Noughts-and-crosses

Most readers will be familiar with the game of Noughts-and-crosses or Tic-tac-toe, a paper-and-pencil game played on a 3×3 square grid of cells. There are two players, who play alternately. The first player places an X in one of the nine cells. The second player then places an O in one of the remaining eight cells. The play continues, each player putting her own symbol in a vacant cell until either (1) one of the players has succeeded in having one row, one column or one diagonal filled with her own symbol, or (2) nine moves have been completed to fill the grid and neither player has three of her symbols in a line. In case (1), the player with a line filled with her symbols is the winner; in case (2), the game ends in a draw.

Most of us have probably played this without thinking about it very much; with experience you probably have realized that with some care, you can prevent your opponent from winning and drive the game to a draw. But actually the game is quite significant mathematically and its structure can be explored in a rigorous way.

Noughts-and-crosses is an example of a two-person zero-sum game of pure strategy with perfect information. This means that there are two players; the gains of one player are equal to the losses of the other; there are no chance elements in the game so that its evolution relies on rational decisions by the players; at each stage in the game, both players have complete knowledge of everything that went on before. In this sense, noughts-and-crosses is in the same category as checkers or chess, but distinct from most card or dice games that involve elements of chance. Furthermore, bridge is not a game of perfect information because there are times in the game where a player has information not available to the others.

To begin with, let us clarify what is meant by a *strategy*. This is simply a prescription for each player to determine her moves. For example, a strategy for the first player in noughts-and-cross might begin: *I will first put X in the middle cell; if my opponent takes a corner cell, I will put my X in an adjacent corner; if she takes the middle of an edge, I will put my X in an adjacent corner;* $\cdots$. Likewise the second player will have a strategy, indicating to begin with how she will respond to each of the three possible moves (centre, corner, edge) of the first player. A *winning* strategy is a strategy by either player that will result in a win for that player regardless of whatever moves the other player makes. If neither player has a winning strategy, then each has a *blocking* strategy that prevents the other player from winning.

Solving the game means determining whether either player has a winning strategy, or whether both players have a blocking strategy.

For a simple game, it is possible to make a list of all possible strategies for the two players. Then we can make an important conceptual reduction in the game: we can consider the game as having one move for each player, namely **Pick a strategy.** A referee can then describe the game that would have been played and declare a result. In principle, we can do the same thing for checkers and chess. The problem is that for these two games the number of strategies is enormous and the largest computer in the world cannot list and examine them all. Thus, in the case of chess, we cannot have a computer look at the strategies and pick out either a winning strategy or a pair of mutually blocking strategies, thus rendering the game trivial. Programming a computer to play chess requires one to look at partial approaches of increasing cleverness, either projecting ahead a few moves and assigning a value to configurations or programming in some kind of "learning" mechanism.

Here are some things about noughts-and-crosses for you to explore:

(1) Does either player have a winning strategy? If so, what is it? If not, explain how each player can block the other.

(2) Suppose we stipulate that the first player cannot take the middle cell on the first move. Does either player have a winning strategy now?

(3) Suppose it is known in advance that the first move of the second player is to put O in the middle of an edge. Does this advantage the first player?

(4) Suppose that the first three moves (two X's and one O) are assigned at random. The O player is to play next. For which configurations does the first player have a winning strategy, the second player have a winning strategy, both players can block?