SIMPLICITY AND POWER

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Knockout tournament

1. Tennis at Wimbledon. Each year, at Wimbledon in England, there is an open tennis tournament. It starts with 64 players who are paired into 32 matches; the 32 winners proceed to the next round who play similarly; 16 proceed to the third round, and so on until the final champoinship round in which two players square off. How many matches are played in the tournament?

Pairing off

2. Dominos on a chessboard. Suppose that we have an 8×8 chessboard and 32 dominos, each of which covers exactly two cells. It is easy to see how you can cover the chessboard with dominos without any two overlapping and leaving no uncovered cells. Now put a coin on two diagonally opposite corner cells. Can the remainder of the chessboard be covered with 31 dominos?

3. Water and wine. You are given a litre of water in Jug A and a litre of wine in Jug B. A small cup of wine is transferred from Jug B to Jug A, and the mixture thoroughly stirred. Then exactly the same amount is transferred back from Jug A to Jug B, so that each jug still contains a litre. Is there more or less wine in Jug A than there is water in Jug B?

Reflection

4. Watering the plants. A man's garden is on the same side of the river as his house, but some distance away. His house is 100 metres from the river, the garden is 50 metres from the river and the garden is 200 metres downstream from the house. Assume the river is straight. Every morning, he takes a pail from his house, fills it at the river bank, and then waters his garden. Where on the bank should he fill his jug so as to minimize the distance that he has to walk?

5. Maintaining the lead. In the 2014 provincial election, Liberal Mike Colle carried the Eglinton-Lawrence riding with 22855 votes against 18836 votes for all of his opponents put together, attaining a majority of 54.82% of the votes cast. Now imagine that it is election night and we are listening to a live telecast that is following the poll numbers as they are counted. It may happen during the evening that the fortunes of the candidates vary with one leading and now another.

So the question is this. What is the chance that as the votes are being tallied, at every stage, Colle has more votes than all of his opponents collectively? For this to happen, the first two ballots have to be for Colle (if for example, he wins the first ballot and loses the second, then there is a tie). The chance that the first ballot is for Colle is 22855/41691 (the denominator being the total number of votes counted). So the chance of his maintaining a lead throughout the count is certainly less than this.

To cut the situation down to size, imagine that there are two candidates X and Y and five votes, candidate X getting three of them and candidates Y getting the other two. Then candidate X maintains the lead through out if the ballots are counted in the order XXXYY, and XXYXY, and loses the lead at some point if the ballots are counted in the order XXYYX, XYXXY, XYXYX, XYYXX, YXXXY, YXXYX, YXYXX, YYXXX. Since each of these orders is equally likely and the lead is maintained for two out of the ten orders, the chance is 1/5, of 1 in 5.

Going into deficit

6. Getting around the track. Image that you have a race car on a circular track. Enough gas to enable the car to get around the track exactly once is distributed into several cans which are placed at arbitrary places along the track. It is always possible to position the car at some starting point so that it will make it around the track, that is, for it to arrive at the next gas can and replenish his tank without running out of gas.

7. Sharing the coconuts. Five pirates on a desert island gather a huge pile of coconuts. When they finish, it is late, so they decide to go to bed and share the coconuts equally among them the next morning.

During the night, one pirate wakes up, and mistrusting his colleagues, splits the coconuts into five equal piles with one left over, which he throws to a nearby monkey, conceals his pile and restacks the rest into a single pile. He goes back to sleep. Later, a second pirate wakes up, and does the same thing to the remaining pile, throwing one coconut left over to the monkey. Throughout the night, each pirate in turn wakes up and goes throw the same process, the monkey getting the odd coconut each time. Finally, they all wake up and partition the (now much smaller) pile of coconuts, which again splits equally except for an odd coconut that goes to the monkey. How many coconuts are there in the pile?

Of course, there are many possible answers, so we are going to ask for the smallest. For the manoeuvre to be possible, there must be one more than a multiple of 5 in the pile: 6, 11, 16, and so on. For the first two rounds to be possible, we need to leave one of these numbers in the pile after the first pirate has taken away his portion. We can check that having 21 coconuts will work: the first pirate takes 4 coconuts, gives one to the monkey and leaves 16. For three rounds, we will need with a larger number, and the problem now seems to be quite complex. However, there are six rounds, so how can we solve the problem?

The pigeonhole principle

The next few problems depends on a very simple idea, that if you take any sets of objects and split them into mutually exclusive categories, and if the number of objects exceeds the number of categories, then some category must have at least two objects assigned to it. We can extend this to say that if the number of objects exceeds two (three, four, ...) times the number of categories, then some category gets at least two (three, four, ...) objects.

