The ace rises to the top

From an ordinary deck of playing cards, remove the thirteen cards of any one of its suits, shuffle them and place them in a pile face up on the table. We use the usual convention that the ace (A) represents 1, the jack (J), 11, the queen (Q), 12, and the king (K), 13. Look at the card at the top; if it is A, then we are done. If it is anything else, remove that many cards from the top and deal them back on the pack in the opposite order.

For example, suppose that the thirteen cards in order from the top are

$$6 \quad 7 \quad 8 \quad J \quad 5 \quad Q \quad 9 \quad A \quad 3 \quad 4 \quad 2 \quad K \quad 10.$$

Taking off the top six cards and dealing them back yields

$$Q \quad 5 \quad J \quad 8 \quad 7 \quad 6 \quad 9 \quad A \quad 3 \quad 4 \quad 2 \quad K \quad 10.$$

Repeat the process. In the example, the next move is to take off the top twelve cards and deal them beck to get

$$K \quad 2 \quad 4 \quad 3 \quad A \quad 9 \quad 6 \quad 7 \quad 8 \quad J \quad 5 \quad Q \quad 10.$$

When you perform this several times, it appears that eventually the ace rises to the top of the pack. Can we see whether this always occurs?

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It might be worth repeating the process several times and watching closely what happens. This will give insight into the general situation. For example, if the king appears at the top, it is shuffled to the bottom at the next move. From then on, it cannot be reached. However, it may happen that the king is never at the top. It appears that the cards with larger numbers tend to congregate towards the bottom of the pack while the smaller numbers tend to rise towards the top.

If perchance the ace never rises to the top, then this means that the process continues indefinitely. We argue that this does not happen. The strategy is to suppose that we can continue making moves forever and deduce from this something that is patently false. This type of argument is called a *contradiction argument*, or, if you want to be snooty, *reductio ad absurdum.*

We make a key observation. There are only finitely many possible arrangements of the thirteen cards (6227020800 to be precise), so if we continue long enough, we come to an arrangement that has occurred before and the process cycles. Look at the cards that come to the top; one has the largest value that occurs during the repeating cycle.

Suppose, for sake of argument, that card is a ten. Then on the next move, the ten will be shuffled into the tenth position from the top. Afterwards, it will never change position. Since a jack, queen or king never appears at the top thereafter, the ten will never be in the shuffled section. It cannot be reached directly because we cannot count down ten cards to it, and if a smaller card is at the top, it again cannot be reached. This means that, contrary to our assumption, it cannot come to the top when the cycle repeats.

This contradiction implies that our assumption that the ace never comes to the top is incorrect.

There are a number of paths for further invbestigation, especially if you can harness the power of a computer. What arragements of the cards will bring the ace to the top on the first move? the second move? the third move? What arrangement of the cards will require the largest possible number of moves to bring the ace to the top?