Matchmaking

In 2012, the Nobel Prize in Economics was awarded to Lloyd Shapley and Alvin Roth for the theory of stable allocations and the practice of market design. One of the publications cited by the Swedish Royal Academy of Sciences was a 1962 paper by David Gale and Lloyd Shapley entitled "College Admissions and the Stability of Marriage", published in the *American Mathematical Monthly*, a widely read expository journal. This paper is nice illustration that mathematical progress does not always involve deep and technical work, but sometimes simple results where the achievement is recognition of its applicability in an important area and its ability to crystallize seminal ideas. It does not involve any equations nor require any background knowledge. It is therefore accessible to ordinary citizens.

Imagine a village in which there are equal numbers of men and women who have to be married off. The matchmaker has to ensure that the resulting marriages are stable; that is, there is never a situation in which a man prefers some woman over his assigned wife while at the same time this woman prefers him over her assigned husband. In other words, any attempts to defect from a marriage are rebuffed. (You can see that college admissions involve a more complicated form of the same thing: candidatee apply to colleges and you want to end up with an assignment where no candidate will be able to turn down an offer to accept one from elsewhere.) Gale and Shapley at first wondered whether such an assignment was indeed possible, but were able to devise a procedure that would achieve it.

The matchmaker asks each man to list in strict order of preference all the women, and each woman to do likewise with the men. The matchmaking proceeds in a number of rounds. In the first round, each man proposes to the woman at the top of his list. If every woman receives a proposal, the marriages are made and each man, having his first choice, will be faithful. However, if not every woman gets a proposal, then some will have more than one proposal. A woman receiving at least one proposal will keep on a string the one she prefers the most and reject all the others. The rejected men will participate in the second round. In this round, each one of them strikes from his list the woman who has rejected him and proposes to the next preferred. Upon receiving a proposal, a woman looks over all the prospects, including anyone that might be on her string, and accepts the most preferred, rejecting all the others. We go on to round three, in which all the loose men strike off their lists those who have rejected them and propose to their next choices. This process continues for as many rounds as necessary for there to be no further rejections.

We need to establish two things. First, we have to be sure that the process terminates. Secondly, we have to argue that it produces a set of stable marriages. For the first, note that there are a finite number of names on all the lists. Every time we need a new round, it is because someone's name gets crossed off of a list. This cannot go on forever. When the last round is reached, each man has finally been accepted by a different woman.

Now we come to the meat of the situation. Suppose that, say, Al is married to Ann, and that Bob is married to Barb. Suppose that Al prefers Barb to his own wife. Then Barb would have been higher on Al's list than Ann, and so Al would have proposed to Barb and been rejected by her before proposing to Ann. Why would Barb have rejected Al? Either she would have already accepted Bob or Bob would have come along later, supplanting any other suitors. In either case, she would have preferred Bob to Al, and so has no incentive to defect.

That's all there is to it. The assignment is not necessarily the only one possible. For example, there is a symmetrical process in which the women do the proposing, and this will in general lead to a different assignment. Notice that the success of this assignment depends on the assumption that the preference orders, once made, are never alterned. Of course, in real life, things are not so cut and dried.