8. Equal sum sets among ten numbers. Pick any ten positive whole numbers less than 100 at random. For example, turing to a page of the telephone directory and taking the last two digits of the first ten telephone numbers, I get: 72, 75, 37, 71, 76, 28, 80, 12, 51, 20. I can say with absolute certainty that there will be two non-overlapping subsets of these numbers that have the same sum. With a little effort we can check this out: 72, 28 and 80, 20 have the same sum. If you think this is a set up, I invite you to change any of the numbers in the set or try it with your own set; you will still find two subsets with the same sum. Why is this so?

9. Increasing or decreasing subsets. Again, take any ten numbers (they do not have to be less than 100 this time). Then you can always delete all but four of them in such a way that the four remaining numbers with either increase from left to right or decrease from left to right. In the example of the previous problem, we can delete 75, 71, 76, 80, 51, 20 and get the decreasing sequence 72, 37, 28, 12. Why does his always happen?

10. The game of SIM. This is a game for two players. Place six dots on a page.

One player has a red pencil and the other a green pencil; they play alternately, each joining a pair of dots that has not previously been joined. The winner of the game is the first player who has managed to have three of the dots, each pair of which is joined by a line segment of his colour. You might try this out with a friend. You will find that the game is never drawn; that is, it is possible to go through a complete set of fifteen moves without one of the players completing a triangle with his own colour. Why is this?

Equatorial weather

11. Temperature on the equator. Suppose that we go around the equator and at each point take the temperature. It turns out that at every instant, there are two antipodal points on the equator that have exactly the same temperature. ("Antipodal" means that they are exactly on opposite sides of the earth, with their meridians differing by 180 degrees.)

The solutions

1. One way to obtain this result is to simply add the number of matches from each round: $32 + 16 + 8 + 4 + 2 + 1 = 63$. But there is a quicker way in. Each match results in one loser, and every player eventually loses once except for the champion. Therefore there are 63 losses and 63 matches.

2. The answer is *no.* Each domino must cover a black and a white cell, so that 31 dominos must cover the same number of cells of each colour. However, the coins cover two cells of the same colour, so that the number of black cells remaining is not equal to the number of white cells remaining.

3. The amount of wine in Jug A is the same as the amount of water in Jug B. Since each jug has the same amount of fluid in the beginning and at the end, the amount of displaced from one jug must be replaced by an equal amount displaced from the other.

4. He locates a point in the opposite side of the river that is the image of the location of the garden reflected in the river. We can pair off paths from the house to this point with paths from the house to the garden that touch the river; the first part of the path is the same for both, but the final part of one path is a reflection of the final part of the other. The shortest path from house to the point on the opposite side of the river is straight, so to find the shortest path to the garden, follow this path to the river, and then go straight to the garden.

5. This looks like it would be a pretty complicated problem, but by a suitable pairing we can get the answer quickly. We divide the counting order into three categories: (1) The winning candidate maintains the lead throughout; (2) The first vote counted is for the winning candidate, but he loses the lead later during the count; (3) The first vote counted is for the losing candidate(s). We claim that the number of possibilities is the same in the second and third categories.

For any count in case (2), there must be a first point at which the winning candidate loses his majority. If X is the winner and Y is the loser, the count will begin XX...Y... and at some point after the first ballot is counted, the candidates will have their first tie. If up to this point, we replace each X by Y and each Y by X, we get a count in category (3).

Likewise, for each path in category (3), while Y is initially leading, there is a first place where they tie. Interchanging X and Y up to this point gives us a count in category (2). Thus there is a one-one relationship between the counts in category (2) and the counts in category (3).

One way to formulate this is, that as the count proceeds, add 1 for each vote for candidate X and -1 for each vote for candidate Y. A tie occurs when the number of occurrences of 1 is equal to the number for -1. What we are doing is interchanging the two numbers up to the stage where the first tie occurs.

Now let x be the number of counts where candidate X is counted first, and y the number of counts where candidate Y is counted first. Then y is the number of counts in each of categories (2) and (3). Then the total number of possible counts is $x + y$, and the number of counts in which X always leads is $x - y$. The chance that X will always lead is $(x - y)/(x + y)$. In the Eglinton-Lawrence example, this turns out to be 4019/41691, or about 0.095, less than one chance in ten.

6. Let us start with any can of gas placed around the track. This of course may not be the right one, and it may be that at some point we run out of gas before the next can is reached. However, let us allow the car to run a deficit and get to the next can, where at least some of the gas can be replenished. We continue in this way around the track until we return to the starting point, where the amount of gas left in the car will be 0 (with any deficits incurred along the way being made up, since we have just enough gas to get around the track once). If we want to know where we really should have started, it should be the place where we had the greatest deficit.

