The richness of mathematics

In response to my December 5, 2013 column on Pythagorean triples, I received an email from Doug Nuttall, a professional engineer in Elphin. He attached to his note a table of Pythagorean triples, sets of three numbers such as $(3, 4; 5)$ for which the square of the largest is equal to the sum of the squares of the smallest two. There is a formula that gives all such triples where the greatest common divisor of the three numbers is 1:

$$(m^2 - n^2, 2mn; m^2 + n^2),$$

where m and n are whole numbers, one of which is even and the other odd.

Nuttall's table reminded me that mathematical discovery, like scientific discovery in general, depends on amassing data, peforming experiments and looking for patterns. However, unlike other sciences, in mathematics we can prove that things are true beyond any doubt.

One of the questions I left the reader with in my column was to find Pythagorean triples other that $(3, 4; 5)$ and $(21, 20; 29)$ for which the smallest numbers differed by 1. One way to do this is to start with two sequences of whole numbers:

$$(0, 1, 2, 5, 12, 29, 70, 169, 408, 985, \cdots)$$

and

$$(1, 1, 3, 7, 17, 41, 99, 239, 577, 1393, \cdots).$$

Both sequences have the property that any term after the second is twice the previous term plus the one before it. For example; $70 = 2 \times 29 + 12$. Moreover, by taking the difference or the sum of two consecutive terms of the first sequence, we get the terms of the second sequence.

Let me zhow you how these sequences are implicated with our Pythagorean examples, and then I will leave it to you to proceed further.

First, $(3, 4; 5)$: $3 = 1 \times 3$; $4 = 2 \times 1 \times 2$; $5 = 1^2 + 2^2$; $5 - 4 = 1^2$; $5 + 4 = 3^2$.

Then, $(21, 20; 29)$: $21 = 3 \times 7$; $20 = 2 \times 2 \times 5$; $29 = 2^2 + 5^2$; $29 - 20 = 3^2$; $29 + 20 = 7^2$.

The next in line is $(119, 120; 129)$: $119 = 7 \times 17$; $120 = 2 \times 5 \times 12$; $169 = 5^2 + 12^2$; $169 - 120 = 7^2$; $169 + 120 = 17^2$.

The two sequences occur in another setting. One of the most ancient mathematical results is that you cannot find a positive square whole number that is double the square of another whole number. But there are many cases where a square differs from twice another square by 1. For instance, $3^2 = 2 \times 2^2 + 1$ and $7^2 = 2 \times 5^2 - 1$. It turns out that such examples fall into a pattern, and the nuumbers involved are corresponding terms of the sequences: $(1, 1)$, $(3, 2)$, $(7, 5)$, $(17, 12)$, $(41, 29)$, and so on.

In the first sequence, the sum of the squares of two consecutive entries is equal to a later term in the sequence. Thus, $5^2 + 12^2 = 169$. If you take any four consecutive terms of this sequence, then the product of the outer two differs from the product of the inner two by 1. The first two sequences are interlinked in a number of ways; for example adding corresponding terms of the two sequences leads to the next term of the first one. There are lots more of these treasures for you to find.

There is a third sequence formed by taking the product of corresponding terms of the foregoing two sequences:

$$(0, 1, 6, 35, 204, 1189, \cdots).$$

I will note simply that $29 = 35 - 6$ and $20 + 21 = 35 + 6$, and leave other discoveries to you.

There is another sequence that is even more brimful of relationships, the Fibonacci sequence, first studied about 800 years ago. It is so fecund that there is a journal, *The Fibonacci Questerly* devoted to it.

In this case, you start with 0 and 1 and add two consecutive terms to get the next:

$$(0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, 987, \cdots).$$

What can you observe about its entries?

Sequences of integers arise in all sorts of mathematical situations, so it was inevitable that they should be tabulated and their properties catalogued along with contexts in which they occur. The Online Encyclopedia of Integer Sequences ($http://oeis.org$) is the standard reference for anyone wanting the dope on any sequence they may encounter.