal and 20 hexagonal patches, $(V, E, F) = (60, 90, 32)$.

Each vertex of a polyhedron has at least three edges emanating from it (since at least three faces must meet there). So it would seem that the number of edges is at least as great as three times the number of vertices. However, when we count the edges by adding together the number of edges at each vertex, each edge gets counted twice, once for each endpoint. So in fact, twice the number of edges is greater than or equal to three times the number of vertices: $2E \geq 3V$. Hence $2 = V - E + F \leq (2/3)E - E + F$ so that $12 \leq -2E + 6F$. We next compute the average number of edges per face. Since each edge gets counted twice as we sum the number of edges over each face, this average is equal to

$$(2E/F) \leq 6 - (12/F) < 6.$$

Since the average number of edges per face is less than 6, there must be a face with fewer than six edges; that is, there must be a triangle, quadrilateral or pentgon.

Suppose, if possible, that the six-colour theorem is false. Then there must be a counterexample with at least 7 faces requiring more than six colours. Let us pick a counterexample with the fewest faces. We know that it contains a face with at most 5 vertices. Let me allow this face to be absorbed by one of its neighbours, thus reducing the number of faces by 1. We now have a polyhedron that can be coloured with at most six colours, so let us do that. Restore the absorbed face. It has at most five neighbours, so one colour is available to colour it differently. This contradicts our initial assumption.

One way to look at this argument is to note that it depends on every map having one of three possible configurations. For the four colour case, Haken and Appel were able to show that each map had at least one of 1936 different configurations. It was necessary to deal with each one of these eventualities to show that there was no counterexample.

One consequence of the work of Haken and Appel was a big debate as to whether such a proof using a computer was legitimate, as it was completely automatic once a program was constructed. But this is a false doubt. First of all, there are many situations in which proofs not using technology proved to be mistaken (Kempe's is one of them); some of them are extremely detailed and complex. Secondly, the solvers still needed to devise a strategy and a means of executing it, so creativity and ingenuity are still needed. Thirdly, proofs that are important tend to get checked over (often by graduate students learning the field) and corrected, and this is certainly true of computer proofs where programs can be refined and checking algorithms implemented. Fourthly, a result once established can often be found in other ways. A simpler proof of this theorem was found in 1997 and in 2005, Georges Gonthier, a Canadian who was on our country's first International Mathematical Olympiad team in 1981, proved the theorem using theorem-proving software.

There is an additional fact of local interest. Haken's daughter, Dorothea Blostein, is a professor of computer science at Queen's University and a reader of the *Frontenac News.*