7. Let us take two possible values for the number of coconuts; call them M and N. Since there is one odd cococut after the first round, $M - N$ must be a multiple of 5. Let us go to the second round. The number of coconuts in the pile to be split is, respectively, $\frac{4}{5}(M - 1)$ and $\frac{4}{5}(N - 1)$. Since each of these numbers is one more than a multiple of 5, their difference $\frac{4}{5}(M - N)$ is a multiple of 5, which means that $M - N$ must be a multiple of 25. Similar to get to the the third round, $M - N$ must be a multiple of $125 = 5^3$. In order to have the right number of coconuts to get to the sixth round, $M - N$ must be a multiple of $5^6 = 15625$. Finally, if M is a possible number, then adding to M a multiple of 15625 will give us another possible number.

So we look for a simple possible number, and the one we pick is -4: that's right, a negative number. Suspend belief for a moment and let us go through the routine. The first pirate sorts the coconuts into piles of -1 coconuts each (for a total of -5 coconuts, and throws the remaining one to the monkey). After he takes his -1 coconut, there remain -4 coconuts for the next round. Since this is what we started with, we can continue indefinitely. However, for the problem to be viable, we need a positive number of coconuts, so we simply add 15625 to this, so that the pirates started with a pile of 15621 coconuts.

Let us check this. The first pirate takes 3124 coconuts and leaves 12496. The second pirate takes 2499 coconuts and leaves 9996. The third pirate takes 1999 coconuts and leaves 7996. The fourth pirate takes 1599 coconuts and leaves 6396. The fifth pirate takes 1279 coconuts and leaves 5116. Finally when they all wake up, they each get 1023 coconuts and throw one to the monkey.

8. To solve this, we need to apply the Pigeonhole Principle. The "pigeonholes" will be the subsets of the ten numbers and we will look at the sum of the numbers in each of these sets. To form a subset of the ten numbers, for each number we have a choice: *Take it or leave it.* There are $2^{10} = 1024$ possible ways of doing this, but this includes the "empty" set which contains no number. So there are 1023 subsets.

Since each set has no more than ten members and each member is less than 100, the sum of the numbers in each set must be less than 1000. Since there are fewer possible sums than sets, there must be two sets with the same sum. However, it is possible that they might overlap. Simply remove the numbers common to both sets to get two disjoint sets with the same sum.

9. Suppose that we have the ten numbers in a row. For each number, we form the longest increasing sequence that begins with that number that we can form by deleting some of the numbers in the set. If there is more than one candidate for this longest sequence, pick any one. In this way we have ten increasing sequence (the last of which will necessarily have only one member as no other number follows the last one in the list). If one of these sequences has four or more members, then we are done.

Otherwise, all the sequences have at most three members. There must be four sequences with the same number of members. The first numbers in each of these four sequences must be in decreasing order; if, say the second is less than the third, then the second number along with the sequence attached to the third would give a longer sequence for the second.

Let us check this out with the sequence: 72, 75, 37, 71, 76, 28, 80, 12, 51, 20. We get the ten subsequences: (72, 75, 76, 80), (75, 76, 80), (37, 71, 76, 80), (71, 76, 80), (28, 80), (12, 51), (51), (20). We have found two increasing sequence with four members, so we are done.

Let us try it with the same sequence with the number 76 replaced by 66: 72, 75, 37, 71, 66, 28, 80, 12, 51, 20. The ten subsequences are now: (72, 75, 80), (75, 80), (37, 66, 80), (71, 80), (66, 80), (28, 51), (80), (12, 51), (51), (20). We find that there are four subsequences of length 2: (75, 80), (71, 80), (66, 80), (28, 51). Their first numbers (75, 71, 66, 28) give us a decrerasing sequence of length 4.

10. Suppose that the game goes to the maximum number, 15, of moves without there being a winner. Then every pair of dots is joined by a line. Pick one of the dots; call it A. It is joined to five other dots. Since there are only two colours, at least three of the joining lines must be of the same colour, say red. Suppose that A is joined to B, C, D by red lines. If any of the pairs (B, C), (C, D), (B, D) are joined by a red line, then the red player is the winner. Otherwise, each of these three pairs musst be joined by a green line, and the green player is the winner.

11. Let us make a graph of the difference in the temperature between any point on the equator and its antipode. Start at the prime meridian (0 degrees) and plot the temperature at 180 degrees minus the temperature at 0. Do this for every meridian up to 180 degrees. At 180 degrees, we are plotting the termperature at 0 degrees minus that at 180 degrees, *i.e.* the negative of what we started with. Since the temperature difference varies continuously, there must be someplace where this difference is 0.