ARITHMETICAL FAMILIES

An important characteristic of successful arithmetic students is an openness to structure. This involves an ability to notice patterns that occur, articulate what they are and to generalize them where possible. Once they are introduced to algebra, then they should be able to describe the pattern by an algebraic equation, and to demonstrate its validity through algebraic manipulation. However, as some of the patterns below demonstrate, it is not always necessary to access algebra to understand what is going on.

The following list is offered to teachers and their students in the hopes that it may whet their appreciation for some of the byways of mathematics and highlight situations in which algebra can be used both as a language to express relationships and as a technique for establishing a result that has infinitely many instances.

§1. Families of products.

1.1. Two sets of squares.

$$4 \times 4 = 16$$
$$34 \times 34 = 1156$$
$$334 \times 334 = 111556$$

$$7 \times 7 = 49$$
$$67 \times 67 = 4489$$
$$667 \times 667 = 44489$$

There are different ways of seeing why things work out. You can compute the squares using long multiplication and pay attention to how the digits in the partical product arrange themselves. Alternatively, you could note that, say, $3 \times 34 = 102$, work out 102^2 and divide by 9; this will also reveal a pattern.

2. Relationships involving consecutive integers.

2.1. Sets of adjacent integers with the same sum.

$$1 + 2 = 3$$
$$4 + 5 + 6 = 7 + 8$$
$$9 + 10 + 11 + 12 = 13 + 14 + 15$$

In adding consecutive numbers, it is helpful to observe that the two numbers the same distance from the ends of the summands have the same sum. When the number of summands is odd, then the sum is the middle term times the number of summands. When the number of summands is even, the sum is the sum of the middle two terms multiplied by half the number of summands.

To understand why this pattern works, we can check a particular case and see that a similar procedure can be carried out for the other cases. For example, the third equation can be verified by noting that

$$(15 - 11) + (14 - 10) - (13 - 9) = 3 \times 4 = 12.$$

2.2. Sets of adjacent squares with the same sum.

$$3^2 + 4^2 = 5^2$$
$$10^2 + 11^2 + 12^2 = 13^2 + 14^2$$
$$21^2 + 22^2 + 23^2 + 24^2 = 25^2 + 26^2 + 27^2$$

A nonalgebraic exegesis is also available here. For the second equation, look at $(13^2 - 12^2) + (14^2 - 11^2)$, and factor the difference of squares.

2.3. Sums of consecutives.

$$\begin{aligned}
1 &= 1 = (0+1)(0^2+1^2) \\
2+3 &= 5 = 1^2+2^2 \\
4+5+6 &= 15 = (1+2)(1^2+2^2) \\
7+8+9+10 &= 34 = 2 \times (1^2+4^2) \\
11+12+13+14+15 &= 65 = (2+3)(2^2+4^2) \\
16+17+18+19+20+21 &= 111 = 3 \times (1^2+6^2)
\end{aligned}$$

$$1 = 1^4 \qquad 1 + 15 = 2^4 \qquad 1 + 15 + 65 = 3^4$$

$$5 = 1 + 2^2 = 1 \times (1^2 + 2^2)$$
$$5 + 34 = 3 + 6^2 = (1+2)(2^2+3^2)$$
$$5 + 34 + 111 = 6 + 12^2 = (1+2+3)(3^2+4^2)$$

§3. Sequences

Sequences of integers determined according to some rule have can have many interesting properties. The grandaddy of them all is the *Fibonacci sequence* which starts with the two terms 0 and 1; each subsequent term is the sum of its two predecessors: 0, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 155, ... There are so many interesting features about this sequence (for example, the sum of the squares of two consecutive entries appears later in the sequence) that there is a journal, *The Fibonacci Quarterly*, in its honour. However, this is not the only sequence that can fascinate.

3.1. A sequential trio.

Below are three related sequences. Determine a rule that governs how each term is found from its predecessors and write the next few terms of the sequence. The sequences are also interrelated with each other. To get into it, check out the sum and difference of consecutive terms as well as the squares of terms in the sequence. Return to them often and you will see more interesting features.

n	a_n	b_n	c_n
0	0	1	0
1	1	1	1
2	2	3	6
3	5	7	35
4	12	17	204
5	29	41	1189
6	70	99	6930