FACTORING DIFFERENCE OF SQUARES

1. Let $a > b > 0$. By means of a diagram, in which one slices and rearranges the gnomon (L-shaped region) when a square of side length b is removed from a square of side length a, illustrate the identity

$$a^2 - b^2 = (a+b)(a-b) \quad .$$

2. The identity $a^2 - b^2 = (a+b)(a-b)$ gives an approach for mental multiplication. Suppose that we wish to multiply two positive integers u and v, particularly if they are of the same parity. Show how we can find suitable positive numbers a and b such that u and v can be written in the form $a+b$ and $a-b$. Then the product of u and v is the difference of squares $a^2 - b^2$. All that is needed is to know how to compute squares.

   (a) Use the suggested method to mentally compute 37×43, 22×28, 46×54.

   (b) By making use of the identity $(10t + u)^2 = 100t^2 + 20tu + u^2$, devise a mental algorithm for quick computation of squares of two-digit numbers.

3. Let n be a positive integer. The number $n!$ is defined to be the product of the first n positive integers.

   (a) Verify that $1! = 1$, $2! = 2$, $3! = 6$, $4! = 24$, $5! = 120$.

   (b) By factoring the difference of squares in each case and then writing each factor as a product of smaller factors (no multiplications are necessary) and recomposing some products where needed, verify the following numerical results:

$$4! = 5^2 - 1^2 = 7^2 - 5^2$$

$$5! = 11^2 - 1^2 = 13^2 - 7^2$$

$$6! = 27^2 - 3^2 = 28^2 - 8^2 = 29^2 - 11^2$$

$$7! = 71^2 - 1^2 = 72^2 - 12^2 = 73^2 - 17^2 \quad .$$

   (It has been conjectured, but never proved, that $4!$, $5!$ and $7!$ are the only numbers of the form $n!$ that are 1 less than a perfect square.)

   (c) Verify that $13! = 78912^2 - 288^2 = 112296^2 - 79896^2$.

   (d) Write $17!$ as the difference of two integer squares.

   (e) Prove that $n!$ cannot be written as the difference of two integer squares if and only if $n = 2$ or $n = 3$.

4. You are equipped with a pocket calculator that can display integers up to eight digits long. Let

$$a = 4565486027761$$

$$b = 1061652293520$$

$$c = 4687298610289 \ .$$

   (a) Verify that (a, b, c) is a pythagorean triple, *i.e.*, $a^2 + b^2 = c^2$. (Hint: Try factoring either $c^2 - b^2$ or $c^2 - a^2$.)

   (b) Verify that c is a square.

   (c) Verify that $a + b$ is a square.

5. (a) Make a list of the numbers between 1 and 20 inclusive, expressing each, when possible, as the difference of the squares of two integers. Make a conjecture as to a criterion under which such a representation of a difference of squares in possible. To formulate or check your conjecture, you might want to extend your list.

(b) Suppose that an integer n can be written in the form $u^2 - v^2$. By factoring the difference of squares, show that it is necessary that n can be written as the product of two integers of the same parity (*i.e.*, both even or both odd).

(c) Suppose that n can be written in the form hk where h and k are integers of the same parity. Prove that there are integers u and v for which $h = u + v$ and $k = u - v$, and deduce that n can be written as the difference of two squares.

(d) Show that any odd number can be written as the product of two odd integers.

(e) Show that each integer divisible by 4 can be written as the product of two even integers.

(f) Show that each integer that leaves a remainder 2 upon division by 4 cannot be written as the product of two integers that are both odd or both even.

(g) Use the previous sections to formulate and prove a necessary and sufficient condition that an integer can be expressed as the difference of two squares.

6. Prove that 4 is the only perfect square that is 1 more than a prime number.

7. Prove that two nonzero perfect squares of integers cannot differ by 1.

8. (a) Make a table listing the numbers from 1 to 20, and beside each entry write the product of that number and the next higher number and four times this product. Formulate and prove a conjecture.

   (b) Prove that the product of two consecutive positive integers is never a perfect square.

9. Prove that the product of two consecutive odd integers is not a perfect square.

10. Prove that the product of two consecutive even integers is not a perfect square.

11. (a) Look at the difference between the squares of two consecutive integers, working up from 1. What do you observe?

    (b) What is the sum of the first million positive odd integers?

12. Prove that

$$1000^2 - 999^2 + 998^2 - 997^2 + \cdots + 2^2 - 1^2$$

    is equal to the sum of the first thousand positive integers. Formulate and prove a generalization.

13. A straight metal rail is 2400 cm long and is firmly fixed at both ends. On a warm day, its length increases to 2402 cm and so it buckles. Assuming that its final shape is closely approximated by an isosceles triangle, determine how far from the ground its midpoint rises.

