Summing cubes

Here is an interesting fact:

$$1^3 + 2^3 + 3^3 + 4^3 + 5^3 = 225 = (1 + 2 + 3 + 4 + 5)^2.$$

We can check this by simply doing the necessary arithmetic, but I want to look at it in a more insightful way. Consider a corner of the standard multiplication table consisting of those numbers in its first five rows and columns:

1	2	3	4	5
2	4	6	8	10
3	6	9	12	15
4	8	12	16	20
5	10	15	20	25

We will sum the numbers in this table in two different ways. The sum of the first row is 15. Each term in the second row is twice the corresponding number in the first row, so the second row sums to 2×15. Similarly, the third row sums to 3×15, the fourth row sums to 4×15 and the last row to 5×15. Altogether, the sum of all the numbers in the five rows is

$$1 \times 15 + 2 \times 15 + 3 \times 15 + 4 \times 15 + 5 \times 15 = (1 + 2 + 3 + 4 + 5) \times 15 = (1 + 2 + 3 + 4 + 5)^2.$$

Alternatively, we can sum along *gnomons*, this is, arrays of numbers in the shape of a backwards L. The first gnomon consists of 1 alone, the second of all the multiples of 2 up to $2^2 = 4$, the third of all multiples of 3 up to $3^2 = 9$, and so on. Thus, we can write the sum of all the entries as

$$
\begin{aligned}
&1 + (2 + 4 + 2) + (3 + 6 + 9 + 6 + 3) + (4 + 8 + 12 + 16 + 12 + 8 + 4) \\
&\quad + (5 + 10 + 15 + 20 + 25 + 20 + 15 + 10 + 5) \\
&= 1 + ((2 + 2) + 4) + ((3 + 6) + (6 + 3) + 9) + ((4 + 12) + (8 + 8) + (12 + 4) + 16) \\
&\quad + ((5 + 20) + (10 + 15) + (15 + 10) + (20 + 5) + 25) \\
&= 1 + 2 \times 4 + 3 \times 9 + 4 \times 16 + 5 \times 25 \\
&= 1^3 + 2^3 + 3^3 + 4^3 + 5^3.
\end{aligned}
$$

We conclude that the equation that opened this article is correct.

The advantage of this way of looking at things is that we can see that the number 5 is not particularly significant. A similar argument is available when 5 is replaced by another whole number, be it 3, 97 or 142857. We can write our discovery as

$$1^3 + 2^3 + 3^3 + \cdots + n^3 = (1 + 2 + 3 + \cdots + n)^2$$

where n is any positive integer whatsoever.

Another way of saying this is that the set $\{1, 2, \cdots, n\}$ has the property that the sum of its cubes is equal to the square of its sum. Are there other sets of integers with this property? If we ask for all the entries to be positive and distinct, the answer is *no*. However, if we allow repetitions, there are many examples. Here are some: $\{3, 3, 3\}.\{1, 2, 2, 4\}, \{1, 1, 1, 1, 1, 1, 5, 5\}$. Can you find others? What happens if you allow some integers to be negative?

Here is a neat way to find some examples. Pick any positive whole number; we will use 12. Write down all of its divisors: 1, 2, 3, 4, 6, 12. Beside each divisor, write the number of divisors that this divisor has: (1: 1), (2: 2), (3: 2), (4: 3), (6: 4), (12: 6). The set of numbers of divisors $\{1, 2, 2, 3, 4, 6\}$ has the property that the sum of cubes of its entries is equal to the square of their sum:

$$1 + 8 + 8 + 27 + 64 + 216 = 324 = (1 + 2 + 2 + 3 + 4 + 6)^2.$$

Try this with a number of your own; it works every time!