THE ARITHMETIC GAMES

In the modern school curriculum, it is important not only for students to be fluent with the standard arithmetic number facts and algorithms, but also for them to have opportunity to express their creativity and ingenuity. Simple memorization is not enough to master mathematics; you need the experience and familiarity with its structure that only investigating a good problem can provide. When students become absorbed in a problem and work towards a goal, then they control the agenda and develop their skills with understanding and context.

This column will provide a few examples of open-ended investigations to encourage number play, either by individuals or groups. In some situations, paper and pencil computations are appropriate, in others, a calculator or computer is better. (Calculator use is not necessarily easier; students still have to figure out the best way for the calculator to deliver the information they want.) In particular, I encourage teachers of all grades to use ones suitable to their students and let me know (at *barbeau@math.utoronto.ca*) how they were received and whether they were valuable. In some cases, teams can compete to see who gets the best answer. I will report on any answers I receive and challenge readers to decide whether they are best possible.

1. The biggest and smallest differences. Using the digits 1, 2, 3, 4, 5, 6, 7, 8 each exactly once, create two numbers and subtract the smaller from the larger. For example, the two numbers might be 35876 and 214, and their difference is 35662. What is the largest possible difference you can obtain in this way? the smallest?

Answer the same question, but now using the nine digits 1, 2, 3, 4, 5, 6, 7, 8, 9.

2. Ratio extremes. As in the first investigation, form two numbers from the eight digits, but this time divide one into the other. What is the largest possible value of the quotient? What is the quotient that is closest to 1? Answer the same question with the nine nonzero digits.

3. Number bagels with products. Write the numbers from 1 up to 6 in a circle in some order, and form the sum of products of adjacent pairs of them. For example, we might arrange them in the order 3, 5, 2, 6, 1, 4 with the 4 next to the 3. The sum of the products of adjacent pairs is $15 + 10 + 12 + 6 + 4 + 12 = 59$. If we do this with different orders, what is the largest possible value of the sum? the smallest? Do the same problem with the numbers from 1 up to other positive integers.

4. Number bagels with differences. This is similar to the last investigation, except that you take the sum of the squares of the differences of adjacent pairs of numbers. In the example above, we would compute

$$(5-3)^2 + (5-2)^2 + (6-2)^2 + (6-1)^2 + (4-1)^2 + (4-3)^2 = 4 + 9 + 16 + 25 + 9 + 1 = 64.$$

What are the largest and smallest possible values of this sum? Is it possible for the sums in the third and fourth investigation to be equal for some ordering of the numbers? Do the same problem replacing 6 by some other positive integer.

5. The $4/n$ problem. We can write the fraction 4/5 as the sum of three distinct reciprocals $1/2 + 1/5 + 1/10$. It has been conjectured that any fraction $4/n$ whose numerator is 4 and whose denominator is any positive integer n exceeding 3 can be written as the sum of three distinct integer reciprocals. For how high a value of n can you do this? Within an hour, a group of secondary students working together should be able to discover general patterns for solving this with any number n which is not one more than a multiple of 24. For numbers of the form $n = 24k + 1$, while individual cases can be solved, no general pattern is known that covers all eventualities.

6. Making up to squares. Start with any positive integer. Multiply it by any number of distinct larger positive integers so that the product of all of them is a perfect square. For example, if we start with 5, we can multiply it by 10 and 18 to get the product

$$5 \times 10 \times 18 = 2^2 \times 3^2 \times 5^2 = 900 = 30^2.$$

We can actually do this with a smaller largest factor:

$$5 \times 8 \times 10 = 2^4 \times 5^2 = 400 = 20^2.$$

If we start with a perfect square, like 4, we can multiply it by the next square up to get a perfect square product: $4 \times 9 = 6^2$.

Taking each positive integer in turn, multiply it by any number of distinct larger integers so that the product is a perfect square in such a way that the largest factor in the product is as small as possible. This needs careful work and you could have teams competing with a point awarded to the team providing an example with the minimum largest factor. Is it possible for this minimum largest factor to be the same when you start with different integers?