14. It is possible to arrive at the factorization of $x^2 - y^2$ by the technique of adding in an extra term and subtracting it out again:

$$x^2 - y^2 = x^2 - xy + xy - y^2 = x(x - y) + y(x - y) = (x + y)(x - y) \quad .$$

    (a) Apply this technique to determine a factorization of $x^3 - y^3$.

    (b) Consider the factorization of $x^4 - y^4$. Use the technique just described to write this polynomial as the product of $x - y$ and another polynomial. Check your answer by factoring $x^4 - y^4$ as a difference of squares, and then factoring a second difference of squares.

    (c) Factor $x^n - y^n$ where n is a positive integer. Check the results of your method for $n = 5, 6, 7$.

    (d) Determine a method for factoring $x^3 + y^3$ and $x^5 + y^5$, and generalize it to a method for factoring $x^n + y^n$ where n is any odd positive integer.

15. (a) By adding to and subtracting from $x^4 + 4$ a term which is a square, factor $x^4 + 4$ as a product of two polynomials with integer coefficients.

(b) Write $x^4 + 1$ as the product of two quadratic factors (in this case, not all of the coefficients will be integers).

16. (a) Look at the product of four consecutive integers in several cases. Do you notice anything of interest?

(b) Determine in general that the product of four consecutive nonzero integers is never a perfect square.

17. (a) Let p and q be two distinct odd primes. Prove that the number

$$(pq + 1)^4 - 1$$

has at least four distinct prime divisors.

(b) Suppose that $p = 2$ and q is an odd prime. Does the conclusion of (a) still hold?

18. An ordered set (a, b, c) of three positive integers a, b, c is called a *pythagorean triple* if it satisfies $a^2 + b^2 = c^2$. The name derives from the fact that pythagorean triples represent side lengths of right triangles. A pythagorean triple is *primitive* if the greatest common divisor of its members is 1.

(a) Verify that $(3, 4, 5)$ is a primitive pythagorean triple, but $(24, 45, 51)$ is a pythagorean triple that is not primitive.

(b) There exist pythagorean triples (a, b, c) for which $c = b + 1$; an example is $(5, 12, 13)$. Prove that each such triple is primitive and that the smallest number of each such triple must be odd. Determine 7 such triples.

(c) Show that we can determine a pythagorean triple (a, b, c) for which a is any number we choose except for 1 and 2.

(d) Prove that, if (u, v, w) is a pythagorean triple, then there is an integer k and a primitive pythagorean triple (a, b, c) for which $u = ka$, $v = kb$ and $w = kc$.

(e) It is possible to give a formula which will churn out all pythagorean triples. Suppose that (a, b, c) is a primitive pythagorean triple. Prove that exactly one of a and b is odd. Without loss of generality, let us suppose that a is even. Verify that $a^2 = (c - b)(c + b)$ and that the greatest common divisor of $c - b$ and $c + b$ is 2. Deduce that $c + b = 2m^2$ and $c - b = 2n^2$ for some integers m and n.

(f) Prove that (u, v, w) is a pythagorean triple if and only if there are integers k, m, n for which

$$u = 2kmn, \qquad v = k(m^2 - n^2), \qquad w = k(m^2 + n^2) \quad .$$

(You have to show, first of all, that if u, v, w have this form, then (u, v, w) is a pythagorean triple. Secondly, you have to argue, possibly using (d) and (e), that if (u, v, w) is a pythagorean triple, then k, m, n can be found as required.)

(g) (For those with trigonometry) Let (m, n) be a pair of integers. Then from (f) it can be seen that

$$\left(\frac{m^2 - n^2}{m^2 + n^2}, \frac{2mn}{m^2 + n^2} \right)$$

is a point on the circumference of the unit circle with centre at the origin. Show that the angle between the radius vector from the origin to this point and the $x-$axis is equal to twice the angle whose tangent is n/m.
On the other hand, if $\tan\theta = n/m$, determine $\sin 2\theta$ and $\cos 2\theta$. deduce that there is a one-one correspondence between primitive pythagorean triples and angles whose tangent is rational.

19. (a) By factoring the left side as a difference of squares, show that

$$\left(\frac{x + y}{2} \right)^2 - \left(\frac{x - y}{2} \right)^2 = xy \quad .$$

(b) Use (a) to show that, when x and y are nonnegative, then

$$\sqrt{xy} \leq \frac{x+y}{2} \quad .$$

When does equality occur?

(c) The sum of two positive integers is 56. What is the largest possible value of their product?

20. Consider the following numerical equations:

$$\begin{gathered}
3^2 + 4^2 = 5^2 \\
10^2 + 11^2 + 12^2 = 13^2 + 14^2 \\
21^2 + 22^2 + 23^2 + 24^2 = 25^2 + 26^2 + 27^2 \\
36^2 + 37^2 + 38^2 + 39^2 + 40^2 = 41^2 + 42^2 + 43^2 + 44^2
\end{gathered}$$

Suggest a generalization and verify these numerical equations in a way that will convince you that the generalization also is valid.